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PERFORMANCE COMPARISON OF SPREAD AND BERNSTEIN BASIS IN THE SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATION VIA COLLOCATION METHOD

Olagunju A. S.[1], Joseph F. L.[2] and Atanyi Y. E. [3]

[1, 3] Department of Mathematics, Federal University of Lafia, Nigeria.
[1, 2] Department of Mathematical Sciences, Bingham University Karu, Nigeria.

Abstract

In this article, basis functions of Spread and Bernstein polynomials are linearly combined with unknown coefficients. These linear combinations are applied in formulating approximate solution for fractional differential equations. Residual equation derived from the fractional differential equation is collocated at equally spaced interval of the boundary where the problem exists. Systems of equation derived from this approach is solved and values of coefficients are obtained. Numerical solution of the problem is arrived at by substituting values of the coefficients into constructed linear combinations. To illustrate the effectiveness of these two polynomials, comparison between the two over a varying degree n of the approximants is carried out. This is done alongside the analytical solution of each problem. The discrepancies obtained speak in favour of the proposed methods.

Mathematics subject classification: 65L10, 65L60
Keywords: Fractional differential equations; Caputo Fractional Derivative; Spread polynomial; Bernstein polynomial.

1.0 Introduction

In the past few decades, a very cogent attention has been given to the development of non-integer order differential equations. This is as a result of growing realization of the importance and diverse use of this equation in mathematical models of problems in physics, regular variation in biophysics, thermodynamics, blood flow phenomena, viscoelasticity, electrical circuits, aerodynamics, astrophysics, biology, control theory, economics and a host of interdisciplinary applications [1, 2]. For solutions to this form of differential equation, a good number of researchers have devised methods that yielded exact solutions, these include [3, 4, 5]. However, the limitation of exact solution methods in practical applications and nonavailability of closed form solutions for some real life problems call for numerical techniques which apart from their versatility, they can easily be captured into computational tools as built-in functions. On a current note, research into numerical methods of solving this equation has enjoyed a great deal of attention, methods that have been effectively deployed into the solution of FDEs include finite difference method, methods of orthogonal function, Chebyshev wavelets, generalized block pulse operational matrix etc [6 – 9].

Methods based on the orthogonal functions are powerful and effective for solving FDEs as they have achieved great success in this field [1, 2]. The direct implications of this statement is that these numerical methods possesess the ability to handle a wide spectrum of problems derived from mathematical models of real life problems. In addition to this, vast majority of available computational tools do not have built-in functions for obtaining numerical solution for FDEs, this can be effectively incorporated into them as numerical tool boxes.

The multi-order FDEs considered in this study is typically of the form:

$$(\beta_n D^{\alpha_n} + \beta_{n-1} D^{\alpha_{n-1}} + \ . \ . \ . \ + \beta_1 D^{\alpha_1})\, y(t) = f(t) \qquad (1.1)$$

With sufficient conditions imposed on the boundary.

Where D^{α} are defined in Caputo sense, $\alpha_n > \alpha_{n-1} > \ . \ . \ . > \alpha_1$, $\beta_n \neq 0$, $\alpha_n \geq 1$, and $\alpha_k \in R^+$, $\beta_k \in R$, $k = 1, \ldots, n$. The function $f(t)$ belongs to the space $L^2(\Omega)$ and $\Omega = [0, T]$, $T \in R^+$ [10].

2. Review of relevant definitions and results from fractional calculus

Fractional derivatives is the derivatives of arbitrary real order α denoted

Correspondence Author: Olagunju A.S., Email: olagunjuas@gmail.com, Tel: +2348032515655

${}_aD_t^{\alpha}\, f(t)$

Where α is the order of the derivative, the subscripts a *and* t are the 2 limits related to the operation of fractional differentiation and are referred to as terminals of the fractional differentiation [8]. The terminals are sometimes omitted for convenience. There are several expressions for fractional derivatives, out of which 3 most used are given as follows.

2.1 Grunwald-Letnikov Derivatives

Grunwald-Letnikov fractional derivatives of function $f(t)$ with order $\alpha > 0$ is defined as:

$$ {}_aD_t^{\alpha}\, f(t) \;=\; {}_{GL}D_{a,t}^{\alpha}\, f(t) \;=\; \sum_{\substack{h\to\infty \\ nh=t-a}} f_n^{(\alpha)}(t) \qquad (2.1)$$

$$ = \sum_{k=0}^{n} \frac{f^{(k)}(a)(t-a)^{-\alpha+k}}{\Gamma(-\alpha+k+1)} + \frac{1}{\Gamma(-\alpha+n+1)} \int_a^t (t-\tau)^{n-\alpha} f^{(n+1)}(\tau)\, d\tau \qquad (2.2)$$

Note that here and elsewhere, Γ denotes the Gamma function.

2.2 Riemann-Liouville Fractional Derivative

It has been observed that the use Grunwald-Letnikov fractional derivative is not convenient, especially for non-integer terms [1]. The most widely known alternative to this is the Riemann-Liouville definition given as the integro-differential expression:

$$ {}_aD_t^{\alpha}\, f(t) \;=\; {}_{RL}D_{a,t}^{\alpha}\, f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \displaystyle\int_a^t \dfrac{f(\tau)}{(t-\tau)^{\alpha+1-n}}\, d\tau\,, & n-1 \le \alpha < n \in \mathrm{N} \\ \dfrac{d^n}{dt^n} f(t) & \alpha = n \in \mathrm{N} \end{cases} \qquad (2.3)$$

where $\alpha > 0,\ \ t > a,\ \alpha,\ a,\ t \in R$. Equation (2.3) is the Riemann-Liouville fractional differential operator of order α. Equation (2.2) is considered as a particular case of the integro-differential equation (2.3) called Riemann-Liouville definition of fractional derivative. The expression of Grunwald-Letnikov goes with an assumption that the function $f(t)$ must be $n+1$ times continuously differentiable, except with very few exceptions, Riemann-Liouville's definition bypasses this condition on the function $f(t)$ as it only demands the integrability of $f(t)$.

2.3 Caputo Fractional Derivative

Mathematical modelling of a good number of physical phenomenal such as in viscoelasticity, solid mechanics, biology etc demands for the utilization of physically interpretable initial conditions such as the ones defined as $f(a),\ f'(a)\ e.t.c.$, but the Riemann-Liouville's definition leads to initial conditions containing the limit values of the Riemann-Liouville fractional derivatives at the lower terminal $t=a$. [1, 6]. For example:

$$\lim_{t\to a} {}_aD_t^{\alpha-1}\, f(t) \;=\; b_1$$

$$\lim_{t\to a} {}_aD_t^{\alpha-2}\, f(t) \;=\; b_2$$

$$\cdot \quad \cdot \quad \cdot$$

$$\lim_{t\to a} {}_aD_t^{\alpha-3}\, f(t) \;=\; b_n$$

Where $b_k\,,\ k = 1,\ 2,\ .\ .\ .,\ n$ are given constants.

As observed by Podlubny [1], despite the fact that initial value problems with such initial conditions can be successfully solved mathematically, their solutions are practically useless because there is no known physical interpretation for such type of initial conditions. To resolve this limitation, M Caputo proposed, a definition for fractional derivative as:

$$ {}_aD_t^{\alpha}\, f(t) \;=\; {}_{C}D_{a,t}^{\alpha}\, f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \dfrac{f^{(n)}(\tau)\, d\tau}{(t-\tau)^{\alpha+1-n}}\,, & n-1 < \alpha < n \in N \\ \dfrac{d^n}{dt^n} f(t)\,, & \alpha = n \in N \end{cases} \qquad (2.4)$$

It is noted that for $\alpha \to n$ the Caputo derivative resulted into integer order n^{th} derivative of the function $f(t)$ i. e.

$$\lim_{\alpha \to n} {}^{C}D^{\alpha} f(t) = \lim_{\alpha \to n} \left(\frac{f^{(n)}(a)(t-a)^{n-\alpha}}{\Gamma(n-\alpha+1)} + \frac{1}{\Gamma(n-\alpha+1)} \int_a^t f(t-\tau)^{n-\alpha} f^{(n+1)}(\tau)\, d\tau \right) \qquad (2.5)$$

$$= f^{n}(a) + \int_a^t f^{(n+1)}(\tau)\, d\tau = f^{(n)}(t) , \qquad n = 1, 2, 3, \ldots$$

As in the case of Grunwald-Letnikov and the Riemann-Liouville's definition, Caputo's definition equally provides an interpolation between integer-order derivatives [1, 2]. The advantage of Caputo's definition is that the initial condition takes the same form as that of integer-order differential equations i. e. it contains the limit values of integer-order derivatives of unknown functions at the lower terminal $t = a$. In this work, Caputo's definition is used for the expression of fractional derivatives.

As illustrated in [1], the above fractional derivatives ${}_{a}D_{t}^{\alpha} f(t)$ are with fixed lower terminal a and moving upper terminal *t*, this is the case when $a < t$ and refered to as left derivatives. For cases with moving lower terminal *t* and fixed upper terminal b, we have the derivatives as ${}_{t}D_{b}^{\alpha} f(t)$ and referred to as right derivatives. More on this can be found in [1, 2].

3.0 Spread Polynomials Collocation Method (SPCM)

In this section we review some properties of spread polynomial which are applied in finding numerical solution of FDE (1.1).
This method entails approximating the unknown function $y(x)$ as:

$$y(x) = \sum_{k=1}^{\infty} c_k\, S_k(s) \qquad (3.1)$$

where c_k, $k = 1, 2, \ldots$ are unknown parameters to be determined and $S_i(s)$ are spread polynomials which can be obtained as follows:

$$S_n(s) = Sin^2\left(n\, arc\, Sin\left(\sqrt{s}\right)\right) \qquad (3.2)$$

$$= \sum_{k=0}^{n} C^{(k)} s^k , \quad k = 0, 1, \ldots n; \qquad 0 \le s \le 1$$

Where n is a fixed positive integer and S_n satisfies the second order linear nonhomogeneous differential equation:

$$s(1-s)y'' + \left(\tfrac{1}{2} - s\right) y' + n^2 \left(y - \tfrac{1}{2}\right) = 0 , \qquad (3.3)$$

together with the orthogonality property:

$$\int_0^1 \left(S_n(s) - \tfrac{1}{2}\right)\left(S_m - \tfrac{1}{2}\right) \frac{ds}{\sqrt{s(1-s)}} = \begin{cases} 0, & m \ne n \\ \frac{\pi}{8}, & m = n \ne 0 \\ \frac{\pi}{4}, & m = n = 0 \end{cases} \qquad (3.4)$$

The ordinary generating function of this polynomial is:

$$\sum_{n=1}^{\infty} S_n(s) t^n = \frac{t\, s(1+t)}{(1-t)^3 + 4t\, s(1-s)} \qquad (3.5)$$

While the exponential generating function is:

$$\sum_{n=1}^{\infty} \frac{S_n(s)}{n!} t^n = \frac{1}{2} e^{t} \left(1 - e^{-2ts}\, Cos\left(2t\sqrt{s(1-s)}\right)\right) \qquad (3.6)$$

The first 5 Spread polynomials $S_n(s)$ are given as follows:

$S_0(s) = 0$

$S_1(s) = s$

$S_2(s) = 4s - 4s^2$

$S_3(s) = 9s - 24s^2 + 16s^3$

$S_4(s) = 16s - 80s^2 + 128s^3 - 64s^4$

$S_5(s) = 25s - 200s^2 + 560s^3 - 640s^4 + 256s^5$

A 3-term fundamental recurrence relation is given as

$$S_n(s) = 2(1-2s) S_{n-1}(s) - S_{n-2}(s) + 2s \qquad (3.7)$$

Other explicit formulae can be found in [11, 12].
The technique involves substituting finite sum of equation (3.1) into (1.1) to obtain:

$$(\beta_n D^{\alpha_n} + \beta_{n-1} D^{\alpha_{n-1}} + \ . \ . \ . \ + \beta_1 D^{\alpha_1}) \sum_{i=0}^{N} c_i \, S_i \ (t) \neq f\,(t) \tag{3.8}$$

The derivatives are expressed in Caputo sense as given in equation section (2.3). Since the substituted equation (3.1) is finite and non-exact, both sides of (1.1) are no longer equal, but as depicted in (3.8). Equation (3.8) is thereafter collocated at points $t_i \in (0\ , T)$, where t_i $\left(i = N+1-\lceil \alpha_n \rceil\right)$ are set of equally-spaced points within the given interval, this process yields a system of $N+1-\lceil \alpha_n \rceil$ equations of the form

$$(\beta_n D^{\alpha_n} + \beta_{n-1} D^{\alpha_{n-1}} + \ . \ . \ . \ + \beta_1 D^{\alpha_1}) \sum_{k=1}^{N+1-\lceil \alpha_n \rceil} \sum_{i=0}^{N} c_i \, S_i \ (t_k) = \sum_{k=1}^{N+1-\lceil \alpha_n \rceil} f\,(t_k) \tag{3.9}$$

The given conditions are also imposed on finite sum of (3.1) to give $\lceil \alpha_n \rceil$ number of equations. These in addition to equation (3.9) give to a system of $N+1$ equations. Solving this system, we obtain of unknown coefficients c_i. These are thereafter substituted into finite sum

$$y_N\,(x) = \sum_{k=1}^{N} c_k \ S_k\,(t) \tag{3.10}$$

to yield a spread polynomial collocation solution of order N to fractional differential equation (1.1).

4.0 Bernstein polynomials Method (BPM)

We approximate the unknown function $y\ (x)$ in form similar to (3.1), i. e. a linear combination of Bernstein polynomials $B_i\ (x)$ and unknown parameters a_i [13].

$$y(t) = \sum_{k=1}^{\infty} a_k \ B_{k,n}\,(t) \tag{4.1}$$

$B_i\ (x)$ of degree *n* on the interval $[\,a\,,\,b\,]$ is defined as

$$B_{k,n}\,(t) = \frac{1}{(b-a)^{\,n}} \binom{n}{k} (t-a)^k\,(b-t)^{n-k} \tag{4.2}$$

$$for \ k = 0,1,\,.\,.\,.,\,n$$

$$where \ \binom{n}{k} = \frac{n!}{n!(n-1)!} \tag{4.3}$$

It satisfies a 3-term recursive relation:

$$B_{k,n}\,(t) = (1-t)\,B_{k,n-1}\,(t) + t\,B_{\,k-1,n-1}\,(t)$$

The first 3 Bernstein polynomials $B_n\,(t)$ within interval (0 1) are given as follows:

$$B_{0,1} = 1-t, \quad B_{1,1} = t$$

$$B_{0,2} = (1-t)^2, \quad B_{1,2} = 2t\,(1-t)\,, \quad B_{2,2} = t^2$$

$$B_{0,3} = (1-t)^3, \quad B_{1,3} = 3t\,(1-t)^2\,, \quad B_{2,3} = 3t^{\,2}\,(1-t), \quad B_{3,3} = t^3$$

Other properties of Bernstein polynomials as applied in this study are found in [13],
substituting equation (4.1) into (1.1), we have:

$$(\beta_n D^{\alpha_n} + \beta_{n-1} D^{\alpha_{n-1}} + \ . \ . \ . \ + \beta_1 D^{\alpha_1}) \sum_{k=0}^{N} a_k \, B_{k,N} \ (t) \neq f\,(t) \tag{4.4}$$

All other steps are as illustrated in section (3), from this we obtain unknown coefficients a_k which are substituted into finite sum

$$y_N\,(x) = \sum_{k=1}^{N} a_k \ B_{k,N}\,(t) \tag{4.5}$$

to yield Bernstein polynomial solution of order N to fractional differential equation (1.1).

5.0 Illustrative examples

In this section we present some numerical examples of fractional differential equation to illustrate the methods.

Example 1. Consider the following inhomogeneous boundary value problem:

$$D^{\alpha}\, y(t) + y(t) = t^{4} - \tfrac{1}{2}t^{3} - \frac{3t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{24t^{4-\alpha}}{\Gamma(5-\alpha)} \quad , \qquad 0 < \alpha < 1 \qquad (5.1)$$

$$y(0) = 0$$

$$y(t) = t^{4} - \tfrac{1}{2}t^{3}$$

A. **Solution via SPCM:** To apply Spread polynomial collocation method (SPCM) in solving equation (5.1), we substitute the finite form of equation (3.1) :

$$y(t) = \sum_{k=1}^{N} c_k\, S_k(t) \qquad (5.2)$$

into equation (5.1), with N = 5 and $\alpha = \frac{1}{2}$. We have:

$$D^{\frac{1}{2}} \sum_{k=1}^{5} c_k\, S_k(t) + \sum_{k=1}^{5} c_k\, S_k(t) \neq t^{4} - \tfrac{1}{2}t^{3} - \frac{3t^{\frac{5}{2}}}{\Gamma(7/2)} + \frac{24t^{\frac{7}{2}}}{\Gamma(9/2)} \quad , \qquad (5.3)$$

Set of 5 equally spaced points within the interval [0 , 1] , i. e. $t_1 = \frac{1}{6}$, $t_2 = \frac{1}{3}$, $t_3 = \frac{1}{2}$, $t_4 = \frac{2}{3}$, $t_5 = \frac{5}{6}$ are substituted into equation (5.3), the condition $y(0) = 0$ is also imposed on equation (5.2). From these, we obtain a system of 6 linear equations with 6 unknowns c_k , $k = 0(1)5$. By solving this system, we obtain the values of coefficients as :

$c_0 = 0$

$c_1 = 4.06250 \times 10^{-01}$

$c_2 = -2.5 \times 10^{-01}$

$c_3 = 9.375 \times 10^{-02}$

$c_4 = -1.5625 \times 10^{-02}$

$c_5 = 1.29943 \times 10^{-20}$

Substituting these values into (5.2), we obtain the numerical solution for equation (5.1) which is equal to exact solution when compared.

B. **Solution via BPCM:** To solve (5.1) for $\alpha = \frac{1}{2}$ via Bernstein polynomial method, we substitute equation (4.5) with N = 5 into (5.1) which gives:

$$D^{\frac{1}{2}} \sum_{k=1}^{5} a_k\, B_{k,n}(t) + \sum_{k=1}^{5} a_k\, B_{k,n}(t) \neq t^{4} - \tfrac{1}{2}t^{3} - \frac{3t^{\frac{5}{2}}}{\Gamma(7/2)} + \frac{24t^{\frac{7}{2}}}{\Gamma(9/2)} \quad , \qquad (5.4)$$

As in SPM, set of 5 equally spaced points within the interval [0 , 1] , are substituted into equation (5.4), the condition $y(0) = 0$ is also imposed on equation (4.5). From these, we obtain a system of 6 linear equations with 6 unknowns a_k , $k = 0(1)5$. We solve this system to obtain the 6 unknown coefficients, whose numerical values are:

$a_0 = 0$

$a_1 = 2.11004 \times 10^{-2}$

$a_2 = -6.39088 \times 10^{-2}$

$a_3 = 5.0 \times 10^{-01}$

$a_4 = 1.19029 \times 10^{-3}$

$a_5 = -5.0 \times 10^{-1}$

Substituting these values into (4.5), we obtain the approximate solution for equation (5.1) which is also equal to exact solution when compared.

The solutions of this example is displayed in figure 1 where SPM, BPM are plotted alongside the exact solution.

Example 2.

$$D^{2}\, y(t) + D^{\alpha} y(t) + y(t) = 8\,, \quad t > 0, \quad , \qquad 0 < \alpha < 2$$

$$y(0) = 1\,, \qquad y'(0) = 0$$

Ffhf

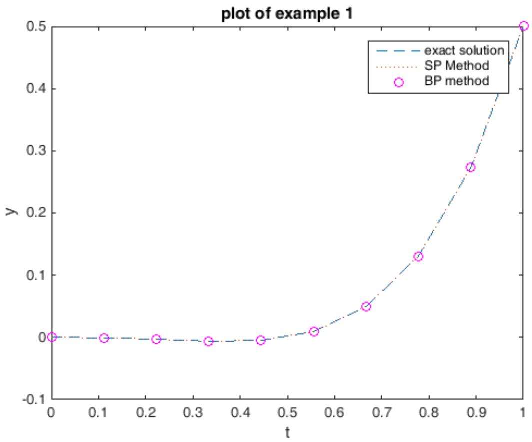

Figure 5.1 Plot of example 1 (For N = 5)

Example 2. Solve the following inhomogeneous boundary value problem. Fakhrodin (2014)

$$D^{\frac{3}{2}}y(t)+y(t)=t^5-t^4+\frac{128t^{3.5}}{7\sqrt{\pi}}-\frac{64t^{2.5}}{5\sqrt{\pi}},$$

$$y(0)=0,\quad y(1)=0$$

$$y(t)=t^4(t-1)$$

A. **Solution via SP method:** Following the same procedure, SP method produces the following values of the coefficients for N= 7.

$c_0 = 0$

$c_1 = -5.46876\times 10^{-01}$

$c_2 = -3.12501\times 10^{-01}$

$c_3 = 5.07814\times 10^{-02}$

$c_4 = -2.34375\times 10^{-02}$

$c_5 = 3.90626\times 10^{-03}$

$c_6 = 2.12657\times 10^{-10}$

$c_7 = -9.16609\times 10^{-12}$

Putting these values into (5.2), we obtain the numerical solution for example 2

B. **Solution via BPCM:** Also for N = 7, The BP method produces the following values of the Coefficients:

$a_0 = 0$

$a_1 = 7.38142\times 10^{-01}$

$a_2 = 1.46624\times 10^{-01}$

$a_3 = 2.085227\times 10^{-02}$

$a_4 = 2.85715\times 10^{-02}$

$a_5 = 9.52383\times 10^{-01}$

$a_6 = 1.42857\times 10^{-02}$

$a_7 = 0$

Substituting these into (4.5), to obtain the solution for example 2

Table 5.1: Table of maximum errors for values of N ranging from 5 - 12.

N	SPCM	BPCM
5	1.9335e-06	5.1721e-07
7	3.8512e-06	6.9194e-07
10	6.1702e-08	2.8491e-10
12	1.0621e-09	4.5712e-10

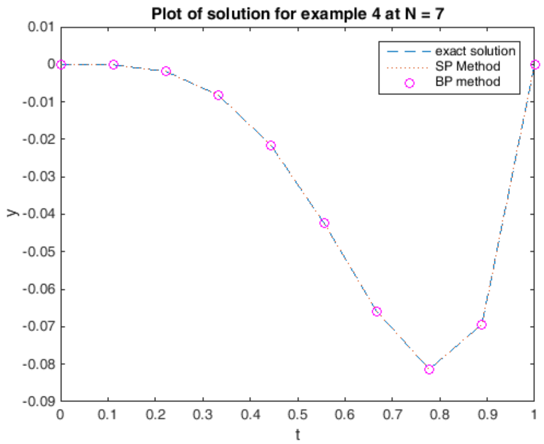

Figure 5.2: Plot of example 2 (N = 7)

Example 3.

Consider the following boundary value problem in the case of the inhomogeneous Bagley-Torvik equation [9].

$D^2 y(t) + D^{\frac{3}{2}} y(t) + y(t) = t^2 + 4\sqrt{\frac{t}{\pi}} + 2$,

$y(0) = 0, \quad y(5) = 25$

$y(t) = t^2$

A. **Solution via SPM:** For this problem, SP method for N = 7 yields the following values of coefficients.

$c_0 = 0$

$c_1 = 2.499908 \times 10^{02}$

$c_2 = -6.24378$

$c_3 = 1.00466 \times 10^{-01}$

$c_4 = -3.71387 \times 10^{-01}$

$c_5 = -8.71658 \times 10^{-02}$

$c_6 = 1.31395 \times 10^{-02}$

$c_7 = 1.24894 \times 10^{-06}$

We obtain approximate solution for this by putting these values into (5.2).

B. **Solution via BPM:** For BP method with N = 7, the following values of the coefficients a_k, $k = 0(1)7$ are obtained.

$a_0 = 0$

$a_1 = -3.56872 \times 10^{-03}$

$a_2 = -1.19745$

$a_3 = -3.57959$

$a_4 = -7.14960$

$a_5 = -1.19090 \times 10^{01}$

$a_6 = -1.78590 \times 10^{01}$

$a_7 = -2.50 \times 10^{01}$

Substituting these into (4.5), to obtain the solution for this example.

Table 5.2: Table of maximum errors for values of N ranging from 5 - 12.

N	SPCM	BPCM
5	1.9604e-03	2.5266e-04
7	1.0275e-03	5.4940e-04
10	7.9214e-05	1.5044e-06
12	3.1633e-06	6.3632e-07

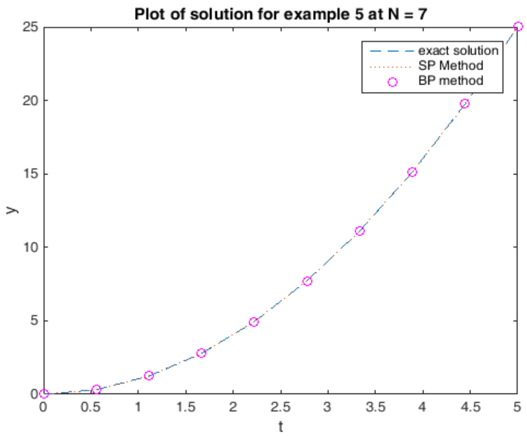

Figure 5.3: Plot of example 3 (Foor N =12)

6.0 Conclusion

Numerical solution of fractional differential equation has been considered in this study, the performance of two basis functions (Spread and Bernstein basis) were compared in implementing collocation method. The numerical results as depicted in figure 5.1 – 5.3 show the efficiency of these two polynomials as basis function, the approach is simple and easily automated. Basis function from the two polynomials also produced results that are so close to the exact solution at varying degree N of the approximation. In all of these, Bernstein polynomial basis yielded better results than Spread polynomials at the same degree N of the approximate solution. Comparative study of different types of collocation points, rather than equally-spaced points is suggested for further investigation.

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