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A MATHEMATICAL ASSESSMENT OF THE EXTENT OF HEAVY RAINDROPS AND MANURE FERTILISATION ON TOPSOIL FERTILITY AND CROP YIELD

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Abstract

*In this paper, a nonlinear mathematical model to study the effect of heavy rain and manure on the fertility of a topsoil and crop-yield is proposed and analysed. The d**ynamical properties, that is, the local and global stabilities, of the model are analysed. The model analyses are basically performed on three physical interacting scenarios – a situation where both manure and crop are absent, a case of continuous farming without any form of nutrient replenishment and the effect of a sustainable manure use on the farm topsoil as well as its effect on crop growth and yield. It is shown that the depth of fertile topsoil is affected by increasing densities of crop and could be adversely exacerbated with increasing stresses of heavy rain. Manure is shown to indicate the potency to improve topsoil quality and increase productivity; thereby suggesting a sustainable pathway for soil sustainability and reclamation. Numerical simulations were carried out to support the analytical results.***

Keywords: Topsoil fertility, heavy rain, manure, sustainability, stability, crop yield.

1. Introduction

Soil fertility is the intrinsic ability of soil to supply all the essential nutrients to plants in readily accessible form and in the appropriate balance [1]. Soil fertility is the natural, sustainable ability of a soil to sufficiently sustain plant live [2]. Soil fertility is therefore an important requirement for crop production. Soil is therefore the most influential factor in crop production [3] as an estimated 95% of all food resource is directly or indirectly produced on it [1]. Thus, a declining fertility has adverse effect on crop yield due tothe resultant shortfalls in the supply of essential soil nutrients[4]. Extreme weather conditions due to global warming [5], unwholesome and unsustainable land use practices [2, 6, 7] are mounting very intense pressure on the This natural ability of the soil to sustain plant life. The delicate and fragile nature of the soil has become its greatest weakness to the degrading impact of these phenomena [2].The degradation of soil organic matter (SOM), soil structure and associated nutrient supply are major factors for yield decline in intensive cereal based cropping system [8]. The combined use of organic manure with inorganic fertilizer has been found to be of far greater sustainable impact on the soil and agriculture than singly using either organic or inorganic fertilisers [8, 9]. Organic farming is gaining popularity due to its underlying similar principle to tradition agriculture and emphasis on human welfare without any harm to environmental health [10].The sustainable application of soil amendments (compost for instance) does significantly increases microbial biomass to such an extent that soil microbial functional diversity is considerably impacted thereby substantially affecting crop productivity [11]. Rain is vital for plant life and crop production. However, on the one hand, an extended period of lack of rain has a devastating consequence as evidence from the resultant famine as a result of the 1984 – 1985 Horn of Africa draught that killed over 750,000 people. While on the other hand, excessive rain and the accompanying flooding can submerge crops in water resulting in potentially devastating loses [12].

2. Model formulation

It is assumed that in the region considered, $R(t)$ is the density of raindrops, which grows with a constant rate Π ,$F(t)$ is depth of fertile topsoil, which grows at a constant rate Γ, $M(t)$ is the density of manure stock, which grows at a constant rate Δ and $C(t)$ be the density of crops on the farm field and is assumed to be governed by a logistic model at time *t*. Thus, the problem being considered is governed by the following system of nonlinear ordinary differential equations

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$$\frac{dR}{dt} = \Pi - \psi R - \psi_0 FR,$$
$$\frac{dF}{dt} = \Gamma - \gamma F - \gamma_0 FR - \gamma_1 CF + \gamma_2 FM, \quad (1)$$
$$\frac{dM}{dt} = \Delta - \alpha M - \alpha_0 FM,$$
$$\frac{dC}{dt} = \phi C\left(1 - \frac{C}{K}\right) + \phi_0 CF + \phi_1 CR,$$

where $R(0) \ge 0, F(t) \ge 0, M(0) \ge 0, C(0) \ge 0.$

In putting up the model equations of the system (1), it is assumed that ψ is the natural depletion rate coefficient of C and ψ_0 is the depletion rate coefficient of *C* due to its interaction with the soil. It is also assumed that Γ is the constant natural growth rate coefficient of fertile topsoil, further, it is considered that γ is the natural depletion rate of *F*, γ_0 is the depletion rate of *F* due to, among others, surface runoff by *R*, γ_1 is the depletion rate coefficient of *F* due to nutrient loss through absorption by growing plants. Again, it is assumed that soil fertility is improved due to manure decomposition reaction because of manure application at a rate γ_2.Manure is assumed to grow at a constant growth rate coefficient Δ, and decreases due to natural depletion rate coefficient α, and application rate coefficient α_0. The constants ϕ and K are the intrinsic growth rate and carrying capacity of the of the cumulative density of crops. The constants ϕ_0 and ϕ_1 are the nourishment rate coefficients of the density of crops due to crops' interaction with topsoil and raindrops, respectively.

3. Equilibrium analysis

It can be checked that model (1) has two nonnegative equilibria, namely $E\left(\frac{\Pi}{\psi}, \frac{\Gamma}{\gamma}, \frac{\Delta}{\alpha}, 0\right)$ and $E^*\left(R^*, F^*, M^*, C^*\right)$. The existence of the E can easily be verified. We therefore show the existence of E^*. In the equilibrium E^*, the values of R^*, F^*, M^* and C^* are the positive solutions of the following algebraic equations:

$$\psi R + \psi_0 FR - \Pi = 0,$$
$$\gamma F + \gamma_0 FR + \gamma_1 CF - \gamma_2 FM - \Gamma = 0, \quad (2)$$
$$\alpha M + \alpha_0 FM - \Delta = 0,$$
$$\phi\left(1 - \frac{C}{K}\right) + \phi_0 F + \phi_1 R = 0.$$

Then from the first and third equations of (2), it follows that

$$R = \frac{\Pi}{\psi + \psi_0 F}, \quad (3)$$

and

$$M = \frac{\Delta}{\alpha + \alpha_0 F}, \quad (4)$$

On using equation (3) in the fourth equation of (2), it can easily be seen that

$$C = \frac{K(\phi\psi + \phi_1 \Pi) + [\phi\psi_0 + (\psi + \psi_0 F)\phi_0]KF}{\phi(\psi + \psi_0 F)}. \quad (5)$$

Similarly, on using equations (3), (4) and (5) in the second equation of (2), we obtain

$$g(F) = \frac{\Gamma(\alpha + \alpha_0 F)(\psi + \psi_0 F)}{\alpha_0 \gamma_1 \psi_0 \phi_0 KF^3 + a_0 F^2 + a_1 F + a_2}, \quad (6)$$

where, $a_0 = \alpha_0 \gamma \phi \psi_0 + \alpha \gamma_1 \psi_0 \phi_0 K + \alpha_0 \gamma_1 K(\phi\psi_0 + \psi\phi_0)$,

$a_1 = \psi + \alpha\gamma\phi\psi_0 + \phi\alpha_0\gamma_0\Pi + \gamma_1 K[\alpha_0(\phi\psi + \phi_1\Pi) + \alpha(\phi\psi_0 + \psi\phi_0)] - \phi\gamma_2\psi_0\Delta$, $a_2 = \alpha\gamma\phi\psi + \phi\alpha\gamma_0\Pi + \alpha\gamma_1 K(\phi\psi + \phi_1\Pi) - \phi\psi\gamma_2\Delta.$

It is easy to see from equation (6) that:

1. $g(0) = \frac{\alpha\psi\Gamma}{a_2} > 0$, provided that $\alpha[(\gamma\phi\psi + \phi\gamma_0\Pi) + \gamma_1 K(\phi\psi + \phi_1\Pi)] > \phi\gamma_2\psi_0\Delta.$

2. $$g'(F)=\frac{\begin{array}{c}(\alpha\psi_0+\psi\alpha_0+2\alpha_0\psi_0 F)(\alpha_0\gamma_1\psi_0\phi_0 KF^3+a_0F^2+a_1F+a_2)\\ -(\alpha+\alpha_0F)(\psi+\psi_0F)(3\alpha_0\gamma_1\psi_0\phi_0KF^2+2a_0F+a_1)\end{array}}{(\alpha_0\gamma_1\psi_0\phi_0KF^3+a_0F^2+a_1F+a_2)^2}<0,$$

provided that $\frac{(a_2+a_1F+a_0F^2+\alpha_0\gamma_1\psi_0\phi_0KF^3)}{(a_1+2a_0F+3\alpha_0\gamma_1\psi_0\phi_0KF^2)}<\frac{(\alpha+\alpha_0F)(\psi+\psi_0F)}{(\alpha\psi_0+\psi\alpha_0+2\alpha_0\psi_0F)}$

thus, there exists a unique positive root $F=F^*$ of equation (6), provided the specified inequalities hold. Using this value of F_3, the positive values of $R=R^*, M=M^*$ and $C=C^*$ can be obtained. This proves the existence of the interior equilibrium $E^*(R^*,F^*,M^*,C^*)$.

4. Stability Analysis

consider the positive definite function $U=\frac{1}{2}\left(k_0r^2+k_1f^2+k_2m^2+\frac{k_3}{C^*}c^2\right)$

the local stability behaviour of the equilibrium$E$ is investigated by determining the eigenvalues of the corresponding Jacobian matrix. The Jacobian matrix, J, for the system (1) is given by

$$J=\begin{pmatrix}-\psi-\psi_0F & -\psi_0R & 0 & 0\\ -\gamma_0F & \gamma_2M-\gamma-\gamma_0R-\gamma_1C & \gamma_2F & -\gamma_1F\\ 0 & -\alpha_1M & -\alpha-\alpha_1F & 0\\ 0 & \phi_0C & \phi_1C & \phi\left(1-\frac{2C}{K}\right)+\phi_0F+\phi_1R\end{pmatrix}$$

To study the behaviour of the equilibrium point, we state the following theorems.

Theorem 1. *The interior equilibrium* E^*, *if it exists, is locally asymptotically stable, provided the following inequalities hold,*

$$(\psi_0R^*+\gamma_0F^*)^2<\frac{4\Pi\Gamma}{3F^*R^*} \quad (7)$$

and

$$\frac{\gamma_1\phi_1^2}{\phi_0}<\frac{16\gamma_2\Delta}{\alpha_1KM^{*2}} \quad (8)$$

then, E^* is locally asymptotically stable.

Proof. To establish the local stability of E^*, it is enough to consider the following positive definite function,

$$U=\frac{1}{2}\left(k_0r^2+k_1f^2+k_2m^2+\frac{k_3}{C^*}c^2\right) \quad (9)$$

where r, f, m and c are small perturbations about E^* such that $R=R^*+r, F=F^*+f, M=M^*+m$ and $C=C^*+c$ $C=C^*+c$ and k_0,k_1,k_2 and k_3 are some positive constants to be appropriately chosen.

Now, the time derivative of $U$is

$$U'=k_0rr'+k_1ff'+k_2mm'+\frac{k_3}{C^*}cc' \quad (10)$$

The using the linearised system of the system (1),

$$\begin{pmatrix}r'\\ f'\\ m'\\ c'\end{pmatrix}=\begin{pmatrix}-\frac{\Pi}{R^*} & -\psi_0R^* & 0 & 0\\ -\gamma F^* & -\frac{\Gamma}{F^*} & \gamma_2F^* & -\gamma_1F^*\\ 0 & -\alpha_1M^* & -\frac{\Delta}{M^*} & 0\\ 0 & \phi_0C^* & \phi_1C^* & -\frac{C^*}{K}\end{pmatrix}\begin{pmatrix}r\\ f\\ m\\ c\end{pmatrix}.$$

in (10), we obtain

$$U'=-k_0\left(\frac{\Pi}{R^*}r+f\psi_0R^*\right)r-k_1\left(r\gamma_0F^*+\frac{\Gamma}{F^*}f-m\gamma_2F^*+c\gamma_1F^*\right)f-k_2\left(f\alpha_1M^*+m\frac{\Delta}{M^*}\right)m$$
$$+\frac{k_3}{C^*}\left(f\phi_0C^*+m\phi_1C^*-c\frac{C^*}{K}\right)c$$

Thus,

$$U' = -\left[k_0 r^2 \frac{\Pi}{R^*} + \left(k_0\psi_0 R^* + k_1\gamma_0 F^*\right)rf + k_1\frac{\Gamma}{F^*}f^2\right] - \left[k_2\frac{\Delta}{M^*}m^2 - k_3\phi_1 cm + k_3\frac{1}{K}c^2\right]$$
$$+\left(k_1\gamma_2 F^* - k_2\alpha_1 M^*\right)mf + \left(k_3\phi_0 - k_1\gamma_1 F^*\right)cf.$$

On choosing $k_0 = k_1 = 1,\ k_2 = \frac{\gamma_2 F^*}{\alpha_1 M^*}$ and $k_3 = \frac{\gamma_1 F^*}{\phi_0}$, then

$$U' = -\left[k_0\frac{\Pi}{R^*}r^2 + \left(k_0\psi_0 R^* + k_1\gamma_0 F^*\right)rf + \frac{k_1\Gamma}{3F^*}f^2\right] - \left[\frac{k_2\Delta}{2M^*}m^2 - k_3\phi_1 cm + \frac{k_3}{2K}c^2\right].$$

Thus, it can be easily verified that $U¢$ is negative definite under conditions (7) and (8) thereby establishing the prove.

Theorem 2.*The interior of E^*, if it exists is globally asymptotically stable in the region , provided the following inequalities are satisfied,*

Proof. Consider the following positive definite function

$$L = \frac{1}{2}g_0\left(R - R^*\right)^2 + \frac{1}{2}g_1\left(F - F^*\right)^2 + \frac{1}{2}g_2\left(M - M^*\right)^2 + \left(C - C^* - C\ln\frac{C}{C^*}\right)^2,$$

where g_0, g_1 and g_2 g_2 are positive constants to be appropriately chosen. The time derivative of L along the solutions of the model system (1) is

$$L' = g_0\left(R - R^*\right)R' + g_1\left(F - F^*\right)F' + g_2\left(M - M^*\right)M' + \left(C - C^*\right)\frac{C'}{C}.$$

Thus, using model system(1) in the algebraic equation above and simplifying, we obtain

$$\frac{dL}{dt} = g_0\left(R - R^*\right)R' + g_1\left(F - F^*\right)F' + g_2\left(M - M^*\right)M' + \left(C - C^*\right)\frac{C'}{C}$$
$$= -g_0\frac{\Pi}{R^*}\left(R - R^*\right)^2 - g_0\psi_0 R^*\left(F - F^*\right)\left(R - R^*\right) - g_1\gamma_0 F^*\left(F - F^*\right)\left(R - R^*\right)$$
$$- g_1\frac{\Gamma}{F^*}\left(F - F^*\right)^2 + g_1\gamma_2 F^*\left(F - F^*\right)\left(M - M^*\right) - g_1\gamma_1 F^*\left(C - C^*\right)\left(F - F^*\right)$$
$$- g_2\alpha_1 M^*\left(F - F^*\right)\left(M - M^*\right) - g_2\frac{\Delta}{M^*}\left(M - M^*\right)^2 + \phi_0\left(C - C^*\right)\left(F - F^*\right)$$
$$+ \phi_1\left(C - C^*\right)\left(M - M^*\right) - \frac{1}{K}\left(C - C^*\right)^2$$

Now choosing $g_0 = g_1 = 1,\ g_2 = M^*$, we obtain

$$\left(\psi_0 R^* + \gamma_0 F^*\right)^2 < \frac{4\Pi\Gamma}{3F^* R^*},\ \left(\gamma_2 F^* - \alpha_1 M^{*2}\right)^2 < \frac{2\Delta\Gamma}{3F^*}, \left(\gamma_1 F^* - \phi_0\right) < \frac{2\Gamma}{3KF^*}, \phi_1^2 < \frac{\Delta}{K}$$

5. Numerical simulation

In this section, we illustrate the results obtained in previous sections through computer simulations. We take thefollowing parameter values

$$\Pi = 10,\ \psi = 8.0, \psi_0 = 1.0, \Gamma = 1.8, \gamma = 1.5, \gamma_0 = 2.01, \gamma_1 = 1.25, \Delta = 1.01, \alpha = 1.05, \alpha_0 = 2.15,$$
$$\phi = 0.6, \phi_0 = 0.02, \phi_1 = 0.02, \phi_2 = 0.05, K = 50.$$

From the foregoing parameter values it can easily be verified that the interior equilibrium E^* exists and is given by $C^* = 1.1872,\ F^* = 0.4192,\ M^* = 0.4619,\ R^* = 0.5773.$ Further, it can easily be checked that the positive equilibrium E^* is both locally and globally asymptotically stable. Finally, to investigate the effect of different parameters on different variables for some interacting possibilities using MATLAB.

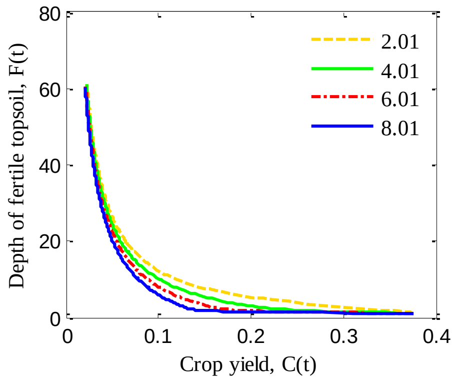

Fig. 1. Plot of $F(t)$ against $C(t)$ for varying values of γ_0 while other parameters are held constant

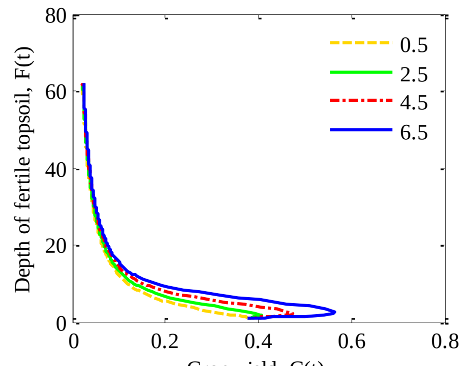

Fig. 2. Plot of $F(t)$ against $C(t)$ for varying values of γ_2 while other parameters are held constant

Fig. 1 shows the dynamics of depth of fertile topsoil against crop yield for different values of γ_0. It can be noted that the continuous depletion of fertile topsoil would naturally lead to decrease in expected yield. It is also observed crop yield would eventually settle down at a steady state. Fig. 2 shows the effect of the increasing consequence of γ_2 on the simultaneous decreases on depth of fertile topsoil and increase in crop yield. However, it is noted that this increase in yield would experience fall with continuous depletion of fertility. Figs. 3 and 4 show the respective impact of γ and ψ on depth of fertile topsoil and crop yield as the former is plotted against the latter. It is observed that yield is respectively decreasing and increasing. However, each production trajectory is noted to eventually settle to a steady state.

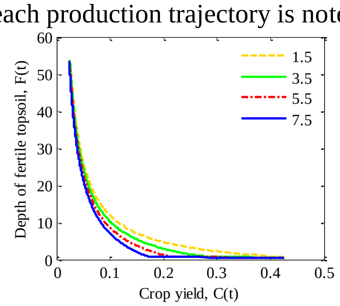

Fig. 3. Plot of $F(t)$ against $C(t)$ for varying values of γ while other parameters are held constant

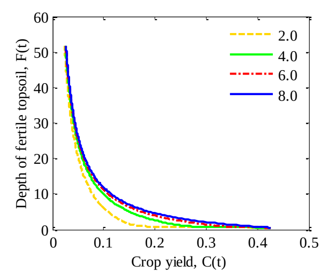

Fig. 4. Plot of $F(t)$ against $C(t)$ for varying values of φ while other parameters are held constant

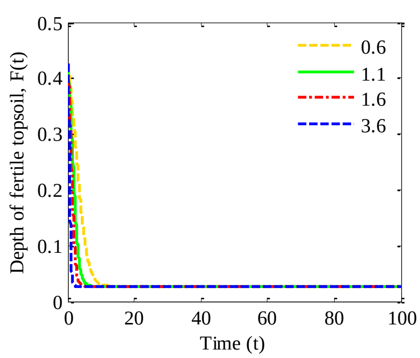

Fig. 5. Plot of $F(t)$ against t for varying values of ϕ while other parameters are held constant

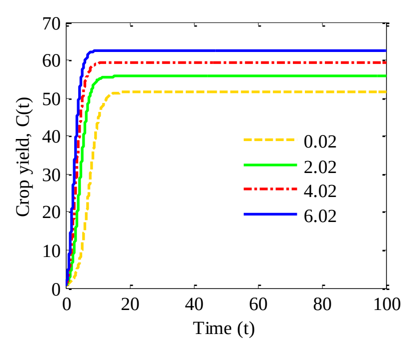

Fig. 6. Plot of $C(t)$ against t for varying values of ϕ_0 while other parameters are held constant

Figs. 5 and 6 show the respective impact of ϕ and ϕ_0 on depth of fertile topsoil and crop yield as the each variable considered against time *t*. the depleting consequence of ϕ is noted from Fig. 5 while the increasing prospect of crop yield is noted with increasing ϕ_0. Each trajectory as observed would eventually settle to a steady state.

6. Conclusion

Available scarce resources are being overstretched by increasing human demands. Soil for instance is substantial pressured by the simultaneous impact of the continuous increase in human population and increasing unpredictability of natural factors; like rainfall due to the adverse effect of global warming. This in turn is exerting phenomenal consequence on the intrinsic growth rate and carrying capacity of crop resources and the farming system in general. In the present study, we have proposed and analysed a nonlinear mathematical model for crop yield on a fertility depleting topsoil farm field by considering the effect of rain drops density and a manure-only topsoil fertilisation cropping system. It is shown that if the topsoil is continually pressured by natural factors, in this case raindrops; leading to decreasing fertility, due possibly to runoff, then crop yield would be significantly diminished to a very worrying level. Again, decomposing manure is seen to have a rather minimal positive impact on yield, suggesting a further investigation is required on the issue of sustainable application procedure. It is also shown that increasing raindrops density would have an adverse effect on both fertility and desirable yield. The model is analysed using the stability theory of differential equations and numerical simulations were performed using MATLAB.

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