Transactions of the Nigerian Association of Mathematical Physics
***Volume 15, (April –June, 2021 Issue*), pp81 – 90**
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A COMPARATIVELY EFFICIENT METHOD OF GENERATING RANDOM SAMPLES FROM THE GAMMA DISTRIBUTION USING ACCEPTANCE-REJECTION METHOD

N. Ekhosuehi and C. O. Odijie

Department of Statistics, University of Benin, Benin City, Nigeria.

Abstract

This study considered generating random samples from the gamma distribution using acceptance-rejection method with the double-exponential (Laplace) distribution as proposal or majorizing density. Parameter estimates of some random samples generated with the method were compared with parameter estimates of random samples generated with the method used for the gamma random variables generator 'gamrnd' in MATLAB® R2010b under simulation study. Efficiency of the method was measured by examining some statistical properties of the estimated parameters from the random samples generated with the method and comparisons were made with these properties, which included average bias, MSE and coverage probability, rather than machine time or speed. Results of the study shows that our method outperformed the method used in MATLAB® R2010b for the algorithm 'gamrnd' in most of the cases with different sample sizes that were considered.

Keywords: gamma distribution, acceptance-rejection method, random samples, efficiency.

1. Introduction

When considering different versions of acceptance-rejection method of generating random variables from a known distribution, authors have commonly compared the efficiency of different methods by the acceptance rate and machine time (e.g. [1 – 7], among others). This efficiency is usually measured by $\frac{1}{c}$, c being the expected number of uniform random deviates required to produce one gamma variate [7], usually computed as the maximum value of the ratio of the target distribution to the majorizing density. $\frac{1}{c}$ indicates the acceptance rate of the method [8]. However, comparison of estimates of the parameters of the distribution using various sample sizes actually generated from the distribution by each method under simulation study is usually not common. Kundu-Gupta compared their method with some existing methods in this regard, using K-S statistic and p-value as basis for comparison of the average values of 10,000 sample replicates of the parameter estimates computed [7]. One of the primary aims of generating random variables is, to a great extent, to obtain a representative sample for a population under study and, as such, the random samples so generated by any method need to be considered or tested for true representation of the population, on the average, via the estimates of parameters obtained from the random samples. It is arguably possible that a method for which the acceptance rate is higher or the machine time is faster may yet not generate random samples whose estimates are closer to the actual parameter values under simulation study than some other method with relatively slower acceptance rate or machine time. To this effect, our interest in this study was to generate random samples from the gamma distribution using an acceptance-rejection method and then compare the parameter estimates from some random samples generated by the method with those estimated from samples generated by an existing method, specifically the one used in MATLAB® R2010b (an algorithm sourced from [6]) whose reported acceptance rate is nearly 100% for all values of the shape parameter of the standard gamma distribution. The method, which was applied for the MATLAB® R2010b (and possibly some other older versions) code 'gamrnd', generates random gamma samples for various chosen values of the shape and scale parameters. The algorithm's implementation in MATLAB® can be viewed by typing 'open gamrnd' in the command window of any of the MATLAB® versions that has it. Specifically, [6] reported acceptance rate of the method for different shape parameter α as high as 0.951, 0.981, 0.992, 0.996 for $\alpha = 1, 2, 4, 8$, respectively. The method involved some power of normal random variates and was designed basically for the standard gamma density (unit scale parameter) with shape parameter $\alpha \geq 1$. The algorithm from [6] is given as follows (with a $\equiv \alpha$):

Correspondence Author: Ekhosuehi N., Email: Cyril.odijie@uniben.edu, Tel: +2348064459101

(1) Setup: d = a-1/3, c = 1/sqrt(9d).
(2) Generate v = (1+cx)ˆ3 with x normal. Repeat if v<=0 [rare;
requires x < -sqrt(9a-3)].
(3) Generate uniform U.
(4) If U < 1-0.0331*xˆ4 return d*v.
(5) If log(U) < 0.5*xˆ2+d*(1-v+log(v)) return d*v.
(6) Go to step 2.

Before [6], extensive studies have been carried out on generating random variables from the gamma distribution (e.g. [1 – 5] and [9], among others). This extensive study may be attributed to the importance of the gamma distribution in Probability and Statistics especially for analyzing positively skewed data. Many of its areas of usefulness can been found in [10] and [11]. Thus, generating a good representative sample from the distribution has been of high importance to researchers. [1], for instance, developed a method of generating gamma random variates using Cauchy distribution as envelope (i.e. the proposal density) but they observed, however, that the algorithm 'has less potential than another algorithm which compares the gamma densities with normal distributions, except in the tails where an exponential distribution is substituted', where they referred to the latter method as 'Modified Normal-Exponential Method'. [2] made use of some root of normal variates to generate gamma variates. Ahrens and Dieter further presented a modified rejection technique where they made use of Laplace density as 'hat' for the standard gamma distribution (i.e. with constant scale parameter $\beta = 1$) with some shifting factor for the Laplace density in [9]. [4] worked on generating gamma variables with shape parameter less than unity. Minh formulated an algorithm that seems a bit more complicated— with long initialization and generation procedures— when compared to the others under reference (see [5]).The method by [6], in terms of high acceptance rate and fast machine time, reportedly bested most of the previous methods (specifically those of [2, 3, 5] and [9] which were compared with it). This suggestively prompted its implementation for the MATLAB® code 'gamrnd' which generates gamma random variables or samples for given values of the shape and scale parameters in MATLAB® (our reference being R2010b specifically and possibly other versions). Therefore, in our interest to compare parameters estimated from samples generated by our method with those estimated from samples generated by a fast method, it suffices to compare with Marsaglia-Tsang's method via the MATLAB® gamma generator, 'gamrnd' or by implementing their algorithm as summarized clearly in [8]. It should be noted that all the methods mentioned in the foregoing section are based on acceptance-rejection technique with different variations.
The remaining part of this paper is organized as follows: section 2 reviews the distributions relevant to the study and the general concept of acceptance-rejection method, section 3 presents our method, section 4 contains the simulation study and comments on the results while section 5 summarizes and concludes the study.

2. Review of relevant distributions

2.1 The gamma distribution

A random variable X is said to follow the gamma distribution if its probability density function (PDF) is defined by:

$$f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}},\ x > 0,\ \alpha,\ \beta > 0 \tag{1}$$

whereα and are the shape and scale parameters, respectively.
For the standard gamma, the scale parameter $\beta = 1$, and the PDF reduces to:

$$f(x;\alpha) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x},\ x > 0,\ \alpha > 0 \tag{2}$$

where, in equations (1) and (2), $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}\, e^{-t} dt$ is the complete *gamma function*.
The cumulative density function (CDF) of the gamma distribution is given by:

$$F(x;\ \alpha,\beta) = \frac{1}{\Gamma(\alpha)} \gamma\left(\alpha, \frac{x}{\beta}\right) \tag{3}$$

where $\gamma(\alpha, x) = \int_0^{x} t^{\alpha-1}\, e^{-t} dt$ is the (lower) *incomplete gamma function.*
Other properties of the gamma distribution includes:
Mean: $\mathbb{E}(X) = \alpha\beta$
Variance: $\mathbb{V}(X) = \alpha\beta^2$
Mode: $(\alpha - 1)\beta$ for $\alpha \geq 1$
Skewness: $\frac{2}{\sqrt{\alpha}}$
Moment Generating Function (MGF): $M(t) = \frac{1}{(1-\beta t)^{\alpha}}$ for $t < \frac{1}{\beta}$
We shall write $X \sim Gamma(\alpha, \beta)$ for short for a random variable X which follows the gamma distribution with shape and scale parameters $\alpha$and β, respectively.

2.2 The Laplace distribution

The PDF of a random variable Y having the Laplace distribution is given by:

$$g(y) = \frac{1}{2b} exp\left(-\frac{|y-\mu|}{b}\right) = \frac{1}{2b}\begin{cases} \exp\left(-\frac{\mu-y}{b}\right), & y < \mu \\ \exp\left(-\frac{y-\mu}{b}\right), & y \geq \mu \end{cases} \quad (5)$$

with location parameter $\mu \in \mathbb{R}$ and scale parameter $b > 0$.
The CDF is given by:

$$G(y) = \begin{cases} \frac{1}{2}\exp\left(\frac{y-\mu}{b}\right), & y < \mu \\ 1 - \frac{1}{2}\exp\left(-\frac{y-\mu}{b}\right), & y \geq \mu \end{cases} \quad (6)$$

The inverse of the CDF or the quantile function of the Laplace distribution can easily be derived by solving the following equation for y as a function of u.

$$G(y) = u \quad (7)$$

where $u = Unif(0,1)$
to obtain:

$$y = G^{-1}(q) = \mu - b\, sgn(q) \ln(1 - 2|q|) \quad (8)$$

wheresgn denotes the signum or sign function and $q = \left(u - \frac{1}{2}\right) \sim U(-1/2, 1/2)$. The proof of equation (8) is given in Appendix.

2.3 Acceptance-rejection (A-R) method

The general idea behind acceptance rejection method is to generate random samples from a target distribution $f(x)$, whose CDF $F(x)$ cannot be inverted into a simple closed quantile function such as the gamma distribution of interest here, using another distribution whose CDF can be inverted into a closed form quantile function. The random variables generated via the other distribution, known variously as proposal distribution, majorizingdensity $g(y)$, etc., are accepted for the target distribution provided that the proposal "majorizes" the target for all the support i.e. $g(y) \backslash f(y)\ \forall\ y$.
The general algorithm for implementing an acceptance rejection method is stated briefly as:

1. Generate a uniform random number u
2. Tranform$y = G^{-1}(u)$
3. Compute $h(y) = \frac{f(y)}{c\, g(y)}$ where $c = \sup_{y}\left\{\frac{f(y)}{g(y)}\right\}$
4. Accept $y = x$ if $u \leq h(y)$

3 Methodology

Our method is a version of the general A-R method highlighted in section 2.
For $\alpha = 1$, the gamma distribution $Gamma(1, \beta)$ coincides with the exponential distribution $exp(1/\beta)$ and, in this case, it is sufficient to generate random variables from the distribution directly by the inverse transform method using the quantile function of the exponential distribution given by:

$$x = F^{-1}(u) = -\frac{1}{\lambda}\ln(1 - u) \quad (4)$$

where$u \sim Uniform(0, 1)$ is a uniformly distributed random variable and $\lambda = 1/\beta$.
When the value of λ is suitably chosen, $\exp(\lambda)$ can still be used as a good majorizing density for the gamma distribution when $\alpha = 1 + \varepsilon$ for some $\varepsilon \in (0,1)$ and even up to $\alpha = 2$. When $\alpha \gg 1$, we make use of the double-exponential (Laplace) distribution as majorizing density for the gamma distribution. The motivation behind this is the fact that the exponential curve no longer lies close enough to the gamma curve at the upper tail and as such will lead to more rejections being made. The Laplace density will help to reduce the widening space between the exponential curve and the gamma curve around this upper tail. This is because, for suitably selected parameter values of the Laplace distribution, the distribution before its mean forms a mirror image of the distribution after its mean (i.e. symmetric about its mean) and thus fits the gamma distribution which becomes more and more symmetric as α increases. For $\alpha = 16$, for instance, this idea is illustrated in Figure 1, which is plotted after setting the parameter values of the majorizing density as will be seen in section 3.1.

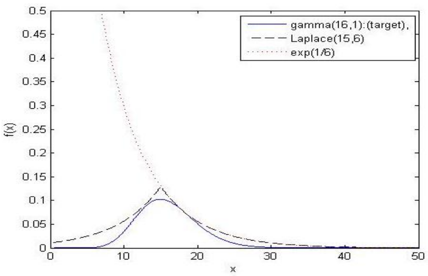


Figure 1: Illustrating the rationale for choosing the Laplace distribution as envelope for the gamma distribution.

3.2 Setting the parameters of the Laplace distribution

Observe that when $\alpha = 1$, we have that:

$$Gamma(1, \beta) = \exp\left(\frac{1}{\beta}\right) = 2 \times Laplace(0, \beta) \quad (9)$$

where$Laplace(0, \beta)$ denotes the Laplace distribution with location parameter $\mu = 0$ and scale parameter $b = \beta$ (for $x > 0$ only, in this case, since the gamma distribution has the support $x > 0$). Following this, simple multiples of the scale parameter b for changing values of α will yield a good scaled density to majorize the gamma density gamma density in question. Specifically, for the location parameter and scale parameters, $\mu$and b, we perform the following initialization.

Initialization

1. Set $\mu = \beta(\alpha - 1)$ [i.e. the mode of $Gamma(\alpha, \beta)$ for $\alpha > 1$],
2. Set $b = 2\beta$ if$1 < \alpha < 5$
3. Set $b = 2n\beta$ if $5n < \alpha < 10n$

where$n = [\alpha/5]$ and $[\ x\ ]$ is the greatest integer function of x, that is, the largest integer less than or equal to x. Note that the initialization is for the general case of $\beta > 0$, while for the standard gamma distribution under study here, we recall that $\beta = 1$.

The algorithm for our method is as follows.

Algorithm

1. Generate $U_1 \sim Unif(0, 1)$ and then transform to $q = U_1 - \frac{1}{2}$ so that $q \sim U\left(-\frac{1}{2}, \frac{1}{2}\right)$
2. Compute $Y = \mu - b\ \mathrm{sgn}(q)\ \ln(1 - 2|q|)$ and $h(Y) = \frac{f(Y)}{cg(Y)}$, where $c = \sup_{x} \{f(x)/g(x)\}$
3. Generate $U_2 \sim U(0,1)$
4. If $U_2 \leq h(Y)$, return $X = Y$, otherwise, go to step 1.

Now, computing c in this case was not analytically tractable because $\sup_{x}[h(x)]$ where

$$h(x) = \frac{f(x)}{g(x)} = \frac{\left(x^{\alpha-1}\exp(-x)/\Gamma(\alpha)\right)}{\left(\frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right)\right)}$$

could not be obtained algebraically. However, we computed the values of c numerically by assigning values to x within the domain of definition of $h(x)$, i.e. $x > 0$, [e.g. **x** = linspace(0.001, 10^4, 10^5) which is a vector of points x] using MATLAB®, given the initialized values of the parameters μ and b as set in section 3.1 for different values of α. This assignment helped us to evaluate $\boldsymbol{h}(\mathbf{x})$, a vector of points evaluated at each element, x of the vector, **x**. Then c was obtained as the maximum value of the vector $\boldsymbol{h}(\mathbf{x})$. Table 1 shows thevalues of c of our method and the corresponding acceptance rates, $1/c$, denoted by $ar(OM)$ alongside with selected efficiency values from Marsaglia and Tsang (2000) denoted by $ar(MM)$ at three different parameter values.

Table 1: ***Values of c from our method, acceptance rates $ar(OM)$ of our method andacceptance rates $ar(MM)$ of Marsaglia & Tsang's Method.***

α	c	$ar(OM)$	$ar(MM)$
2	1.4715	0.6796	0.9817
4	1.5997	0.6251	0.9920
8	2.3037	0.4341	0.9963

4. Simulation Study

In this section we present the steps taken in performing the simulation study of the two methods of our comparison: namely, Our Method (OM) and Marsaglia& Tsang's Method (MM). For the comparison, a total of $M = 10{,}000$ samples, each of sizes $n = 25, 50, 100, 200$ were generated from the standard gamma distribution using each of the two methods under study, with the same uniform random random number U generated and used in each case, using MATLAB® uniform random numbers generator, rand. Parameter values were chosen from those for whichefficiencies values were reported in Marsaglia and Tsang (2000) for the purpose of comparison. Maximum likelihood estimates (MLEs) of the samples generated were computed using the MATLAB code, mle, which returns simultaneously, the confidence intervals (CIs) for the parameter estimates. Hence we easily computed the CI widths as the difference between the upper and lower CI bounds. Apart from the Average CI widths, the CIs were also used to compute the coverage probabilities (CPs). The main statistics computed as basis for comparison (see [12]) were:

(a) Average bias of parameter estimates of $\hat{\alpha}_i, i = 1,2 \ldots, M$, is given by:
$\frac{1}{M}\sum_{i=1}^{M}(\hat{\alpha}_i - \alpha)$

(b) Average Mean Square Error (MSE) of parameter estimates of $\hat{\alpha}_i,\ i = 1,2 \ldots, M$, is given by:
$\frac{1}{M}\sum_{i=1}^{M}(\hat{\alpha}_i - \alpha)^2$

(c) Average width of 95% confidence intervals (CIs) of parameter estimates, given by:
$\frac{1}{M}\sum_{i=1}^{M}(\widehat{U}_i - \widehat{L}_i)$

where$\widehat{U}_i$and $\widehat{L}_i$ are the upper and lower limits of the ith CI of the parameter estimate $\hat{\alpha}_i, i = 1,2, \ldots, M$, respectively.

(d) Coverage Probability (CP) which is computed as the proportion of $M = 5{,}000$ estimated CIs that contained the true parameter value, α.

4. Results and discussion:

In Table 2a, we present the results of simulation of four sample sizes ($n = 25, 50, 100, 200$) of the standard gamma distribution with $\alpha = 2$, that is, $Gamma(2,1)$. In Table 2b, the corresponding CI widths and CPs of the estimates in Table 2a are displayed. Please note that the results in Tables 2a – 4b are displayed here in the format in which they were output in MATLA® R2010b.

In Table 2a, for $\alpha = 2$, we can easily see that the sample estimates for is better (having closer average MLEs to actual parameters values and hence smaller biases and MSEs) for all the sample sizes considered using our method (O-M) than using Marsaglia and Tsang's method (M-M). In the same vein, the average width of confidence intervals (Ave. Width) for O-M in Table 2b, are smaller than those of M-M and the coverage probabilities (Cov. Prob) are higher for O-M than for M-M for all the sample sizes considered. Yet at this value (i.e. $\alpha = 2$), the acceptance rate is just about 68% compared to that of M-M which is about 98%.

Table 2a: Ave. MLE, Ave. Bias and Ave. MSE of 10000 parameter estimates of Gamma(2,1) for n = 25, 50, 100, 200.

Sample size(n)		Our Method(O-M)		M&T Method (M-M)	
		alpha	beta	alpha	beta
	25				
Ave. MLE:		2.2361	0.9658	2.2574	0.9552
Ave. Bias:		0.2361	-0.0342	0.2574	-0.0448
Ave. MSE:		0.0558	0.0012	0.0663	0.0020
	50				
Ave. MLE:		2.1138	0.9812	2.1277	0.9744
Ave. Bias:		0.1138	-0.0188	0.1277	-0.0256
Ave. MSE:		0.0129	0.0004	0.0163	0.0007
	100				
Ave. MLE:		2.0549	0.9903	2.0701	0.9828
Ave. Bias:		0.0549	-0.0097	0.0701	-0.0172
Ave. MSE:		0.0030	0.0001	0.0049	0.0003
	200				
Ave. MLE:		2.0256	0.9972	2.0388	0.9884
Ave. Bias:		0.0256	-0.0028	0.0388	-0.0116
Ave. MSE:		0.0007	0.0000	0.0015	0.0001

Table 2b: Ave. Width of 10000 estimated 95% CIs and CPs for parameter estimates of Gamma(2,1) for n = 25, 50, 100, 200.

Sample size(n)	Our Method (O-M) alpha	Our Method (O-M) beta	M&T Method(M-M) alpha	M&T Method(M-M) beta
25				
Ave.Width:	2.4258	1.1986	2.4502	1.1848
Cov.Prob(%):	93.25	93.13	92.99	92.99
50				
Ave.Width:	1.5797	0.8358	1.5907	0.8297
Cov.Prob(%):	94.31	93.99	94.17	94.17
100				
Ave.Width:	1.0718	0.5878	1.0802	0.5830
Cov.Prob(%):	94.26	94.17	94.05	94.05
200				
Ave.Width:	0.7422	0.4155	0.7473	0.4116
Cov.Prob(%):	94.72	94.73	94.69	94.69

Table 3a shows the results of average estimates of 10,000 replicates of sample sizes $n = 25, 50, 100, 200$ under simulation study. Table 3b shows the corresponding CI widths and CPs of the estimates in Table 3a.

Table 3a: Ave. MLE, Ave. Bias and Ave. MSE of 10000 parameter estimates Gamma(4,1)for n = 25, 50, 100, 200.

Sample size(n)	Our Method(O-M) alpha	Our Method(O-M) beta	M&T Method (M-M) alpha	M&T Method (M-M) beta
25				
Ave. MLE:	4.4933	0.9632	4.5054	0.9641
Ave. Bias:	0.4933	-0.0368	0.5054	-0.0359
Ave. MSE:	0.2433	0.0014	0.2555	0.0013
50				
Ave. MLE:	4.2525	0.9778	4.2544	0.9773
Ave. Bias:	0.2525	-0.0222	0.2544	-0.0227
Ave. MSE:	0.0638	0.0005	0.0647	0.0005
100				
Ave. MLE:	4.1033	0.9929	4.1308	0.9873
Ave. Bias:	0.1033	-0.0071	0.1308	-0.0127
Ave. MSE:	0.0107	0.0001	0.0171	0.0002
200				
Ave. MLE:	4.0508	0.9959	4.0673	0.9927
Ave. Bias:	0.0508	-0.0041	0.0673	-0.0073
Ave. MSE:	0.0026	0.0000	0.0045	0.0001

Table 3b: Ave. Width of 10000 estimated 95% CIs and CPs for parameter estimatesof Gamma(4,1) for n = 25, 50, 100, 200.

Sample size(n)	Our Method (O-M) alpha	Our Method (O-M) beta	M&T Method(M-M) alpha	M&T Method(M-M) beta
25				
Ave.Width:	5.0424	1.1550	5.0564	1.1561
Cov.Prob(%):	93.55	93.36	93.43	93.43
50				
Ave.Width:	3.2891	0.8069	3.2905	0.8064
Cov.Prob(%):	93.98	93.99	94.14	94.14
100				
Ave.Width:	2.2148	0.5716	2.2301	0.5683
Cov.Prob(%):	94.23	94.63	93.95	93.95
200				
Ave.Width:	1.5362	0.4026	1.5426	0.4013
Cov.Prob(%):	94.77	94.71	94.41	94.41

Again, in Tables 3a and 3b, Our Method O-M seems to have outperformed M-M in terms of smaller biases and MSEs, smaller CIs and higher CPs for the estimates of α, except for the CPs under the sample size 50 where that of M-M exceeds

O-M. Yet again, this happened even though the acceptance rate of our method, $ar(OM)$, has reduced from roughly 68% to 63% whereas that of Marsaglia and Tsang (2000), $ar(MM)$, has increased from roughly 98% to 99% as shown in Table 1

Table 4a: Ave. MLE, Ave. Bias and Ave. MSE of 10000 parameter estimates of Gamma(100,1)for n = 25, 50, 100, 200.

Sample size(n)		Our Method(O-M) alpha	Our Method(O-M) beta	M&T Method (M-M) alpha	M&T Method (M-M) beta
25					
	Ave. MLE:	114.0807	0.9579	113.3447	0.9600
	Ave. Bias:	14.0807	-0.0421	13.3447	-0.0400
	Ave. MSE:	198.2655	0.0018	178.0808	0.0016
50					
	Ave. MLE:	106.0085	0.9816	106.4552	0.9792
	Ave. Bias:	6.0085	-0.0184	6.4552	-0.0208
	Ave. MSE:	36.1021	0.0003	41.6698	0.0004
100					
	Ave. MLE:	102.2162	0.9985	103.0975	0.9904
	Ave. Bias:	2.2162	-0.0015	3.0975	-0.0096
	Ave. MSE:	4.9116	0.0000	9.5942	0.0001
200					
	Ave. MLE:	101.4403	0.9961	101.5630	0.9949
	Ave. Bias:	1.4403	-0.0039	1.5630	-0.0051
	Ave. MSE:	2.0746	0.0000	2.4429	0.0000

Table 4b: Ave. Width of 10000 estimated 95% CIs and CPs for parameter estimatesof Gamma(100,1) for n = 25, 50, 100, 200.

Sample size(n)		Our Method (A-M) alpha	Our Method (A-M) beta	M&T Method(M-M) alpha	M&T Method(M-M) beta
25					
	Ave.Width:	132.9293	1.1186	131.9908	1.1208
	Cov.Prob(%):	92.66	92.63	93.20	93.20
50					
	Ave.Width:	85.1370	0.7903	85.4727	0.7882
	Cov.Prob(%):	94.43	94.42	93.73	93.73
100					
	Ave.Width:	57.3063	0.5612	57.7922	0.5566
	Cov.Prob(%):	94.63	94.61	94.11	94.11
200					
	Ave.Width:	39.9579	0.3933	40.0009	0.3928
	Cov.Prob(%):	94.18	94.30	94.68	94.68

Finally, in Table 4a, a very large value of $\alpha = 100$ was considered. The results yet again are better for O-M than M-M. Except for the smallest sample size of n =25, the average estimates of the samples generated were closer to the actual value for O-M than M-M in all cases. As a result, they have smaller average biases and MSEs. The coverage probabilities (CPs) are higher for O-M for sample sizes $n = 50, 100$. However, the CPs are higher for sample sizes $n = 25, 100$ for M-M.

Figures 2a, 2b, 2c and 2d show the distributions of the estimates of the parameters of $Gamma(2,1)$for sample sizes 25, 50, 100 and 200, respectively, whose MLEs are shown in Table 2a. It can easily be noticed that the distributions are spread around the true parameter values ($\alpha = 2$and $\beta = 1$) for both methods. The distributions are more symmetric about the true parameters values as the sample size increases.

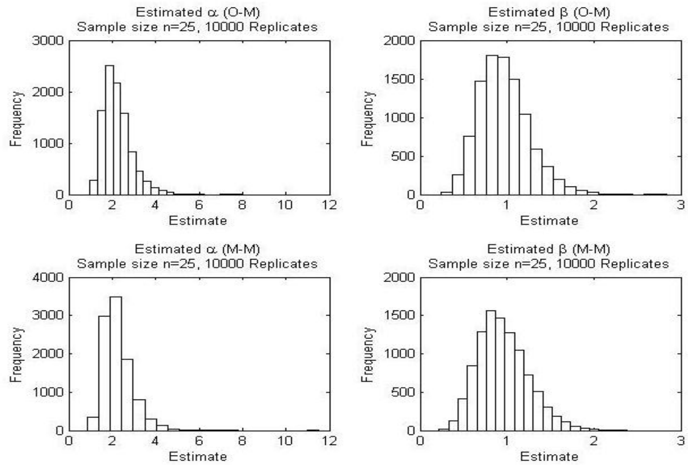
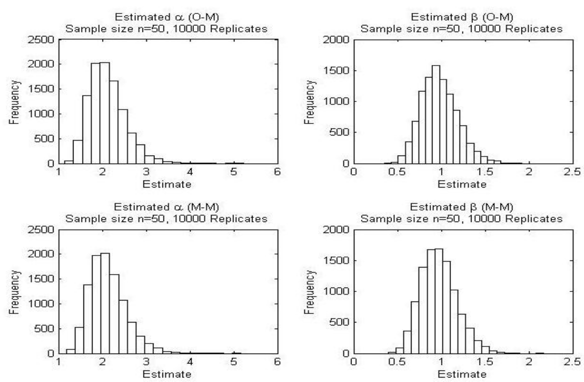


Figure 2a: ***Distribution of 10,000 estimates of*** $Gamma(2, 1)$ ***for sample size*** $n = 25$



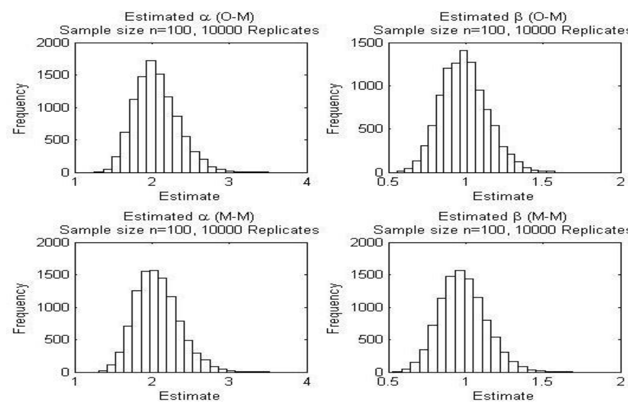


Figure 2b: ***Distribution of 10,000 estimates of*** $Gamma(2, 1)$ ***for sample size*** $n = 50$ **Figure 2c:** ***Distribution of 10,000 estimates of*** $G(2, 1)$ ***for sample size*** $n = 100$

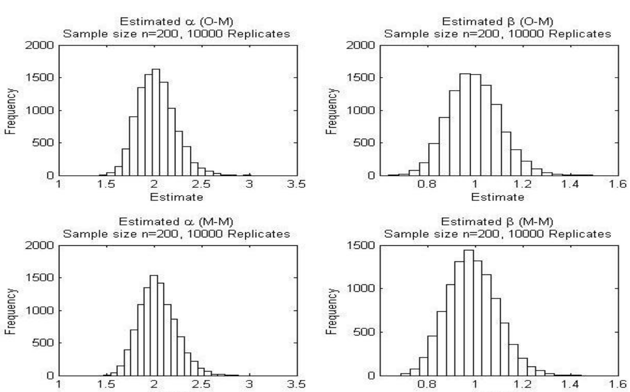


Figure 2d: ***Distribution of 10,000 estimates of*** $G(2, 1)$ ***for sample size*** $n = 200$

In a similar manner, Figures 3a – 3d, show the distributions of the estimates of the parameters of $Gamma(4,1)$for sample sizes 25 – 200 whose MLEs are shown in Table 3a. Again the distributions have the true parameter values approximately as their mode and the distributions become more symmetric about the true parameter values as the sample size increases.

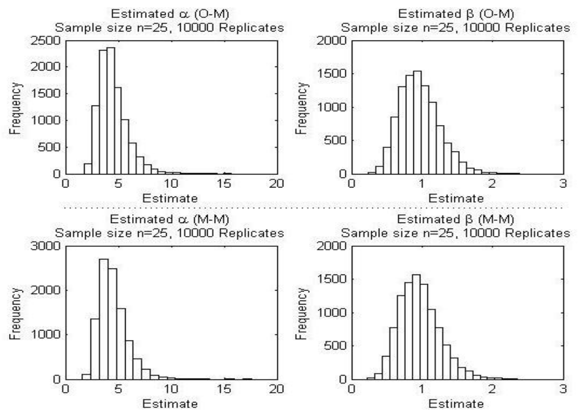


Figure 3a: ***Distribution of 10,000 estimates of*** $G(4,1)$ ***for sample size*** $n = 25$

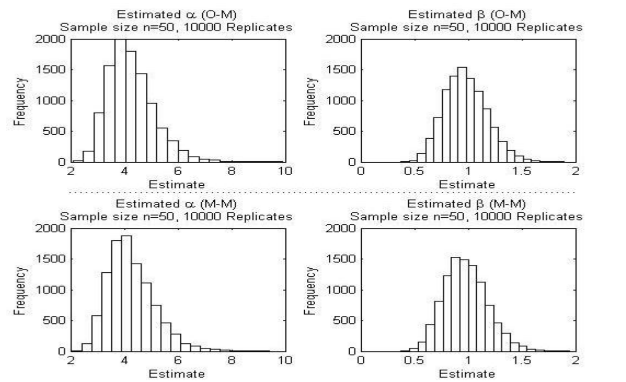


Figure 3b: ***Distribution of 10,000 estimates of*** $G(4,1)$ ***for sample size*** $n = 50$

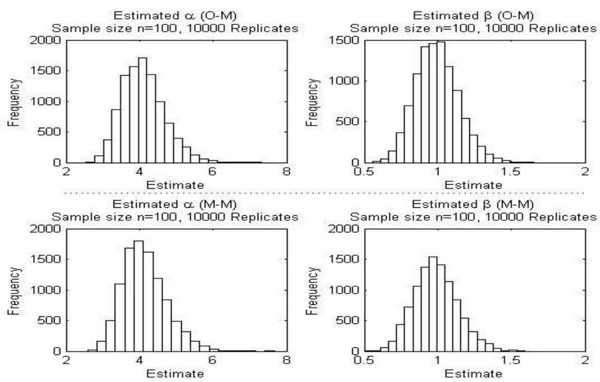


Figure 3c: ***Distribution of 10,000 estimates of*** $G(4,1)$***for sample size*** $n = 100$

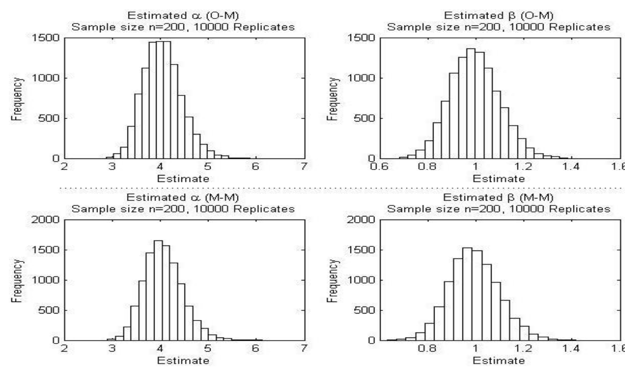


Figure 3d: ***Distribution of 10,000 estimates of*** $G(4,1)$***for sample size*** $n = 100$

Finally, the last set of figures (Figures 4a – 4d) show the distribution of the estimates of the parameters corresponding to Table 4a for $Gamma(100,1)$. A similar thing to the cases of $Gamma(2,1)$and $Gamma(4,1)$can be inferred from Figures 4a – 4d.

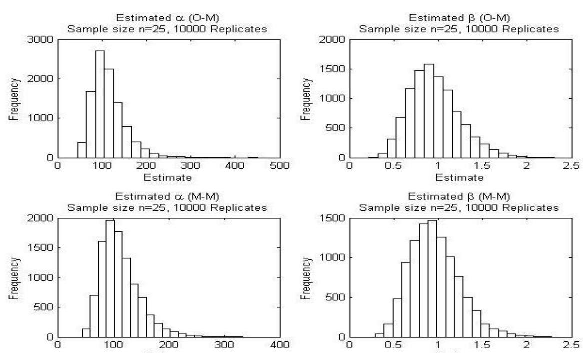


Figure 4a: ***Distribution of 10,000 estimates of*** $Gamma(100,1)$***for sample size*** $n = 100$

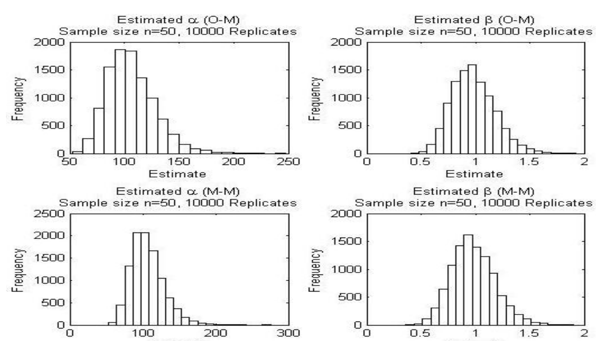


Figure 4b: ***Distribution of 10,000 estimates of*** $Gamma(100,1)$***for sample size*** $n = 100$

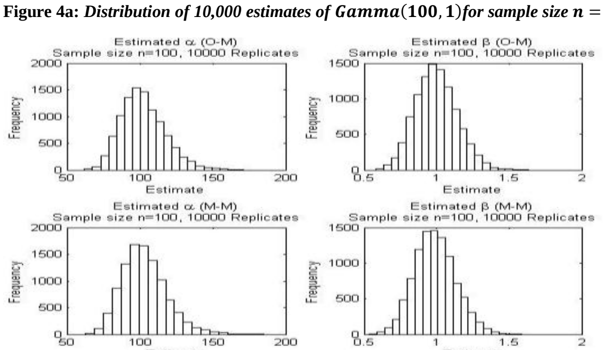


Figure 4c: ***Distribution of 10,000 estimates of*** $Gamma(100,1)$***for sample size*** $n = 100$

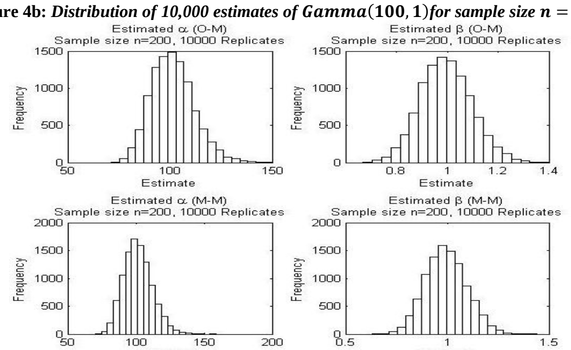


Figure 4d: ***Distribution of 10,000 estimates of*** $Gamma(100,1)$***for sample size*** $n = 200$

5. Conclusion

The fact that a method has a high acceptance rate is plausible but does not necessarily imply that it generates samples with the best statistical properties for a given distribution under simulation. This claim has been justified to a great extent by this study which compared the fast method by [6] for generating gamma variables with our method for which the acceptance rate is relatively smaller. Although the acceptance rate differ very much (e.g. only approximately 63% for O-M as against nearly 100% for M-M at $\alpha = 4$), the estimates of 10,000 samples have better properties (smaller biases, MSEs and CIs, higher coverage probabilities, etc.) for most of the sample values using O-M than using M-M. Since there seems to be a tradeoff between high acceptance rate and better sample estimates, the choice for any of the two methods compared may be determined by the researcher's immediate need. If the need is to perform a bulk of time-efficient simulations, Marsaglia and Tsang's method may prove more useful because of its speed of generating gamma random variables, whereas if the need is a good representative sample for a simple study which does not require a large number of simulations, our method may be preferable.

Appendix

Solving the equation:

$$u = G(y) = \begin{cases} \frac{1}{2}\exp\left(\frac{y-\mu}{b}\right), & y < \mu \\ 1 - \frac{1}{2}\exp\left(-\frac{y-\mu}{b}\right), & y \geq \mu \end{cases}$$

for y in terms of u, we have:

$$y = G^{-1}(u) = \begin{cases} \mu + b\ln(2u), & y < \mu \\ \mu - b\ln[2(1-u)], & y \geq \mu \end{cases}$$

Now, we express $y = G^{-1}(u)$ in a single equation with the signum (sgn) function as:

$$y = G^{-1}(u) = \mu - \text{sgn}(y-\mu)b\ln\{1 + \text{sgn}(y-\mu)[1-2u]\}$$

Where $\text{sgn}(y-\mu) = \begin{cases} -1, & y < \mu \\ 0, & y = \mu \\ 1, & y > \mu \end{cases}$

Since the random variable $Y \sim Laplace(\mu, b)$ is symmetric about its mean, μ, and the random variable, $U \sim unif(0,1)$ is also symmetric about its mean, $1/2$, it follows that:

$$\text{sgn}(y-\mu) = \text{sgn}\left(u - \frac{1}{2}\right)$$

Thus,

$$y = G^{-1}(u) = \mu - \text{sgn}\left(u-\frac{1}{2}\right) b \ln\left\{1 + sgn\left(u-\frac{1}{2}\right) - 2u\, sgn\left(u-\frac{1}{2}\right)\right\}$$

Finally, we know that $a + (b-a)u \sim unif(a,b)$ if $u \sim unif(0,1)$.

Hence, $-\frac{1}{2} + \left[\frac{1}{2} - \left(-\frac{1}{2}\right)\right]u = u - \frac{1}{2} \sim unif\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Let $q = u - \frac{1}{2}$, so that $q \sim unif\left(-\frac{1}{2}, \frac{1}{2}\right)$. Then,

$$\begin{aligned} y = G^{-1}(q) &= \mu - b\,\text{sgn}(q)\ln\left\{1 + sgn(q) - 2\left(q + \frac{1}{2}\right)sgn(q)\right\} \\ &= \mu - b\,\text{sgn}(q)\ln\{1 - 2q\, sgn(q)\} \\ &= \mu - b\,\text{sgn}(q)\ln(1 - 2|q|) \blacksquare \end{aligned}$$

We have used the fact that $= q + \frac{1}{2}$, $\text{sgn}(q) = \begin{cases} -1, & q < 0 \\ 0, & q = 0 \\ 1, & q > 0 \end{cases}$ and $|q| = \begin{cases} -q, & q < 0 \\ q, & q \geq 0 \end{cases}$

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