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IRREVERSIBILITY ANALYSIS FOR A THIN FILM OF THIRD-GRADE FLUID OVER HEATED INCLINE PLATE

***J.C. Ukaegbu*[1], *D.M. Okewole*[2], *T.A. Yusuf*[1] *and S.O. Adesanya*[2]**

[1]Department of Mathematics, Faculty of Science, Adeleke University, Ede, Nigeria.
[2]Department of Mathematical Sciences, Faculty of Natural Sciences, Redeemer's University, Ede, Nigeria.

Abstract

The present article investigates the effect of flow and heat transfer characteristics of a thin layer of third grade fluid that is flowing steadily over a heated incline plate. The mathematical formulations for the physical system are based on the combination of momentum together with heat conservation laws that forms the theoretical framework for the present investigation. The exact solutions of the dimensionless nonlinear problems are obtained and valid for both small and large parameter values. The effects of viscous dissipation and heat transfer are balanced in the energy equation and play a major role in the entropy generation procedure. The result of the mathematical analysis revealed the significance of thermal effects together with the entropy built-up due to incomplete energy conversion.

Keywords: third-grade fluid; thin film; entropy generation; heat irreversibility

Nomenclature

(E_G, N_S)	Dimensional and dimensionless entropy generation rate
(u', u)	Dimensional and dimensionless fluid velocity
(T, θ)	Dimensional and dimensionless fluid temperature
(β', β)	Dimensional and dimensionless non-Newtonian parameter
(T_0, T_f)	Initial and final fluid temperature
Br	Brinkman number
δ	Plate distance
Ω	Temperature difference
Be	Bejan Number
μ	Fluid viscosity
k	Fluid thermal conductivity
ρ	Fluid density
g	Acceleration due to gravity

1. Introduction

In a recent paper, Hayat *et al* [1] provided a broader approach in obtaining the exact solution to nonlinear third-grade fluid flow model which is valid for both small and large parameter values of the nonlinear term. The same problem was earlier investigated by Siddiqui *et al* [2] and the solution obtained was valid for only small parameter values. Moreover, Abbasbandy *et al* [3] and Kumaran *et al* [4] were able to validate the accuracy of the exact solution by comparing with solutions obtained by using Homotopy Analysis Method (HAM) and asymptotic techniques respectively. However, in many practical situations, thin layer of third grade fluid are witnessed on hot solid boundaries on a nanometer or micrometer

Correspondence Author: Adesanya S.O., Email: adesanyas@run.edu.ng, Tel: +2348055161181

scale. For instance, on solid surfaces in which thin film are applied to reduce abrasion, corrosion and adsorption at very high temperature. Aziz and Mahomed [5] constructed a class of closed-form solutions with the principle of Lie group method for boundary value problems arising in the study of time-dependent third- grade fluids thorough a porous material.
Under these conditions, a new set of governing equation must be introduced to monitor heat transfer along the heated inclined plane. Having such vital implications of heat transfer analysis in view, excellent critical review in heat transfer analysis of third grade fluid can be seen in [6-7]. Similarly, it is a known fact that many third-grade fluid flows under intense heat due to the non-Newtonian nature, for example, polymer extrusion, molten steel and in-situ combustion during oil recovery or by thermal cracking. In all these applications, heat cannot be neglected. In fact, Okoya [8-11] showed that some non-Newtonian fluid such as the third-grade fluid undergoes a strong exothermic chemical reaction that follows Arrhenius kinetics.

In view of the foregoing studies, the goal is to extend the work done in [1-4] above to include the first and second law of thermodynamics based on Bejan's approach [12-13]. The thermal analysis presented here is based on the exact solution procedure in [1]. Following Bejan, the approach reveals component and subcomponent encouraging energy wastage. When this is taken care of, thermo-fluid systems can be easily upgraded so that efficiency can be improved. When industrial machines produce optimally, economic growth and industrialization is encouraged in accordance with the eighth and ninth united nation's sustainable development goals. In the following section, the problem formulation is introduced.

2. Problem Formulation

The steady flow of thin film of third-grade fluid down an infinitely long inclined heated plate is considered. The fluid is non-Newtonian and follows the third-grade constitutive model as presented in [1-5]. Considering a constant thickness of the thin film, equation for the flow velocity and the appropriate boundary conditions in the differential form are given by [1-4]:

$$0=\rho g Sin\alpha+\mu\frac{d^2u'}{dy'^2}+6(\beta_2+\beta_3)\left(\frac{du'}{dy'}\right)^2\frac{d^2u'}{dy'^2} \tag{1}$$

With

$$u'(0)=0,\quad \left(\frac{du'}{dy'}+2(\beta_2+\beta_3)\left(\frac{du'}{dy'}\right)^3\right)_{y=\delta}=0 \tag{2}$$

Since most engineering applications involving non-Newtonian fluid are always accompanied with thermal effect as seen in many metallurgical, manufacturing and industrial processes, in such cases, the velocity interaction with the heat transfer from the heated inclined plate cannot be neglected. Thus the modified problem presented here balances the heat generated by frictional interaction with heat transfer. In accordance with the first law of thermodynamics, the differential form for the balanced energy equation is given by:

$$0=k\frac{d^2T}{dy'^2}+\mu\left(\frac{du'}{dy'}\right)^2\left(1+\frac{2(\beta_2+\beta_3)}{\mu}\left(\frac{du'}{dy'}\right)^2\right) \tag{3}$$

and

$$T(0)=T_0\,,\quad \frac{\partial T}{\partial y'}(\delta)=0 \tag{4}$$

From statistical mechanics viewpoint, sustainability of the thermo-fluid set up depends largely on optimal conversion, efficiency of equipment and waste management. As a result, the randomness and the disorder linked with the flow and heat transfer can be monitored using the second law of thermodynamics:

$$E_G=\frac{k}{T_0^2}\left(\frac{dT}{dy'}\right)^2+\frac{\mu}{T_0}\left(\frac{du'}{dy'}\right)^2\left(1+\frac{2(\beta_2+\beta_3)}{\mu}\left(\frac{du'}{dy'}\right)^2\right) \tag{5}$$

accurate dimensionless procedure for (1) - (5) by using the following

$$y=\frac{y'}{\delta},u'=\frac{\rho g\delta^2\sin\alpha}{\mu}u,\beta=\frac{(\beta_2+\beta_3)}{\mu^3}(\rho g\delta\sin\alpha)^2,\ \theta=\frac{T-T_0}{T_f-T_0},\ \Omega=\frac{T_f-T_0}{T_0} \tag{6}$$

$$Br=\frac{\mu\left(\rho g\delta^2\sin\alpha\right)^2}{k\left(T_f-T_0\right)}$$

leads to the following boundary value differential problems

$$0=1+\frac{d}{dy}\left(\frac{du}{dy}+2\beta\left(\frac{du}{dy}\right)^3\right);u(0)=0=\left(\frac{du}{dy}+2\beta\left(\frac{du}{dy}\right)^3\right)_{y=1} \tag{7}$$

also, the energy equation becomes

$$0=\frac{d^2\theta}{dy^2}+Br\left(\frac{du}{dy}\right)^2\left(1+2\beta\left(\frac{du}{dy}\right)^2\right);\theta(0)=0=\frac{d\theta}{dy}(1) \tag{8}$$

While the entropy profile can be obtained from the dimensionless form

$$Ns=\left(\frac{d\theta}{dy}\right)^2+\frac{Br}{\Omega}\left(\frac{du}{dy}\right)^2\left(1+2\beta\left(\frac{du}{dy}\right)^2\right) \tag{9}$$

The Bejan number is given by the ratio

$$Be=\frac{\left(\frac{d\theta}{dy}\right)^2}{\left(\frac{d\theta}{dy}\right)^2+\frac{Br}{\Omega}\left(\frac{du}{dy}\right)^2\left(1+2\beta\left(\frac{du}{dy}\right)^2\right)} \tag{10}$$

that is, the ratio of heat transfer irreversibility to the total entropy generated.

3. Method of Solution

A simple integration of (7) yields with the relevant boundary condition gives

$$\frac{du}{dy}+2\beta\left(\frac{du}{dy}\right)^3=1-y \tag{11}$$

The solution of the cubic nonlinear differential equation (11) is by

$$\frac{du}{dy}=\frac{2^{\frac{1}{3}}}{\left(108\beta^2(1-y)+\sqrt{864\beta^3+\left(108\beta^2(1-y)\right)^2}\right)^{\frac{1}{3}}}+\frac{\left(108\beta^2(1-y)+\sqrt{864\beta^3+\left(108\beta^2(1-y)\right)^2}\right)^{\frac{1}{3}}}{6\times 2^{\frac{1}{3}}\beta} \tag{12}$$

The solution obtained in (12) is in perfect agreement with the results in [3]-[4]. Further simplification leads to

$$\frac{du}{dy}=\frac{-6^{2/3}\beta+6^{1/3}\left(-9(-1+y)\beta^2+\sqrt{3}\sqrt{\beta^3\left(2+27(-1+y)^2\beta\right)}\right)^{2/3}}{6\beta\left(-9(-1+y)\beta^2+\sqrt{3}\sqrt{\beta^3\left(2+27(-1+y)^2\beta\right)}\right)^{1/3}} \tag{13}$$

It is easy to note that $y=1$ is a singular point, therefore, by change of variable, it is convenient to set $\eta=y-1$ so that we have the following smooth first derivative

$$\frac{du}{d\eta}-\frac{-6^{2/3}\beta+6^{1/3}\left(-9\eta\beta^2+\sqrt{3}\sqrt{\beta^3\left(2+27(\eta)^2\beta\right)}\right)^{2/3}}{6\beta\left(-9\eta\beta^2+\sqrt{3}\sqrt{\beta^3\left(2+27(\eta)^2\beta\right)}\right)^{1/3}}=0;u(1)=0 \tag{14}$$

The integral of (14) leads to

$$\begin{aligned}
u(\eta)=&\frac{1}{A_8}2^{1/3}3^{1/6}\left(A_1\left(3^{1/6}\left(-1+54\beta\eta^2+1458\beta^2\eta^4\right)A_3\right)\right)\begin{pmatrix}\sqrt{3}\beta\left(2+135\beta+1458\beta^2\right)\\-27(1+18\beta)A_6\end{pmatrix}\\
&+3\beta\left(1623^{1/6}\beta^2\eta^3\left(2+27\beta\eta^2\right)A_4\right)\left(-\beta\left(2+135\beta+1458\beta^2\right)+9\sqrt{3}(1+18\beta)A_6\right)\\
&+2^{1/3}\begin{pmatrix}9\beta^2\eta\left(2+99\beta\eta^2+972\beta^2\eta^4\right)\begin{pmatrix}\sqrt{3}\beta\left(2+135\beta+1458\beta^2\right)\\-27(1+18\beta)A_6\end{pmatrix}-\\\left(1+189\beta\eta^2+2916\beta^2\eta^4\right)A_5\left(\beta\left(2+135\beta+1458\beta^2\right)-9\sqrt{3}(1+18\beta)A_6\right)\end{pmatrix}\\
&+A_2\left(27\eta\left(1+18\beta\eta^2\right)A_5\begin{pmatrix}27\beta^3(2+27\beta)\left(2^{1/3}(1+36\beta)-63^{2/3}A_7\right)\\-3^{1/6}A_6\begin{pmatrix}6^{1/3}\beta\left(1+189\beta+2916\beta^2\right)\\+\left(1-54\beta-1458\beta^2\right)A_7\end{pmatrix}\end{pmatrix}\right)\\
&-\beta\left(2+135\beta\eta^2+1458\beta^2\eta^4\right)\begin{pmatrix}273^{1/6}\beta^3(2+27\beta)\left(6^{1/3}(1+36\beta)-18A_7\right)\\-A_6\begin{pmatrix}32^{1/3}\beta\left(1+189\beta+2916\beta^2\right)\\+3^{2/3}\left(1-54\beta-1458\beta^2\right)A_7\end{pmatrix}\end{pmatrix}
\end{aligned} \tag{15}$$

also substituting (15) in and the transformation $\eta = y-1$ in (8), we obtain

$$\theta(\eta) = (3^{1/6}\beta^7(2975389355523^{1/6}\beta^8\eta^{13} + B_1\left(143^{2/3} - 717\eta B_2\right) + 2\beta\begin{pmatrix} 10533^{2/3}\eta^2 B_1 + \\ 7\sqrt{3}\left(-18\beta^2\eta + 2\sqrt{3}B_1\right)^{1/3} \\ -86265\eta^3 B_1 B_2 \end{pmatrix}$$
$$-472392\beta^5\eta^7\begin{pmatrix} -10433^{1/6} + 187923^{2/3}\eta^3 B_1 \\ -2238\sqrt{3}\eta B_2 + 20412\eta^4 B_1 B_2 \end{pmatrix} - 26244\beta^4\eta^5(-3563^{1/6} + 311043^{2/3}\eta^3 B_1$$
$$-2139\sqrt{3}\eta B_2 + 102060\eta^4 B_1 B_2) - 486\beta^3\eta^3\begin{pmatrix} -233^{1/6} + 612363^{2/3}\eta^3 B_1 \\ -2757\sqrt{3}\eta B_2 \\ +534600\eta^4 B_1 B_2 \end{pmatrix} - 3\beta^2\eta(2363^{1/6}$$
$$+1030323^{2/3}\eta^3 B_1 - 3819\sqrt{3}\eta B_2 + 3560436\eta^4 B_1 B_2) - 765275043^{1/6}\beta^6\eta^9(-132 + 432\sqrt{3}\eta^3 B_1$$
$$-119\eta\left(-54\beta^2\eta + 6\sqrt{3}B_1\right)^{1/3}) + 41324852163^{1/6}\beta^7\eta^{11}\left(22 + 7\eta\left(-54\beta^2\eta + 6\sqrt{3}B_1\right)^{1/3}\right))\lambda) \tag{16}$$
$$/\left(1402^{2/3}\left(-9\beta^2\eta + \sqrt{3}B_1\right)^{17/3}\left(-2\beta - 27\beta^2\eta^2 + 3\sqrt{3}\eta B_1\right)\left(-\beta - 27\beta^2\eta^2 + 3\sqrt{3}\eta B_1\right)\right) + K + \eta L$$

The expressions for the integration constants K and L together with the constant $A_i\left(i=1,2...8\right)$, $B_i\left(i=1,2\right)$ are defined in the appendix. Finally, the solutions of $\theta(y), u(y)$ are used in (9)-(10) to compute the entropy generation and Bejan number respectively, the graphical results are presented in section four of the paper.

4. Discussion of Results

In this section, the results of the simulation are presented for both large and small parameter values. The errors encountered in the result in [1] are corrected for the velocity profile. The exact solutions are then utilized for the dimensionless energy equation with the associated boundary conditions for both small and large values of the flow parameters. Table 1 highlights the excellent agreement between the exact and numerical results obtained using Spectra-collocation method for fixed values of non-Newtonian parameter and Brinkman number. Figures 1-4 represent the solutions in the large parameter values while the results in the small parameter values are presented in Figures 5-8. As seen in both Figures 1 and 2, an increase in the values of the third grade material parameter leads to decrease in the fluid velocity profiles for both small and large parameter values. This is physically correct due to increasing boundary layer thickness along the inclined plate; since fluid velocity has an inverse relationship with the resistive nature of viscosity is the fluid on solid boundaries. Therefore, both velocity profiles are well behaved. Moreover, the fluid temperature is observed to decreases with the increase in the third grade material parameter due to thickening nature of the fluid and the reduced fluid-particles interaction along the heated inclined plate. It is important to note that the reduction becomes more pronounced at the free end as shown in Figures 3-4.
Figures 5 and 6 give an insight into the entropy generation within the thin fluid layers. As seen from the two Figures, the entropy value is much higher with small third grade fluid parameter than the entropy values for the large third grade parameter values. This simply shows that entropy generation decreases with increasing values of the material parameter. The behavior is physically true since the fluid velocity and the kinetic energy decreases with higher values of the third grade material parameter.

Table 1: Result validation for $\beta = 0.1, Br = 0.5$

y	$u(y)$		**Absolute Error**	$\theta(y)$		**Absolute Error**
	Exact	**Numerical method**		**Exact**	**Numerical method**	
0	0	0	0	0	0	0
0.1	0.083384	0.083384	4.07494×10^{-9}	0.0131673	0.0131673	3.2502×10^{-9}
0.2	0.159532	0.159532	5.44270×10^{-9}	0.0227377	0.0227377	9.76313×10^{-9}
0.3	0.228072	0.228072	1.86924×10^{-9}	0.0294067	0.0294068	2.02877×10^{-8}
0.4	0.288613	0.288613	9.46191×10^{-9}	0.0338091	0.0338091	3.5334×10^{-8}
0.5	0.340758	0.340758	3.12249×10^{-8}	0.0365131	0.0365130	5.48168×10^{-8}
0.6	0.384111	0.384111	6.45161×10^{-8}	0.0380152	0.0380153	7.76915×10^{-8}
0.7	0.418297	0.418297	1.06621×10^{-7}	0.0387337	0.0387339	1.01572×10^{-7}
0.8	0.442985	0.442985	1.50242×10^{-7}	0.0390024	0.0390025	1.23037×10^{-7}
0.9	0.457911	0.457911	1.83639×10^{-7}	0.0390647	0.0390649	1.37983×10^{-7}
1	0.462908	0.462908	1.95416×10^{-7}	0.0390689	0.039069	1.43116×10^{-7}

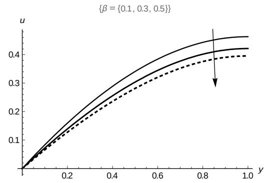

Figure 1: Small third grade material parameter effect on the flow velocity profile

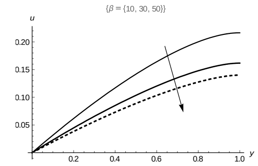

Figure 2: Large third grade material parameter effect on the flow velocity profile

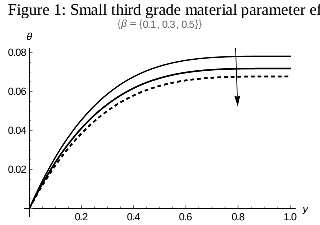

Figure 3: Small third grade material parameter effect on the flow temperature profile

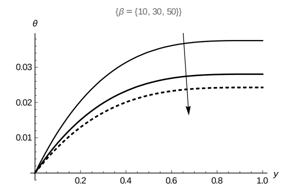

Figure 4: Large third grade material parameter effect on the flow temperature profile

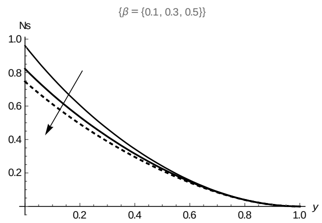

Figure 5: Small third grade material parameter effect on the flow entropy generation profile

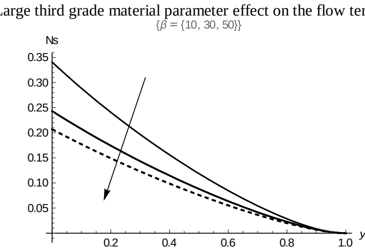

Figure 6: Large third grade material parameter effect on the flow entropy generation profile

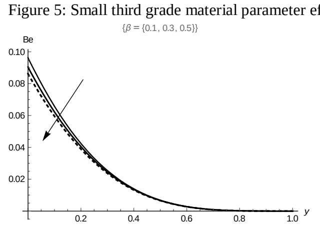

Figure 7: third grade material parameter effect on the flow Bejan number.

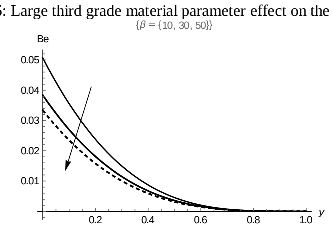

Figure 8: third grade material parameter effect on the flow Bejan number

Finally, the graphical results in Figure 7 and 8 show complimentary roles in describing the heat irreversibility in the thin fluid flow. By using the Bejan relationship, It is easy to see that the Bejan value is much higher when the third grade material parameter value is lower and it reduces further in the case of large parameter values. This is so due to the fact that frictional interaction increases with increasing fluid thickness. Therefore heat irreversibility due to fluid friction is expected to dominate over heat transfer irreversibility and the value of Be is expected to tends towards zero as shown in Figure 8.

5. Conclusion

In this work, we have investigated the steady flow of thin film of third grade fluid over an inclined heated plate with the analysis of entropy generation. The exact solution of the nonlinear problem is obtained and utilized to obtain the temperature field and the entropy profile. The outcomes of the study have shown that an upsurge in the strength of non-Newtonian parameter is noticed to decelerate the flow velocity and fluid temperature towards the free surface. Further, the net impact of small and lager values of third-grade material parameter decrease the rate of entropy generation and Bejan number.

Appendix

$$A_1=\left(-9\beta^2+\sqrt{3}\sqrt{\beta^3\left(2+27\beta\right)}\right)^{1/3},\quad A_2=\left(-9\beta^2\eta+\sqrt{3}\sqrt{\beta^3\left(2+27\beta\eta^2\right)}\right)^{1/3}$$

$$A_3=\sqrt{\beta^3\left(2+27\beta\eta^2\right)}\left(-9\beta^2\eta+\sqrt{3}\sqrt{\beta^3\left(2+27\beta\eta^2\right)}\right)^{2/3}\quad A_4=\left(-9\beta^2\eta+\sqrt{3}\sqrt{\beta^3\left(2+27\beta\eta^2\right)}\right)^{2/3},$$

$$A_5=\sqrt{\beta^3\left(2+27\beta\eta^2\right)}\ ,\ A_6=\sqrt{\beta^3\left(2+27\beta\right)},\ A_7=\left(-9\beta^2+\sqrt{3}A_6\right)^{2/3}$$

$$A_8=72\beta A_1\left(-2\beta-135\beta^2-1458\beta^3+9\sqrt{3}A_6+162\sqrt{3}\beta A_6\right)$$

$$A_2\left(-2\beta-27\beta^2\eta^2+3\sqrt{3}\eta A_5\right)\left(-\beta-27\beta^2\eta^2+3\sqrt{3}\eta A_5\right)$$

$B_1 = \sqrt{\beta^3\left(2+27\beta\eta^2\right)}$, $B_2 = \left(-18\beta^2\eta + 2\sqrt{3}B_1\right)^{1/3}$

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