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ANOTHER EXTENSION OF THE LOMAX DISTRIBUTION: PROPERTIES AND APPLICATION

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Abstract

The extension of classical distributions has become necessary as they provide distributions with better fits for existing datasets. In this paper, we provide an extension to the classical Lomax distribution named Generalized Odd Generalized Exponential Lomax Distribution (GOGELD). The GOGELD is a positively skewed distribution with four parameters; three shape parameters and one scale parameter. The structural and mathematical properties of the GOGELD including the asymptotic behaviour, characteristic function, order statistics and quantile function have been obtained. The model parameters are estimated by the method of maximum likelihood while the performance of the distribution has been evaluated through application to two real datasets. The GOGELD outperforms other models in fitting to the two datasets used as it provides the lowest values for the different goodness-of-fit statistics measures considered.

Keywords**:** Lomax Distribution, Order Statistics, Quantile Function, Moments, Maximum Likelihood Estimation, Asymptotic Behaviour.

1.0 Introduction

The Lomax distribution is one of the distributions for modelling heavy tailed dataset like income distribution. It has also found wide applications in the analysis of the business failure life time data, income and wealth inequality, size of cities, actuarial science, medical and biological sciences, engineering, lifetime and reliability modelling [1]. The Lomax distribution is also a special case of the generalized beta of the second kind which has been used in describing the returns of indices because it allows a direct representation of different degrees of fat tails in the distribution. It has two parameters – one shape parameter and one scale parameter. The need to improve the flexibility of the classical Lomax distribution has resulted in other extensions of the distribution. For instance, a three parameter distributions were proposed by [2] and [3]; the exponentiated Lomax and Marshall-Olkin extended Lomax respectively. These distributions each have two shape parameters and a scale parameter. Similarly, a four parameters distributions known as beta Lomax and Kumaraswamy Lomax each having two shape parameters and two scale parameters were introduced and discussed by [4]. Other extensions of the Lomax distribution that have been proposed in the literature include: gamma Lomax distribution by [5]; exponentiated Weibull Lomax distribution by [6]; Weibull Lomax by [7]; Double Lomax distribution by [8]; Lomax-Gumbel distribution by [9]; Power Lomax distribution by [10]; Rayleigh Lomax distribution by [11], Odd Generalized Exponential Lomax by [12]; etc. These variants of the Lomax distribution have been used extensively in economics, engineering, medicine, among others.

In order to model dataset that is skewed, a major characteristic of most dataset, flexible distributions with most desirable properties are useful. Thus, in this paper, we proposed the Generalized Odd Generalized Exponential Lomax Distribution (GOGELD), study some of its properties, estimate its parameters by means of maximum likelihood estimation and evaluate the performance of the distribution by means of application to real datasets.

The Generalized Odd Generalized Exponential-G (GOGE-G) family proposed by [13] is one of several generators that have been developed to generate distributions with most desirable properties. The GOGE-G like the OGE-G has two parameters.

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However, the GOGE-G differ from the OGE-G in that it has two shape parameters. The attractiveness of the GOGE-G family of distributions lies in its ability among others to provide better fits than other generated distributions with the same underlying model. The GOGE-G is found to outperform no fewer than ten other families each having the same number of parameters [13].
The Cumulative Distribution Function (CDF) and Probability Density Function (PDF) of the GOGE-G family of distribution are given in (1) and (2) respectively;

$$F(x)=\left[1-\exp\left(\frac{-G(x;\phi)^{\alpha}}{1-G(x;\phi)^{\alpha}}\right)\right]^{\beta} \tag{1}$$

$$f(x)=\frac{\alpha\beta g(x;\phi)G(x;\phi)^{\alpha-1}}{\left(1-G(x;\phi)^{\alpha}\right)^{2}}\exp\left(\frac{-G(x;\phi)^{\alpha}}{1-G(x;\phi)^{\alpha}}\right)\left[1-\exp\left(\frac{-G(x;\phi)^{\alpha}}{1-G(x;\phi)^{\alpha}}\right)\right]^{\beta-1} \tag{2}$$

where, $G(x;\phi)$ and $g(x;\phi)$ are respectively the baseline CDF and PDF of a random variable X with parameter vector ϕ, and α, β are shape parameters.

2.0 Proposed Distribution

Given that the baseline distribution is the Lomax distribution, then the CDF and PDF of the new distribution (known as Generalized Odd Generalized Exponential Lomax Distribution [GOGELD]) are respectively:

$$F(x)=\left[1-\exp\left(\frac{-\left[1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right]^{\alpha}}{1-\left[1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right]^{\alpha}}\right)\right]^{\beta} \tag{3}$$

and

$$f(x)=\frac{\alpha\beta\lambda\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha-1}\exp\left(\frac{-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}\right)\left[1-\exp\left(\frac{-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}\right)\right]^{\beta-1}}{\tau\left(1+\frac{x}{\tau}\right)^{(\lambda+1)}\left(1-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}\right)^{2}} \tag{4}$$

The PDF of the GOGELD has four parameters; α, β, and λ are shape parameters while τ is a scale parameter. The domain of the CDF and PDF is $[0,\infty)$. The hazard function and survival function of the GOGELD are respectively;

$$h(x)=\frac{\alpha\beta\lambda\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha-1}\exp\left(\frac{-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}\right)\left[1-\exp\left(\frac{-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}\right)\right]^{\beta-1}}{\tau\left(1+\frac{x}{\tau}\right)^{(\lambda+1)}\left(1-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}\right)^{2}\left\{1-\left[1-\exp\left(\frac{-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}\right)\right]^{\beta}\right\}} \tag{5}$$

and

$$R_x=1-F(x)=1-\left[1-\exp\left(\frac{-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}\right)\right]^{\beta} \tag{6}$$

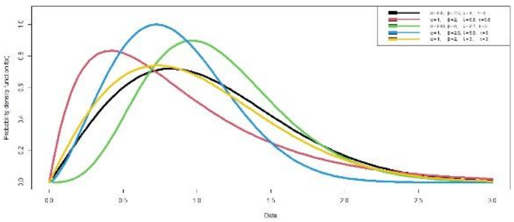


Figure 1: GOGELD Density Function

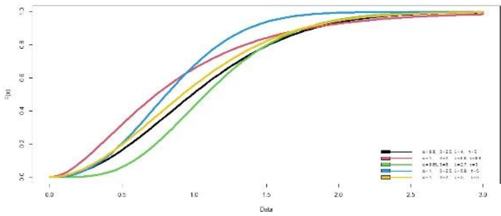


Figure 2: GOGELD Distribution Function

2.1 Investigation of the Proposed Distribution being a PDF

We showed that the proposed distribution is a PDF by showing that $\int_{-\infty}^{+\infty} f(x)dx = 1$. In this case,

$$\int_0^{+\infty} f(x)dx = \frac{\alpha\beta\lambda}{\tau}\int_0^{\infty}\left(1+\frac{x}{\tau}\right)^{-(\lambda+1)}\left\{1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right\}^{\alpha-1}\left\{1-\left[1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right]^{\alpha}\right\}^{-2}\exp\left(\frac{-\left[1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right]^{\alpha}}{1-\left[1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right]^{\alpha}}\right)\left\{1-\exp\left(\frac{-\left[1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right]^{\alpha}}{1-\left[1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right]^{\alpha}}\right)\right\}^{\beta-1} dx \qquad (7)$$

let, $p = 1-\left(1+\frac{x}{\tau}\right)^{-\lambda}$; $dx = \frac{\tau\left(1+\frac{x}{\tau}\right)^{\lambda+1}}{\lambda}dp$

Substituting the above in (7) and simplifying gives:

$$\int_0^{\infty} f(x)dx = \alpha\beta\int_0^{\infty} p^{\alpha-1}\left[1-p^{\alpha}\right]^{-2}\exp\left(\frac{-p^{\alpha}}{1-p^{\alpha}}\right)\left\{1-\exp\left(\frac{-p^{\alpha}}{1-p^{\alpha}}\right)\right\}^{\beta-1}dp \qquad (8)$$

Also, let, $q = \frac{p^{\alpha}}{1-p^{\alpha}}$; $dp = \frac{\left(1-p^{\alpha}\right)^2}{\alpha p^{\alpha-1}}dq$,

when we substitute the above into (8) we have:

$$\int_0^{\infty} f(x)dx = \beta\int_0^{\infty}\exp(-q)\left\{1-\exp(-q)\right\}^{\beta-1}dq \qquad (9)$$

Letting $r = 1 - \exp(-q)$, after few substitutions, we obtain:

$$\int_0^{\infty} f(x)dx = \beta\int_0^{\infty} r^{\beta-1}dr = \beta\left[\frac{r^{\beta}}{\beta}\right]_0^{\infty} = \left[r^{\beta}\right]_0^{\infty}$$

where, $r = 1 - \exp(-q)$, then:

$$\int_0^{\infty} f(x)dx = \left[r^{\beta}\right]_0^{\infty} = \left[\left(1-\exp(-q)\right)^{\beta}\right]_0^{\infty} = \left[\left(1-\frac{1}{\exp(q)}\right)^{\beta}\right]_0^{\infty} = \left[(1-0)^{\beta}-(1-1)^{\beta}\right] = 1-0 = 1$$

The above expression shows that the proposed distribution is a pdf

3.0 Some Statistical Properties

In this section, some of the statistical properties of the GOGELD will be studied.

3.1 Shapes of the PDF

The shapes of the random variable X that follows a GOGELD with pdf (4) is obtained by taking the limiting value of $f(x)$ as $x \to 0$ and $x \to \infty$. Thus,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\alpha\beta\lambda\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha-1}\exp\left(\frac{-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}\right)\left[1-\exp\left(\frac{-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}\right)\right]^{\beta-1}}{\tau\left(1+\frac{x}{\tau}\right)^{(\lambda+1)}\left(1-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}\right)^2}$$

$$= 0$$

and

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{\alpha\beta\lambda\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha-1}\exp\left(\frac{-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}\right)\left[1-\exp\left(\frac{-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}\right)\right]^{\beta-1}}{\tau\left(1+\frac{x}{\tau}\right)^{(\lambda+1)}\left(1-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}\right)^{2}}$$

$$=0$$

Since, $\lim_{x\to 0} f(x)=\lim_{x\to\infty} f(x)=0$, the GOGELD is unimodal (has only one mode).

3.2 Asymptotic Behaviour of GOGELD

The limiting distribution of the GOGELD; $\lim_{x\to 0}F(x)=0$ and $\lim_{x\to 0}F(x)=1$

Proof:

$$\lim_{x\to 0}F(x)=\lim_{x\to 0}\left[1-\exp\left(\frac{-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}\right)\right]^{\beta}$$

$$=\left\{1-\exp\left(\frac{-\left(1-\left(1+\frac{0}{\tau}\right)^{-\lambda}\right)^{\alpha}}{1-\left(1-\left(1+\frac{0}{\tau}\right)^{-\lambda}\right)^{\alpha}}\right)\right\}^{\beta}=\left\{1-e^{-0}\right\}^{\beta}$$

$$=\left\{1-1\right\}^{\beta}=0^{\beta}=0 \qquad \textit{prove.}$$

Similarly,

$$\lim_{x\to 0}F(x)=\lim_{x\to 0}\left\{1-\exp\left(\frac{-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}\right)\right\}^{\beta}$$

$$=\left\{1-\exp\left(\frac{-\left(1-\left(1+\frac{\infty}{\tau}\right)^{-\lambda}\right)^{\alpha}}{1-\left(1-\left(1+\frac{\infty}{\tau}\right)^{-\lambda}\right)^{\alpha}}\right)\right\}^{\beta}=\left\{1-e^{-\infty}\right\}^{\beta}$$

$$=\left\{1-0\right\}^{\beta}=1^{\beta}=1 \qquad \textit{prove}$$

3.3 Series Expansion

Using the expression in (2),

$$\left\{1-\exp\left(\frac{-G(x)^{\alpha}}{1-G(x)^{\alpha}}\right)\right\}^{\beta-1}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta-1}{i}\left[\exp\left(\frac{-G(x)^{\alpha}}{1-G(x)^{\alpha}}\right)\right]^{i} \tag{10}$$

substituting (10) into (2) gives;

$$f(x)=\frac{\alpha\beta g(x;\phi)G(x;\phi)^{\alpha-1}}{\left(1-G(x;\phi)^{\alpha}\right)^{2}}\exp\left(\frac{-G(x;\phi)^{\alpha}}{1-G(x;\phi)^{\alpha}}\right)\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta-1}{i}\left[\exp\left(\frac{-G(x)^{\alpha}}{1-G(x)^{\alpha}}\right)\right]^{i} \tag{11}$$

$$=\frac{\alpha\beta g(x;\phi)G(x;\phi)^{\alpha-1}}{\left(1-G(x;\phi)^{\alpha}\right)^{2}}\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta-1}{i}\exp\left[-(i+1)\left(\frac{G(x)^{\alpha}}{1-G(x)^{\alpha}}\right)\right] \tag{12}$$

simplifying the last expression in (11) gives

$$f\left(x\right)=\alpha\beta\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(1+i)^{j}\binom{\beta-1}{i}}{j!}g(x;\phi)G(x;\phi)^{\alpha+\alpha j-1}\left(1-G(x;\phi)^{\alpha}\right)^{-(j+2)} \quad (13)$$

After further simplification,

$$f\left(x\right)=\alpha\beta\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j}(1+i)^{j}\binom{\beta-1}{i}\Gamma(2+j+k)}{j!k!\Gamma(2+j)}g(x;\phi)G(x;\phi)^{\alpha(1+j+k)-1} \quad (14)$$

where, $g(x;\phi)=\frac{\lambda}{\tau}\left(1+\frac{x}{\tau}\right)^{-(\lambda+1)}$ and $G(x;\phi)=1-\left(1+\frac{x}{\tau}\right)^{-\lambda}$ are the PDF and CDF of the Lomax distribution respectively.

Thus, the series expansion of the GOGELD is

$$f\left(x\right)=\alpha\beta\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j}(1+i)^{j}\Gamma(2+j+k)}{j!k!\Gamma(2+j)}\binom{\beta}{i}\left[\frac{\lambda}{\tau}\left(1+\frac{x}{\tau}\right)^{-(\lambda+1)}\right]\left[1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right]^{\alpha(1+j+k)-1} \quad (15)$$

$$=\frac{\alpha\beta\lambda}{\tau}\sum_{i,j,k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j}(1+i)^{j}\Gamma(2+j+k)(-1)^{m}}{j!k!\Gamma(2+j)}\binom{\beta}{i}\binom{\alpha(1+j+k)-1}{m}\left(1+\frac{x}{\tau}\right)^{-(\lambda+1)-m\lambda} \quad (16)$$

$$=\xi_{i,j,k,m}\left(1+\frac{x}{\tau}\right)^{-(\lambda+1)-m\lambda}$$

where, $\xi_{i,j,k}=\frac{\alpha\beta\lambda}{\tau}\sum_{i,j,k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j}(1+i)^{j}\Gamma(2+j+k)(-1)^{m}}{j!k!\Gamma(2+j)}\binom{\beta}{i}\binom{\alpha(1+j+k)-1}{m}$

3.4 Moments

If X is a random variable with density function given in (16), the n^{th} non central moment of the GOGELD is obtained as follows:

$$\mu_n^{'}=E\left(X_n\right)=\int_0^{\infty}x^n f(x)dx \quad (17)$$

$$=\frac{\alpha\beta\lambda}{\tau}\sum_{i,j,k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j}(1+i)^{j}\binom{\beta-1}{i}\Gamma(2+j+k)(-1)^{m}\binom{\alpha(1+j+k)-1}{m}}{j!k!\Gamma(2+j)}\int_0^{\infty}x^n\left(1+x/\tau\right)^{-(\lambda+1)-m\lambda}dx \quad (18)$$

from (18), let, $y=x/\tau\Rightarrow x=y\tau\Rightarrow dx=\tau dy$ so that

$$\mu_n^{'}=\int_0^{\infty}(y\tau)^n\left(1+y\right)^{-(\lambda+1)-m\lambda}\tau dy=\tau^{n+1}\int_0^{\infty}y^n\left(1+y\right)^{-(\lambda+1)-m\lambda}dy$$

$$=\tau^{n+1}\int_0^{\infty}\frac{y^n}{(1+y)^{\lambda+1+m\lambda}}dy \quad (19)$$

$$\mu_n^{'}=\tau^{n+1}\int_0^{\infty}\frac{y^{n+1-1}}{(1+y)^{n+1+[\lambda(1+m)-n]}}dy$$

$$=\tau^{n+1}B\left(n+1,\lambda(1+m)-n\right) \quad (20)$$

where $B(.,.)$ is the beta function of the second type.

Substituting (20) into (18) gives the moment of the GOGELD as

$$\mu_n^{'}=\tau^{n+1}\frac{\alpha\beta\lambda}{\tau}\sum_{i,j,k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j}(1+i)^{j}\binom{\beta-1}{i}\Gamma(2+j+k)(-1)^{m}\binom{\alpha(1+j+k)-1}{m}}{j!k!\Gamma(2+j)}B\left(n+1,\lambda(1+m)-n\right)$$

$$=\alpha\beta\lambda\tau^{n}\sum_{i,j,k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j}(1+i)^{j}\binom{\beta-1}{i}\Gamma(2+j+k)(-1)^{m}\binom{\alpha(1+j+k)-1}{m}}{j!k!\Gamma(2+j)}B\left(n+1,\lambda(1+m)-n\right) \quad (21)$$

From above, the mean and variance of the GOGELD can be computed as follows;

Mean = $\mu_1^{'}=\alpha\beta\lambda\tau\sum_{i,j,k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j}(1+i)^{j}\binom{\beta-1}{i}\Gamma(2+j+k)(-1)^{m}\binom{\alpha(1+j+k)-1}{m}}{j!k!\Gamma(2+j)}B\left(2,\lambda(1+m)-1\right)$ (22)

$$\mu_2' = \alpha\beta\lambda\tau^2 \sum_{i,j,k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{i+j}(1+i)^j \binom{\beta-1}{i}\Gamma(2+j+k)(-1)^m \binom{\alpha(1+j+k)-1}{m}}{j!k!\Gamma(2+j)} B\left(3, \lambda(1+m)-2\right) \quad (23)$$

Variance = $\mu_2' - \left(\mu_1'\right)^2$

$$= \alpha\beta\lambda\tau^2 \sum_{i,j,k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{i+j}(1+i)^j \binom{\beta-1}{i}\Gamma(2+j+k)(-1)^m \binom{\alpha(1+j+k)-1}{m} B\left(3, \lambda(1+m)-2\right)}{j!k!\Gamma(2+j)}$$

$$-\left(\alpha\beta\lambda\tau\right)^2 \left[\sum_{i,j,k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{i+j}(1+i)^j \binom{\beta-1}{i}\Gamma(2+j+k)(-1)^m \binom{\alpha(1+j+k)-1}{m} B\left(2, \lambda(1+m)-1\right)}{j!k!\Gamma(2+j)}\right]^2 \quad (24)$$

3.5 Moments Generating Function

The moment generating function of a random variable *X*, $M_X(t)$ is given as follows;

$$M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{+\infty} e^{tx} f(x)dx \quad (25)$$

According to [14],

$$M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\mu_n' \quad (26)$$

where, μ_n' is the non-central moment.

The moment generating function of the GOGELD is:

$$M_X(t) = \alpha\beta\lambda \sum_{i,j,k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{i+j}(1+i)^j \binom{\beta-1}{i}\Gamma(2+j+k)(-1)^m \binom{\alpha(1+j+k)-1}{m}}{j!k!\Gamma(2+j)} \frac{(t\tau)^n}{n!} B\left(n+1, \lambda(1+m)-n\right) \quad (27)$$

3.6 Characteristic Function

The characteristic function $\phi_X(t)$ of the GODELD is:

$$\phi_X(t) = \alpha\beta\lambda \sum_{i,j,k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{i+j}(1+i)^j \binom{\beta-1}{i}\Gamma(2+j+k)(-1)^m \binom{\alpha(1+j+k)-1}{m}}{j!k!\Gamma(2+j)} \frac{(it\tau)^n}{n!} B\left(n+1, \lambda(1+m)-n\right) \quad (28)$$

3.7 Quantile Function

The quantile function of a distribution is the solution of the cumulative distribution function with respect to *x*. For the GOGELD, it is the solution of

$$u = F(x) = \left[1 - \exp\left(\frac{-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}\right)\right]^{\beta} \quad (29)$$

which is:

$$x_u = \tau\left\{\left[1-\left(\frac{-\ln(1-u^{\frac{1}{\beta}})}{1-\ln(1-u^{\frac{1}{\beta}})}\right)^{\frac{1}{\alpha}}\right]^{-\frac{1}{\lambda}} - 1\right\} \quad (30)$$

The above quantile function can be used to generate random observations from the GOGELD. In (29) if u = 0.5, we have the median of the GOGELD.

3.8 Order Statistics

Let X_1, X_2, X_3, …, X_n be random samples of size *n* from a PDF ($f(x)$) and CDF ($F(x)$). Suppose $X_{1:n}$, $X_{2:n}$, $X_{3:n}$, …, $X_{n:n}$ denotes corresponding order statistic derived from the samples, then, the p^{th} order statistic is defined by:

$$f_{p:n}(x) = \frac{n!f(x)}{(p-1)!(n-p)!} F(x)^{p-1}\left[1-F(x)\right]^{n-p} \quad (31)$$

From the series expansion,

$$(1-x)^r = \sum_{t=0}^{r} \frac{(-1)^t r!}{(r-t)!t!} x^t$$

Then,

$$[1-F(x)]^{n-p} = \sum_{t=0}^{n-p} \frac{(-1)^t (n-p)!}{(n-p-t)!t!} F(x)^t$$

and (31) becomes

$$f_{p:n}(x) = \sum_{t=0}^{n-p} (-1)^t \frac{n!}{(p-1)!(n-p-t)!} f(x) F(x)^{t+p-1} \quad (32)$$

where, $f(x)$ and $F(x)$ are the PDF and CDF of the GOGELD

$$[F(x)]^{k+p-1} = \sum_{s=0}^{\beta(t+p-1)} (-1)^s \binom{\beta(t+p-1)}{s} \exp\left(\frac{-s\left[1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right]^{\alpha}}{1-\left[1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right]^{\alpha}} \right) \quad (33)$$

So that the order statistics for the GOGELD becomes;

$$f_{p:n}(x) = \sum_{t=0}^{n-p} \sum_{s=0}^{\beta(t+p-1)} (-1)^{s+t} \frac{n!}{(p-1)!(n-p-t)!} \binom{\beta(t+p-1)}{s} \exp\left(\frac{-s\left[1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right]^{\alpha}}{1-\left[1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right]^{\alpha}} \right) f(x) \quad (34)$$

where,

$$f(x) = \frac{\alpha\beta\lambda}{\tau} \sum_{i,j,k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+j}(1+i)^j \binom{\beta-1}{i} \Gamma(2+j+k)(-1)^m}{j!k!\Gamma(2+j)} \binom{\alpha(1+j+k)-1}{m} \left(1+x/\tau\right)^{-(\lambda+1)-m\lambda}$$

4.0 Estimation

Different methods are employed in the estimation of parameters. The maximum likelihood is widely used in the estimation of parameters of distribution because they are asymptotically consistent, asymptotically efficient and asymptotically unbiased among other properties. Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample of size n from the GOGELD with parameter vector $\psi = (\alpha, \beta, \lambda, \tau)$, the likelihood function for ψ (i.e. $L(x;\psi) = L$) is given as:

$$L = \prod_{i=1}^{n} \frac{\alpha\beta\lambda\left(1-\left(1+\frac{x_i}{\tau}\right)^{-\lambda}\right)^{\alpha-1} \exp\left(\frac{-\left(1-\left(1+\frac{x_i}{\tau}\right)^{-\lambda}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x_i}{\tau}\right)^{-\lambda}\right)^{\alpha}} \right) \left[1-\exp\left(\frac{-\left(1-\left(1+\frac{x_i}{\tau}\right)^{-\lambda}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x_i}{\tau}\right)^{-\lambda}\right)^{\alpha}} \right)\right]^{\beta-1}}{\tau\left(1+\frac{x_i}{\tau}\right)^{(\lambda+1)} \left(1-\left(1-\left(1+\frac{x_i}{\tau}\right)^{-\lambda}\right)^{\alpha}\right)^2} \quad (35)$$

and the log likelihood function (l) is

$$\begin{aligned} l = {} & n\ln\alpha + n\ln\beta + n\ln\lambda - n\ln\tau - (\lambda+1)\sum_{i=1}^{n}\ln\left(1+\frac{x}{\tau}\right) - 2\sum_{i=1}^{n}\ln\left\{1-\left[1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right]^{\alpha}\right\} \\ & + (\alpha-1)\sum_{i=1}^{n}\ln\left[1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right] - \sum_{i=1}^{n}\left[\frac{\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}} \right] \\ & + (\beta-1)\sum_{i=1}^{n}\ln\left\{1-\exp\left[\frac{-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}}{1-\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\alpha}} \right]\right\} \end{aligned} \quad (36)$$

The maximum likelihood estimators of parameters $\left(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\tau}\right)$ are the solution of the simultaneous equations obtained when (36) is differentiated partially with respect to each parameter and equated to zero, i.e.

$$\frac{\partial l}{\partial \alpha}=\frac{n}{\alpha}-2\sum_{i=1}^{n}\frac{p_i}{q_i}+\sum_{i=1}^{n}r_i-\sum_{i=1}^{n}s_i+(\beta-1)\sum_{i=1}^{n}\frac{s_i\exp\left[-t_i\right]}{1-\exp\left[-t_i\right]} \quad (37)$$

$$\frac{\partial l}{\partial \beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\left(1-\exp\left[-t_i\right]\right) \quad (38)$$

$$\frac{\partial l}{\partial \lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}\ln\left(1+{x_i}/{\tau}\right)-2\sum_{i=1}^{n}\frac{u_i}{q_i}+(\alpha-1)\sum_{i=1}^{n}\frac{v_i}{r_i}-\sum_{i=1}^{n}w_i+(\beta-1)\sum_{i=1}^{n}\frac{w_i\exp\left[-t_i\right]}{1-\exp\left[-t_i\right]} \quad (39)$$

$$\frac{\partial l}{\partial \tau}=-\frac{n}{\tau}-(\lambda+1)\sum_{i=1}^{n}\frac{\left(1-{x_i}/{\tau^2}\right)}{\left(1+{x_i}/{\tau}\right)}-2\sum_{i=1}^{n}\frac{z_i}{q_i}+(\alpha-1)\sum_{i=1}^{n}\frac{m_i}{r_i}-\sum_{i=1}^{n}d_i+(\beta-1)\sum_{i=1}^{n}\frac{d_i\exp\left[-t_i\right]}{1-\exp\left[-t_i\right]} \quad (40)$$

where, $q_i=1-\left[1-\left(1+{x_i}/{\tau}\right)^{-\lambda}\right]^{\alpha}$; $r_i=1-\left(1+{x_i}/{\tau}\right)^{-\lambda}$; $t_i=\dfrac{\left[1-\left(1+{x_i}/{\tau}\right)^{-\lambda}\right]^{\alpha}}{1-\left[1-\left(1+{x_i}/{\tau}\right)^{-\lambda}\right]^{\alpha}}$ and

$p_i=\frac{\partial}{\partial\alpha}\left(q_i\right)$; $s_i=\frac{\partial}{\partial\alpha}\left(t_i\right)$; $u_i=\frac{\partial}{\partial\lambda}\left(q_i\right)$;$v_i=\frac{\partial}{\partial\lambda}\left(r_i\right)$; $w_i=\frac{\partial}{\partial\lambda}\left(t_i\right)$; $z_i=\frac{\partial}{\partial\lambda}\left(q_i\right)$; $m_i=\frac{\partial}{\partial\tau}\left(r_i\right)$; $d_i=\frac{\partial}{\partial\tau}\left(t_i\right)$

The solution of equations (37) to (40) is obtained using numerical methods found in statistical packages such as R.

5.0 Application

To evaluate the performance of the proposed model (GOGELD), we have used two datasets described below

Dataset 1: This data represents the life of fatigue fracture of Kevlar 373/epoxy subjected to constant pressure at 90 % stress level until all had failed. The data has also been used by [15], [16], among others. The data are given below;

0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960

Dataset 2: This data is a subset of the original data found in [17] which represent the survival times (in years) of 45 patients given chemotherapy treatment alone. The data are given below;

0.047, 0.115, 0.121, 0.132, 0.164, 0.197,0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033

For both datasets, the goodness-of-fit statistics such as the Akaike Information Criterion (AIC), Consistent AIC (CAIC), Bayesian information criterion (BIC) and the Hannan-Quinn Information Criterion (HQIC) are computed to compare the fitted model. The model with the smallest of these statistics is often adjudged as the best among the competing models. The mathematical expressions for these measures are:

$$AIC = -2 + 2k \quad (41)$$

$$CAIC=-2l+\frac{2nk}{(n-k-1)} \quad (42)$$

$$BIC = -2l + k\log(n) \quad (43)$$

$$HQIC = -2l + k\log(\log(n)) \quad (44)$$

where l stands for the log likelihood, k is the number of model parameters, and n is the sample size. We have compared the GOGELD with six other extensions including the Lomax distribution. These distributions and their density functions are given as follows:

Lomax Distribution (LD):

$$f_{LD}(x)=\frac{\lambda}{\tau}\left(1+\frac{\lambda}{\tau}\right)^{-(\lambda+1)} \quad (45)$$

Rayleigh Lomax Distribution (RLD):

$$f_{RLD}(x)=\frac{\beta\lambda}{\tau}\left(\frac{\tau}{x+\tau}\right)^{-2\lambda+1}e^{-\frac{\beta}{2}\left(\frac{\tau}{x+\tau}\right)^{-2\lambda}} \tag{46}$$

Exponentiated Lomax Distribution (ELD):

$$f_{ELD}(x)=\frac{\beta\lambda}{\tau}\left(1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right)^{\beta-1}\left(1+\frac{x}{\tau}\right)^{-(\lambda+1)} \tag{47}$$

Gumbel Lomax Distribution (GLD):

$$f_{GLD}(x)=\frac{\alpha\lambda\left(1+\frac{x}{\tau}\right)^{-\frac{\lambda}{\beta}-1}e^{-\lambda\left\{\left(1+\frac{x}{\tau}\right)^{-\lambda}-1\right\}^{-\frac{1}{\beta}}}}{\beta\tau\left[1-\left(1+\frac{x}{\tau}\right)^{-\lambda}\right]^{\frac{1}{\beta}+1}} \tag{48}$$

Odd Generalized Exponential Inverse Lomax Distribution (OGEILD):

$$f_{OGEILD}(x)=\frac{\alpha\beta\lambda\tau\left(1+\frac{\tau}{x}\right)^{\lambda-1}\exp\left[-\alpha\left(\frac{\left(1+\frac{\tau}{x}\right)^{-\lambda}}{1-\left(1+\frac{\tau}{x}\right)^{-\lambda}}\right)\right]\left(1-\exp\left[-\alpha\left(\frac{\left(1+\frac{\tau}{x}\right)^{-\lambda}}{1-\left(1+\frac{\tau}{x}\right)^{-\lambda}}\right)\right]\right)^{\beta-1}}{x^{2}\left[\left(1+\frac{x}{\tau}\right)^{\lambda}-1\right]^{2}} \tag{49}$$

Odd Generalized Exponential Inverse Lomax Distribution (OGELD):

$$f_{OGELD}(x)=\frac{\alpha\beta\lambda}{\tau}\left(1+\frac{x}{\tau}\right)^{\lambda-1}\exp\left[-\alpha\left(\left(1+\frac{x}{\tau}\right)^{\lambda}-1\right)\right]\left(1-\exp\left[-\alpha\left(\left(1+\frac{x}{\tau}\right)^{\lambda}-1\right)\right]\right)^{\beta-1} \tag{50}$$

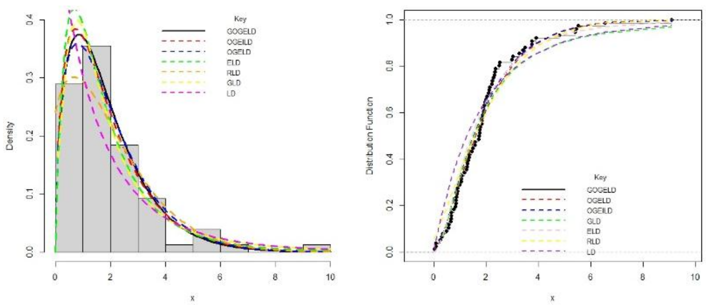
Table 1: Descriptive statistics of dataset 1

Min.	Median	Mean	Variance	Max.	Skewness	Kurtosis	N
0.0251	1.7362	1.9592	2.4774	9.0960	1.9796	5.1608	76

Table 1 summarizes the data on the life of fatigue fracture of Kevlar 373/epoxy subjected to constant pressure at 90% until all has failed. The data is positively skewed and leptokurtic, thus can be modelled by the GOGELD. The estimation of the different models is indicated in Table 2; the GOGELD produces the least values of the different measures indicating its suitability over other models considered. In addition, the estimated densities and distribution functions of the different distributions have been displayed in the plots given in Figure 3. The plots also support the results obtained in Table 2.

Table 2: Parameter estimates and goodness-of-fit statistics for dataset 1

Distribution	Parameter	Estimate	- Log likelihood	AIC	CAIC	BIC	HQIC
LD	λ	6.1613	129.8079	263.6157	263.7801	268.2772	265.4787
	τ	10.6555					
RD	α	0.1116	127.3179	260.6359	260.9692	267.6281	263.4303
	λ	0.6937					
	τ	0.3044					
ELD	β	1.9597	123.3102	252.6204	252.9537	259.6126	255.4148
	λ	6.0668					
	τ	6.5726					
GLD	α	4.5653	123.7920	255.5841	256.1475	264.9070	259.3100
	β	3.8034					
	λ	9.2224					
	τ	1.3027					
OGE_ILD	α	1.1516	122.2107	252.4214	252.9848	261.7443	256.1473
	β	1.7766					
	λ	0.9160					
	τ	1.7539					
OGELD	α	0.5571	122.1853	252.3707	252.9340	261.6936	256.0966
	β	1.5202					
	λ	1.1967					
	τ	1.1697					
GOGELD	α	0.4725	**121.9423**	**251.8847**	**252.4481**	**261.2076**	**255.6106**
	β	2.9122					
	λ	1.1370					
	τ	3.3850					



(a) Estimated Density (b) Estimated Distribution Function

Figure 3: Plots of the estimated density function and distribution function for dataset 1

The basic descriptive of survival times (in years) of 45 patients given chemotherapy treatment alone is given in Table 3. The table shows the dataset is asymmetry and in particular, positively skewed and the data has a negative kurtosis. Furthermore, the GOGELD outperform other competing models with fewer and same number of parameters (4) – this is represented by the smallest values of the goodness-of-fit statistics measures given in Table 4. In addition, Figure 4 gives the plots of the estimated density function and distribution for the second dataset.

Table 3: Descriptive statistics of dataset 2

Min.	Median	Mean	Variance	Max.	Skewness	Kurtosis	n
0.0470	0.8410	1.3414	1.5540	4.0330	0.9722	-0.3362	45

Table 4: Parameter estimates and goodness-of-fit statistics for dataset 2

Distribution	Parameter	Estimate	- Log likelihood	AIC	CAIC	BIC	HQIC
LD	λ	3.7523	59.5791	123.1581	123.4439	126.7715	124.5052
	τ	3.9659					
RLD	α	0.1179	61.8092	129.6187	130.2038	135.0385	131.6390
	λ	0.5880					
	τ	0.1159					
ELD	β	1.3327	58.5650	123.1299	123.7153	128.5499	125.1504
	λ	4.6889					
	τ	4.1576					
GLD	α	1.7051	58.2337	124.4675	125.4675	131.6941	121.1615
	β	1.9499					
	λ	6.2129					
	τ	2.2222					
OGE_ILD	α	1.1531	57.4443	122.8886	123.8886	130.1152	125.5826
	β	0.5875					
	λ	2.9999					
	τ	0.5349					
OGELD	α	0.5437	58.0105	124.0210	125.0210	131.2477	126.7151
	β	1.3414					
	λ	0.7624					
	τ	0.3444					
GOGELD	α	4.8952	**56.8644**	**121.7289**	**122.7289**	**128.9555**	**124.4229**
	β	0.2241					
	λ	4.8996					
	τ	6.1018					

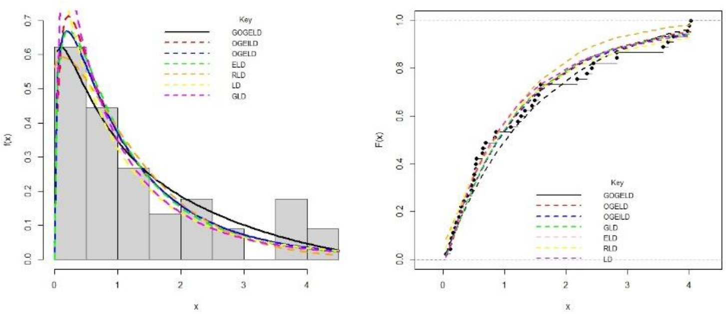


Figure 4: Plots of the estimated density function and distribution function for dataset 2

6. Conclusion

In this work, we have proposed another four parameter extension of the Lomax distribution; Generalized Odd Generalized Exponential Laplace Distribution (GOGELD). Some mathematical properties of the distribution have been obtained including moment, quantile function and order statistics. The parameters of the model are estimated using the maximum likelihood method. The performance of the GOGELD is determined using two positively skewed datasets. The GOGELD is found to give the best fit among all models fitted. Thus, GOGELD can be used to model data that are asymmetric.

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