Transactions of the Nigerian Association of Mathematical Physics
Volume 15, (April – June, 2021 Issue), **pp7 – 12**

SOLUTION OF FOURTH ORDER BOUNDARY VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS USING HYBRID FINITE DIFFERENCE BLOCK METHODS

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Abstract

We propose some symmetric hybrid finite difference methods for the solution of Boundary value problems of general and special fourth order Ordinary Differential Equations. The three members block schemes of the Central, Forward and Backward finite difference methods derived were used simultaneously to obtain their solutions. Two numerical experiments were used to demonstrate the efficiency of the proposed methods.

Keywords: Symmetric, hybrid, finite difference method continuous coefficients Boundary value problems.

1.0 Introduction

Yahaya and Onumanyi [1] proposed a symmetric hybrid finite difference scheme with continuous coefficents for the solution of Boundary value problems of general second order ODEs of the form:

$$y'' = f(x, y, y'), \qquad y(a) = y_0,\ y(b) = y_N,\ a \in [a, b] \tag{1.1}$$

The method was based on interpolation at the grid points and at two other off-grid points x_{n-1} and x_{n+1}. They asserted that the central difference discrete schemes for the first and second derivatives approximations emerging from the continuous scheme are of order four and both have small error constants.

Soladoye and Yahaya [2] also proposed a symmetric hybrid block finite difference scheme with continuous coefficients for the solution of boundary-value problems for both general and special third-order ODEs of the form:

$$y''' = f(x, y, y', y'') \quad y(a) = y_0,\ \ y'(a) = y_1,\ \ y''(b) = y_2 \tag{1.2}$$

and $y''' = f(x, y,) \quad y(a) = y_0,\ \ y(b) = y_1,\ \ y(c) = y_2$ (1.3)

The method was based on interpolation at the grid points and at four other off-grid points $x_{n-\frac{1}{2}}, x_{n-\frac{3}{2}}, x_{n+\frac{1}{2}}$ and $x_{n+\frac{3}{2}}$. They demonstrated that the three members block schemes of central, forward and backward derived were used simultaneously for the solution of the equation.

In this paper, we proposed for the solution of Boundary value problems of both general and special fourth order ODEs of the form:

$$y^{iv}(x) = f(x, y),\ \ y(a) = y_0, y'(a) = y_1, y(b) = y_2, y'(b) = y_3 \tag{1.4}$$

$$y^{iv}(x) = f(x, y),\ \ y(a) = y_0, y'(a) = y_1, y(b) = y_2, y''(b) = y_3 \tag{1.5}$$

Recently some scholars [3-5] have proposed some methods of numerical solution of higher order Boundary value problems of ordinary differential equations. However, all these methods are limited to special and general second order Boundary value problems of ordinary differential equations.

2.0 Methodology

We consider an approximation of the form:

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$$y(x_m) = \sum_{j=0}^{8} a_j Q_J(x_m), \qquad x_{i-2} \le x \le x_{i+2}$$

$$k = 8,\ \ m = i-2, i-\frac{3}{2}, i-1, i-\frac{1}{2}, i, i+\frac{1}{2}, i+1, i+\frac{3}{2}, i+2 \qquad (2.1)$$

where $Q_j(x)$ are canonical polynomials which are used as the basis functions in the approximation and a_j are coefficients to be determined. To generate $Q_j(x)$, we write equation (1.4) and (1.5) in the form

$$y^n(x) + y(x) = y(x) + f(x, y, y', \dots, y^{n-1}) \qquad (2.2)$$

And define a differential operator

$$L^* = \frac{d^3}{dx^3} + 1 \qquad (2.3)$$

Then define the canonical polynomial $Q_j(x)$ by

$$L^* Q(x) = x^j, \qquad j = 0,1, k+1 \qquad (2.4)$$

We generate the canonical polynomials $Q_j(x)$ by starting with the generating polynomial

$$L^* x^j = j(j-1)(j-2)j^{j-3} + x^j \qquad (2.5)$$

Using equations (2.4) and (2.5) it gives

$$L^* x^j = L^*\{j(j-1)(j-2)Q_{j-3}(x) + Q_j\} \qquad (2.6)$$

Assume L^{*-1} exists, then we have

$$Q_j(x) = x^j - j(j-1)(j-2)Q_{j-3}(x), \qquad j = 0,1, k+1 \qquad (2.7)$$

Equation (2.7) is the recursive relation for generating the canonical polynomials. See [5].

Thus from equation (2.7), we obtain recursively for $j = 0,1,2,\dots,8$ and substituting the result in equation (2.1), we have

$$\begin{aligned} y_8(x) = {} & a_0 + a_1 x + a_2 x^2 + a_3(x^3 - 6) + a_4(x^4 - 24x) + a_5(x^5 - 60x^2) \\ & + a_6(x^6 - 120x^3 + 720) + a_7(x^7 - 420x^4 + 340x) \\ & + a_8(x^8 - 240x^5 + 120x^2) \end{aligned} \qquad (2.8)$$

Where a_j's, $j = 0,1,2,\dots,8$ are the parameters to be determined. We interpolate at $x = x_m$, where $m = i-2, i-\frac{3}{2}, i-1, i-\frac{1}{2}, i, i+\frac{1}{2}, i+1, i+\frac{3}{2}, i+2$,which gives the system of non-linear equations of the form:

$$y(x_m) = \sum_{j=0}^{8} a_j Q_J(x_m), \qquad (2.9)$$

Evaluating the values of $a_0, a_1, a_2, \dots, a_8$ into equation (2.8) and substituting them in equation (2.9) then simplifying, the resulting equation will be of the form:

$$\begin{aligned} y_8(x) = {} & A(x)y_{i-2} + B(x)y_{i-\frac{3}{2}} + C(x)y_{i-1} + D(x)y_{i-\frac{1}{2}} + E(x)y_i + F(x)y_{i+\frac{1}{2}} \\ & + G(x)y_{i+1} + H(x)y_{i+\frac{3}{2}} + I(x)y_{i+2} \end{aligned} \qquad (2.10)$$

where $A(x), B(x), \dots, I(x)$, are known functions to be determined.

We take the first, second,…, fourth derivative of equation (2.10) and interpolating each at $x = x_i$, yield the first, second, third and fourth order derivatives central difference schemes of the form:.

$$\begin{aligned} y'_8(x_i) = {} & \frac{1}{840h}\Big(6y_{i-2} - 64y_{i-\frac{3}{2}} + 336y_{i-1} - 1344y_{i-\frac{1}{2}} + 1344y_{i+\frac{1}{2}} - 336y_{i+1} \\ & + 64y_{i+\frac{3}{2}} - 6y_{i+2}\Big) \end{aligned}$$

$$\begin{aligned} y''_8(x_i) = {} & \frac{1}{2520h^2}\Big(-18y_{i-2} + 256y_{i-\frac{3}{2}} - 2016y_{i-1} + 16128y_{i-\frac{1}{2}} - 28700y_i \\ & + 16128y_{i+\frac{1}{2}} - 2016y_{i+1} + 256y_{i+\frac{3}{2}} - 18y_{i+2}\Big) \end{aligned}$$

$$\begin{aligned} y'''_8(x_i) = {} & \frac{1}{30h^3}\Big(-7y_{i-2} + 72y_{i-\frac{3}{2}} - 338y_{i-1} + 488y_{i-\frac{1}{2}} - 488y_{i+\frac{1}{2}} + 338y_{i+1} \\ & - 72y_{i+\frac{3}{2}} + 7y_{i+2}\Big) \end{aligned}$$

$$y^{iv}{}_8(x_i) = \frac{1}{15h^4}\Big(7y_{i-2} - 96y_{i-\frac{3}{2}} + 676y_{i-1} - 1952y_{i-\frac{1}{2}} + 2730y_i - 1950y_{i+\frac{1}{2}} + 676y_{i+1} - 96y_{i+\frac{3}{2}} + 7y_{i+2}\Big) \qquad (2.11)$$

Equation (2.11) is of order $[8,8,6,6]^T$ with error constants $\left[\frac{1}{161280}, \frac{1}{806400}, \frac{-41}{193536}, \frac{-41}{483840}\right]^T$

Also evaluating the first, second,…, fourth derivative of equation (2.10) and interpolating each at $x = x_{i-2}$, and choosing $i = i + 2$ yield the first, second, third and fourth order derivatives forward difference schemes of the form:.

$$y'_8(x_i) = \frac{1}{420h}\Big(-2283y_i + 6720y_{i+\frac{1}{2}} - 11760y_{i+1} + 15680y_{i+\frac{3}{2}} - 14700y_{i+2} + 9408y_{i+\frac{5}{2}} - 3920y_{i+3} + 960y_{i+\frac{7}{2}} - 105y_{i+4}\Big)$$

$$y''_8(x_i) = \frac{1}{1260h^2}\Big(29531y_i - 138528y_{i+\frac{1}{2}} + 312984y_{i+1} - 448672y_{i+\frac{3}{2}} + 435330y_{i+2} - 284256y_{i+\frac{5}{2}} + 120008y_{i+3} - 29664y_{i+\frac{7}{2}} + 3267y_{i+4}\Big)$$

$$y'''_8(x_i) = \frac{1}{30h^3}\Big(-2403y_i + 13960y_{i+\frac{1}{2}} - 36706y_{i+1} + 57384y_{i+\frac{3}{2}} - 58280y_{i+2} + 39128y_{i+\frac{5}{2}} - 16830y_{i+3} + 4216y_{i+\frac{7}{2}} - 469y_{i+4}\Big)$$

$$y^{iv}{}_8(x_i) = \frac{1}{15h^4}\Big(3207y_i - 21056y_{i+\frac{1}{2}} + 61156y_{i+1} - 102912y_{i+\frac{3}{2}} + 109930y_{i+2} - 76352y_{i+\frac{5}{2}} + 33636y_{i+3} - 8576y_{i+\frac{7}{2}} + 967y_{i+4}\Big) \qquad (2.12)$$

Equation (2.12) is of order $[8,7,6,5]^T$ with error constants

$\left[\frac{1}{2304}, \frac{-761}{161280}, \frac{29531}{967680}, \frac{-267}{1920},\right]^T$ respectively.

Also evaluating the first, second,…, fourth derivative equation of (2.10) and interpolating each at $x = x_{i+2}$, and choosing $i = i - 2$ yield the first, second, third and fourth order derivatives backward difference schemes of the form:.

$$y'_8(x_i) = \frac{1}{420h}\Big(105y_{i-4} - 960y_{i-\frac{7}{2}} + 3920y_{i-3} - 9408y_{i-\frac{5}{2}} + 14700y_{i-2} - 15680y_{i-\frac{3}{2}} + 11760y_{i-1} - 6720y_{i-\frac{1}{2}} + 2283y_i\Big)$$

$$y''_8(x_i) = \frac{1}{1260h^2}\Big(3267y_{i-4} - 29664y_{i-\frac{7}{2}} + 120008y_{i-3} - 284256y_{i-\frac{5}{2}} + 435330y_{i-2} - 448672y_{i-\frac{3}{2}} + 312984y_{i-1} - 138528y_{i-\frac{1}{2}} + 29531y_i\Big)$$

$$y'''_8(x_i) = \frac{1}{30h^3}\Big(469y_{i-4} - 4216y_{i-\frac{7}{2}} + 16830y_{i-3} - 39128y_{i-\frac{5}{2}} + 58280y_{i-2} - 57384y_{i-\frac{3}{2}} + 36706y_{i-1} - 13960y_{i-\frac{1}{2}} + 2403y_i\Big)$$

$$y^{iv}{}_8(x_i) = \frac{1}{15h^4}\Big(967y_{i-4} - 8576y_{i-\frac{7}{2}} + 33636y_{i-3} - 76352y_{i-\frac{5}{2}} + 109930y_{i-2} - 102912y_{i-\frac{3}{2}} + 61156y_{i-1} - 21056y_{i-\frac{1}{2}} + 3207y_i\Big) \qquad (2.13)$$

Equation (2.13) is of order $[8,7,6,5]^T$ with error constants $\left[\frac{1}{2304}, \frac{-761}{161280}, \frac{29531}{967680}, \frac{-267}{1920}\right]^T$ respectively.

4.0 Implementation Strategies

Our strategy for implementing the method is such that given an nth order Boundary value problem of ordinary differential equation, we replace the derivatives terms in the equation with an equivalent finite central difference schemes and write the difference equation where the function value is not known. In addition, we replace the derivative Boundary conditions with an equivalent finite forward difference schemes and an equivalent finite backward difference schemes, satisfying the given Boundary conditions. And finally we replace the derivatives terms in the equation with an equivalent finite forward difference schemes and an equivalent finite backward difference schemes, and write the difference equation where the function value is not known, until the number of our unknown functions equals the number of equations. The resulting linear system of simultaneous equations are then solved using MAPLE 17, a mathematical software.

5.0 Numerical Examples

The following problems were used to demonstrate the efficiency and accuracy of the proposed methods.

Example 1.

The linear fourth order Boundary value problem

$y^{(iv)}(x) + xy(x) = -(8 + 7x + x^3), \qquad 0 \le x \le 1$

$y(0) = y(1) = 0, y'(0) = 1, \qquad y'(1) = -e$

Theoretical solution: $y(x) = x(1 - x)e^x$

Example 2.

The linear fourth order Boundary value problem

$y^{iv}(x) - y''(x) - y(x) = e^x(x - 3), \qquad 0 \le x \le 1$

$y(0) = 1, y'(0) = 0, y(1) = 0, \ y'(1) = -e$

Theoretical solution: $y(x) = (1 - x)e^x$

KEY:

SFDM: Standard finite Difference Method

HFDBM: Hybrid Finite Difference Block Method

Table 1: Approximate solution and absolute error of problem 1

x	y-exact	SFDM	HFDBM	Error in SFDM	Error in HFDBM
0	0.0	0.0	0.0	0.0	0.0
0.1	0.09946538263	0.09946552401	0.09946538368	1.4138 E(-7)	1.05 E(-9)
0.2	0.1954244413	0.1954247303	0.1954244426	2.8900 E(-7)	1.30 E(-9)
0.3	0.2834703496	0.2834705948	0.2834703498	2.4520 E(-7)	2.00 E(-10)
0.4	0.3580379274	0.3580379533	0.3580379257	2.5900 E(-8)	1.70 E(-9)
0.5	0.4121803177	0.4121800458	0.4121803138	2.7190 E(-7)	3.90 E(-9)
0.6	0.4373085121	0.4373079729	0.4373085064	5.3920 E(-7)	5.70 E(-9)
0.7	0.4228880686	0.4228874009	0.4228880620	6.6770 E(-7)	6.30 E(-9)
0.8	0.3560865486	0.3560859853	0.3560865427	5.6330 E(-7)	5.90 E(-9)
0.9	0.2213642800	0.2213640396	0.2213642770	2.4040 E(-7)	3.00 E(-9)
1.0	0.0	0.0	0.0	0.0	0.0

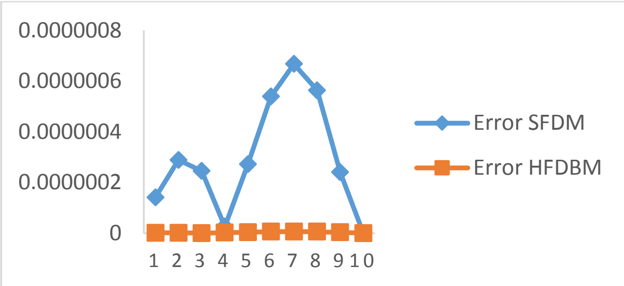

Figure 1: Error graph of Example 1.

Table 2: Approximate solution and absolute error of problem 2

x	y-exact	SFDM	HFDBM	Error in SFDM	Error in HFDBM
0	1.0	1.0	1.0	0.0	0.0
0.1	0.9946538263	0.9946538413	0.9946538261	1.50 E(-8)	2.0 E(-10)
0.2	0.9771222065	0.9771222363	0.9771222061	2.98 E(-8)	4.0 E(-10)
0.3	0.9449011653	0.9449011881	0.9449011647	2.28 E(-8)	6.0 E(-10)
0.4	0.8950948186	0.8950948151	0.8950948177	3.50 E(-9)	9.0 E(-10)
0.5	0.8243606354	0.8243605975	0.8243606343	3.79 E(-8)	1.1 E(-9)
0.6	0.7288475202	0.7288474522	0.7288475191	6.80 E(-8)	1.1 E(-9)
0.7	0.6041258122	0.6041257307	0.6041258112	8.15 E(-8)	1.0 E(-9)
0.8	0.4451081857	0.4451081179	0.4451081849	6.78 E(-8)	8.0 E(-10)
0.9	0.2459603111	0.2459602824	0.2459603107	2.87 E(-8)	4.0 E(-9)
1.0	0.0	0.0	0.0	0.0	0.0

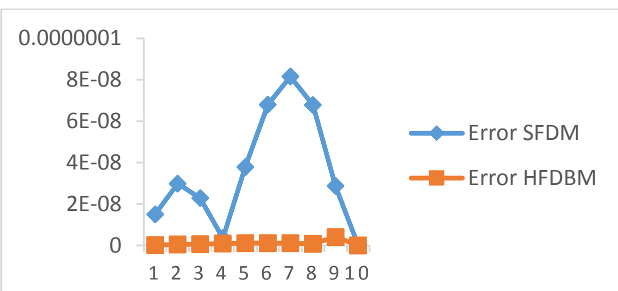

Figure 2: Error graph of Example 2.

6.0 Conclusion

The numerical experiments in this paper shows that the results from HFDBM are consistent and convergent to the theoretical solution and also compete favourably with standard finite difference method. (see figure 1 and 2)

References

[1] Yahaya Y. A. and Onumanyi P. (1982). A Symmetric Hybrid Finite Difference Schemes with Continuous Coefficients and its Applications, *Spectrum Journal Volume 4, No 1 and 2, pages198-205*

[2] S.O. Soladoye and Y.A.Yahaya (2013). An Application of Block Finite Difference Method for Boundary value Problems of Third order Ordinary Differential Equations.*Journal of the Nigerian Association of Mathematical Physics vol 23, pages 109-112.*

[3] Onumanyi P. and Ortiz E. I. (1982): Numerical solution of Higher order Boundary value problems for ordinary differential equations with an Estimation of the Error. *International Journal of Numerical Method for Engineering. Volume 18, pages 775-781*

[4] Yusuph Y. and Onumanyi P. (2002). New Multiple Finite Difference Method Schemes through Multi-step Collocation for $y'' = f(x, y)$. *Abacus Volume 29, pages 92-96.*

[5] Yahaya Y. A. (1995). An Off-grid Finite Difference Method for Two –point Boundary-value Problems. MSc. Thesis (Unpublished) University of Jos, Nigeria.