Transactions of the Nigerian Association of Mathematical Physics
***Volume 15, (April – June, 2021 Issue*), pp1 – 6**
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MODIFIED BOUNDARY VALUE METHODS FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

In this research, modified boundary value method of step-sizes three is applied to solve two dimensional hyperbolic partial differential equations. Using the method of lines, these PDEs are converted into systems of ordinary differential equations by replacing the spatial derivatives with fourth-order central difference method. The resulting systems of ODEs are then solved by applying the derived method. The derived method are analyzed and found to be consistent, zero stabile and convergent. The accuracy of this method over others in the literature has been demonstrated as presented in the table via two examples.

Keywords: Boundary value method, Partial Differential Equations, Power series, Finite difference method

1. Introduction

Many practical problems in applied science, engineering and applied mathematics are modelled mathematically into Differential equations. Differential equations are mathematical equations that relate a function with one or more of its derivatives. These type of equations are useful in the area of applied elasticity, rigid body dynamics such as the theory of plates and shells, hydrodynamics, quantum mechanics and many other. Many of the resulting problems from these fields of research may not have analytical solutions, hence, numerical techniques are applied to obtain the approximate solutions.
Generally, a linear second order partial differential equation can be written as:

$$A(x,y)\frac{\partial^2 u}{\partial x^2}+B(x,y)\frac{\partial^2 u}{\partial x\partial y}+C(x,y)\frac{\partial^2 u}{\partial y^2}+D(x,y)\frac{\partial u}{\partial x}+E(x,y)\frac{\partial u}{\partial y}+F(x,y)U(x,y)=G(x,y). \quad (1)$$

Equation (1) can also be expressed in simple form as

$$AU_{xx}+BU_{xy}+CU_{yy}+DU_x+EU_y+FU=G \quad (2)$$

This equation is said to be homogeneous if $G = 0$, and non-homogeneous if otherwise.
In equation (1) above, x and y are the independent variable (otherwise called spartial variables) A,B,C,D,E,F and G are known functions of the independent variables, while U is the dependent variable and is an unknown function of the independent variables. Partial derivative are denoted by
$U_y=\frac{\partial u}{\partial y}, U_x=\frac{\partial u}{\partial x}, U_{yy}=\frac{\partial^2 u}{\partial y^2}$ e.t.c
Boundary Value methods for the direct solution of systems of the general second order ordinary and partial differential equations with initial or boundary condition were developed by [1]. Some new difference formulas for finite difference approximation based on Taylor series to solve partial differential equations were presented by [2]. Author in [3] presented one way dissection of higher order compact scheme for the solution of two- dimension Poisson equations for solving partial differential equations. Block unification scheme for elliptic telegram and sine-Gordon partial differential equations was developed by [4]. [5] developed a matrix approach to solve Hyperbolic partial differential equations using Bernouui Polynomials while [6] developed legendary approximation for solving linear hyperbolic partial differential equations. Spectral based computational methods for solutions of fourth variable coefficients parabolic partial differential equations were developed by [7]. This research is motivated by the need to develop more accurate method for solving hyperbolic partial differential equations in two-dimensions. The stability properties of the new method are verified.

2. Derivation of three-step modified boundary value method.

The proposed method is derived by considering an approximate solution of the form

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$$U(y)=\sum_{j=0}^{2c+r-1} a_j y^j \tag{3}$$

Where c and r are collocation and Interpolation point, y are continuously differentiable function
The partial sum of (3) is given as

$$U(y)=\sum_{j=0}^{a} a_j y^j \tag{4}$$

Where k=2c+r-1. Setting collocation and Interpolation points to four (4) and two (2) respectively yields

$$U(y)=\sum_{j=0}^{9} a_j y^j \tag{5}$$

Whose first, second and third derivatives are

$$U'(y)=\sum_{j=0}^{9} j a_j y^{j-1} \tag{6}$$

$$U''(y)=\sum_{j=-2}^{9} j(j-1) a_j y^{j-2} \tag{7}$$

$$U'''(y)=\sum_{j=-3}^{9} j(j-1)(j-2) a_j y^{j-3} \tag{8}$$

Note that the approximate solution (5) and its second & third derivatives (7) & (8) coincide with theoretical solution, the differential system and the derivative. Interpolating (5) and collocating (7) at grid points y_{n_+i} , i=0,1,2,3.

$$U(y_{n+1})=U_{n+1}=a_0+ha_1+h^2a_2+h^3a_3+h^4a_4+h^5a_5+h^6a_6+h^7{}_7+h^8a_8+h^9a_9 \tag{9}$$

$$U''(y_n)=f_n=2a_2 \tag{10}$$

$$U''(y_{n+1})=f_{n+1}=72h^7a_9+56h^6a_8+42h^5a_7+30h^4a_6+20h^3a_5+12h^2a_4+6ha_3+2a_2 \tag{11}$$

$$U''(y_{n+1})=f_{n+2}=9216h^7a_9+3584h^6a_8+1344h^5a_7+480h^4a_6+160h^3a_5 +48h^2a_4+12ha_3+2a_2 \tag{12}$$

$$U''(y_{n+1})=f_{n+3}=157464h^7a_9+40824h^6a_8+10206h^5a_7+2430h^4a_6 +540h^3a_5+108h^2a_4+18ha_3+2a_2 \tag{13}$$

Collocating the third derivatives function (8) at the points $y_{n+i}, i=0,1,2,3$ gives

$$U'''(y_n)=g_n=6a_3 \tag{14}$$

$$U'''(y_{n+1})=g_{n+1}=504h^6a_9+336h^5a_8+210h^4a_7+120h^3a_6+60h^2a_5+24ha_4+6a_3 \tag{15}$$

$$U'''(y_{n+1})=g_{n+2}=32256h^6a_9+10752h^5a_8+3360h^4a_7+960h^3a_6+240h^2a_5+48ha_4+6a_3 \tag{16}$$

$$U'''(y_{n+1})=g_{n+3}=367416h^6a_9+81648h^5a_8+17010h^4a_7+3240h^3a_6+540h^2a_5+72ha_4+6a_3 \tag{17}$$

Equations (8) - (17) are combined to form a system of equations which is solved using Gaussian Elimination method to obtain the parameter of $"a_j s"$

$$a_0=u_n \tag{18}$$

The developed continuous linear Multi step Method (LMM) is constructed by substituting the parameters aj's into the approximate solution (5) After simplification with $s=\left(\frac{x-x_{n+k-1}}{h}\right), k=3$ it is then expressed in the for

$$U(y)=\alpha_0u_n+\alpha_1u_{n+1}+h^2(\beta_0f_n+\beta_1f_{n+1}+\beta_2f_{n+2}+\beta_3f_{n+3})+h^3(\psi_0g_n+\psi_1g_{n+1}+\psi_2g_{n+2}+\psi_3g_{n+3}) \tag{19}$$

Where α_k, β_k and Ψ_k are the parameters that defined the method. (see the appendices)
Evaluating at *s* = 0 and *s* = 1 gives the discrete schemes below

$$\begin{aligned} u_{n+2} \;=\; & -u_n+2\,u_{n+1}+\frac{233}{15120}\,h^3g_n-\frac{29}{420}\,h^3g_{n+1}-\frac{43}{560}\,h^3g_{n+2}-\frac{29}{7560}\,h^3g_{n+3} \\ & +\frac{227}{2268}\,h^2f_n+\frac{229}{336}\,h^2f_{n+1}+\frac{17}{84}\,h^2f_{n+2}+\frac{145}{9072}\,h^2f_{n+3} \end{aligned} \tag{20}$$

$$
\begin{aligned}
u_{n+3} &= -2\,u_n + 3\,u_{n+1} + \frac{131}{3780}\,h^3 g_n - \frac{103}{1680}\,h^3 g_{n+1} - \frac{71}{840}\,h^3 g_{n+2} - \frac{349}{15120}\,h^3 g_{n+3} \\
&+ \frac{1961}{9072}\,h^2 f_n + \frac{263}{168}\,h^2 f_{n+1} + \frac{365}{336}\,h^2 f_{n+2} + \frac{599}{4536}\,h^2 f_{n+3}
\end{aligned}
\quad (21)
$$

The first derivatives can also be expressed as follow

$$U'(y) = \alpha'_0 u_n + \alpha'_1 u_{n+1} + h^2\left(\beta'_0 f_n + \beta'_1 f_{n+1} + \beta'_2 f_{n+2} + \beta'_3 f_{n+3}\right) + h^3\left(\psi'_0 g_n + \psi'_1 g_{n+1} + \psi'_2 g_{n+2} + \psi'_3 g_{n+3}\right) \quad (22)$$

Evaluating at $s = -2, -1, 0$ and 1 gives the following

(23)

$$
\begin{aligned}
u'_n &= -\frac{1}{272160h}\,(7791\,h^3 g_n - 33804\,h^3 g_{n+1} - 12015\,h^3 g_{n+2} - 822\,h^3 g_{n+3} \\
&+ 25319\,h^2 f_n + 91638\,h^2 f_{n+1} + 16497\,h^2 f_{n+2} + 2626\,h^2 f_{n+3} \\
&+ 272160\,u_n - 272160\,u_{n+1} \\
u'_{n+1} &= \frac{1}{272160h}\,(3756\,h^3 g_n - 35127\,h^3 g_{n+1} - 9774\,h^3 g_{n+2} - 645\,h^3 g_{n+3} \\
&+ 27876\,h^2 f_n + 35127\,h^2 f_{n+1} + 19548\,h^2 f_{n+2} + 3329\,h^2 f_{n+3} \\
&- 272160\,u_n + 272160\,u_{n+1}
\end{aligned}
\quad (24)
$$

$$
\begin{aligned}
u'_{n+2} &= \frac{1}{272160h}\,(6060\,h^3 g_n + 14121\,h^3 g_{n+1} + 31698\,h^3 g_{n+2} - 13029\,h^3 g_{n+3} \\
&+ 34919\,h^2 f_n + 260118\,h^2 f_{n+1} + 275697\,h^2 f_{n+2} + 109666\,h^2 f_{n+3} \\
&- 272160\,u_n + 272160\,u_{n+1}
\end{aligned}
\quad (25)
$$

$$
\begin{aligned}
u'_{n+3} &= \frac{1}{272160h}\,(4593\,h^3 g_n - 7668\,h^3 g_{n+1} - 37233\,h^3 g_{n+2} - 1482\,h^3 g_{n+3} \\
&+ 28964\,h^2 f_n + 224073\,h^2 f_{n+1} + 148932\,h^2 f_{n+2} + 6271\,h^2 f_{n+3} \\
&- 272160\,u_n + 272160\,u_{n+1}
\end{aligned}
\quad (26)
$$

Equations (20), (21), (25) - (26) form the block method

Expanding this in Taylor series we have

$$
\begin{aligned}
&2\sum_{j=0}^{\infty} \frac{((3^j)(h))^j {y_n}^j}{j!} - 3\,y_n - \sum_{j=0}^{\infty} (3\,h)^{j+2} {y_n}^{j+2} [\frac{19709}{72576}\,0^{j-2} + \frac{9925}{4536} + \frac{767}{336}\,2^{j-2} \\
&+ \frac{5179}{4536}\,3^{j-2}]((j+2)!)^{-1} - \sum_{j=0}^{\infty} h^{j+3} {y_n}^{j+3} [\frac{467}{12096}\,0^{j-3} - \frac{232}{945} - \frac{83}{240}\,2^{j-3} \\
&- \frac{269}{3780}\,3^{j-3}]((j+3)!)^{-1} = 0
\end{aligned}
\quad (27)
$$

Collecting the like terms in power of h and y yields the following $C_0 = C_1 = C_2, \ldots C_{p+1} = 0$

3.1. Consistency of the three-step Modified Boundary Value Method

. The equation $\Pi(r, \bar{h}) = p(r) - \bar{h}(r)$ (28)

where $p(r)$ and $\alpha(r)$ are the first and second characteristics polynomial of the method respectively. [8,9] states that a linear multistep method is consistent if it satisfies the following conditions.

1. The order is $p \geq 1$
2. $\sum_{j=0}^{k} \alpha_j = 0 \quad \alpha_j = 0, \sigma_j = 0$
3. $\rho(1) = \rho''(1) = 0$
4. $\rho\,(1) = 2!\sigma(1)$

3.2. Convergence of the three-step Modified Boundary Value Method

The convergence of our methods with respect to properties discussed in conjunction with the fundamental theorem of Dahlquist for linear multistep methods. We state the theorem without proof. A linear multistep method is convergence, if it is consistence and zero stable.

3.3. Region of absolute stability of the three-step Boundary Value Method

The region of absolute stability is found according to [11], using the formula\

$$p(z) = A0 - B0 \cdot z - C0 \cdot z^2)^1 . A1 + B1 \cdot z + C1 \cdot z^2) \quad (29)$$

where

$$A0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad B0 = \begin{bmatrix} \frac{1301}{10080} & \frac{181}{2520} & \frac{3329}{272160} \\ \frac{296}{315} & \frac{109}{315} & \frac{344}{8505} \\ \frac{2187}{1120} & \frac{729}{560} & \frac{27}{160} \end{bmatrix} B1 = \begin{bmatrix} 0 & 0 & \frac{19519}{68040} \\ 0 & 0 & \frac{5731}{8505} \\ 0 & 0 & \frac{603}{560} \end{bmatrix}$$

$$C0 = \begin{bmatrix} -\frac{313}{2520} & -\frac{89}{2016} & -\frac{137}{45360} \\ -\frac{20}{63} & -\frac{52}{315} & -\frac{4}{405} \\ -\frac{243}{560} & -\frac{243}{1120} & -\frac{9}{280} \end{bmatrix} C1 = \begin{bmatrix} 0 & 0 & \frac{371}{12960} \\ 0 & 0 & \frac{206}{2835} \\ 0 & 0 & \frac{27}{224} \end{bmatrix}$$

using matlab to plot the give the graph below

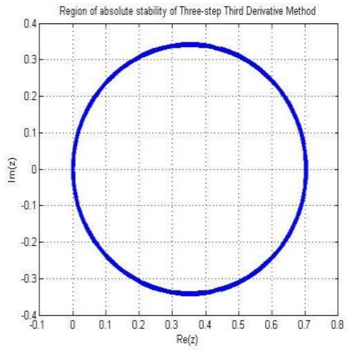

Figure 2: Region of Absolute Stability of the Three-Step Modified Boundary Value Method

4. Numerical experiment and discussion of results

The implementation strategy for the methods is discussed in this chapter. Moreover, the performance of the methods is tested on two numerical examples on second order hyperbolic partial differential equations. The absolute error of the approximate solutions are computed and compared with results from existing methods particularly those proposed by [6].

5. Implementation

The strategy adopted for the implementation of the methods is such that all the discrete methods obtained from the continuous method as well as their derivatives, which have the same order of accuracy, with very low error constants for fixed h, are combined as simultaneous integrators. We proceed by explicitly obtaining initial conditions at x_{n+1} using values from the independent solutions of the simultaneous integrators over non-overlapping subintervals; $[0, x_1],...[x_{N-1},x_N]$. [12]; to implement the respective methods proposed.

5.1. Numerical Examples

In order to study the efficiency of the developed methods, we present some numerical experiments with the following two second order hyperbolic partial differential equations. The Three-step Method is applied to solve the following test problems:

Problem 1

Consider the second order hyperbolic partial differential equation below

$Utt + 4Ut + 2U = Uxx$

$U(x,0) = Sin(x); U_t(x,0) = -Sin(x)$

with the exact solution $U(x,t) = e^{-t} sin(x)$ Source; [6]

Problem 2

Consider the second order hyperbolic equation

$U_{tt} - U_{xx} = -2(x-t)e^{-x-t}, 0 \leq x \leq 1, 0 \leq t \leq 1$

$U(x,0) = 0, 0 \leq x \leq 1$

$U(0,t) = 0, 0 \leq t \leq 1$

Neuman boundary condition

$U_t(x,0) = xe^{-x}, 0 \leq x \leq 1$

Exact Solution
$U(x,0) = xte^{-x-t}sin(x)$
Source: [5].
The following Notations were used in the tables x —– Value of the independent variables where numerical value is taken y exact —– Exact solution at x y-computed —- Computed solution at x
Error = | y-exact - y-computed | at x
3SMBVM —- Three-Step Modified Boundary Value Method

5.2 Numerical Results

Table 1: Result of Problem 1, for Three-Step Modified Boundary Value Method (3SMBVM)

$x-value$	$y-exact$	$y-computed$	Error in 3 SMBVM
1.1000	0.1179637203193835	0.11805529592844832	$9.16*10^{-5}$
1.2000	0.12371170373441585	0.12378233085287307	$7.069*10^{-5}$
1.3000	0.12793395282001335	0.12798257875432284	$4.86*10^{-5}$
1.4000	0.13057839465031007	0.1306044197686899	$2.60*10^{-5}$
1.5000	0.1312979732299798	0.13131255996132923	$1.46*10^{-5}$
1.6000	0.1315207508176514	0.13151236118085563	$8.39*10^{-5}$
1.7000	0.12881820180749187	0.12877562979146104	$4.26*10^{-5}$
1.8000	0.12502442851468534	0.12495988559651186	$6.45*10^{-5}$
1.9000	0.12254564481147873	0.12246931761103175	$7.63*10^{-5}$
2.0000	0.11968873108014735	0.11968873108014735	

Table 2: Result of Problem 2, for Three-Step Modified Boundary Value Method (3SMBVM)

$x-value$	$y-exact$	$y-computed$	Error in 3 SMBVM
0.1000	0.023981195132664963	0.023981195132664963	
0.2000	0.047617493455920604	0.047679341064287846	$6.18476*10^{-5}$
0.3000	0.06003621218638595	0.06013863849937781	$1.02426*10^{-4}$
0.4000	0.07446086776931489	0.07461621062004745	$1.55343*10^{-4}$
0.5000	0.08168178732645898	0.08186765010984308	$1.85863*10^{-4}$
0.6000	0.08727198062707602	0.08748500185821362	$2.13021*10^{-4}$
0.7000	0.09306251354119603	0.09331070924813839	$2.48196*10^{-4}$
0.8000	0.0954632142771206	0.09573159229568343	$2.68378*10^{-4}$
0.9000	0.09727372146347202	0.09756749281991457	$2.93771 *10^{-4}$
1.0000	0.09748687127648628	0.09779873479002195	$3.11864*10^{-4}$

Table 3: Comparison of Error, for Three-Step Modified Boundary Value Method (3SMBVM) with Error in [6]

$x-value$	*Error in 3SM*	Error in 3 SMBVM
1.1000	$9.16*10^{-5}$	$9.90*10^{-4}$
1.2000	$7.06*10^{-5}$	$3.19*10^{-4}$
1.3000	$4.86*10^{-5}$	$9.36*10^{-4}$
1.4000	$2.60*10^{-5}$	$2.54*10^{-3}$
1.5000	$1.46*10^{-5}$	$6.44*10^{-3}$
1.6000	$8.39*10^{-5}$	$1.54*10^{-2}$
1.7000	$4.26*10^{-5}$	$3.50*10^{-2}$
1.8000	$6.45*10^{-5}$	$7.61*10^{-2}$
1.9000	$7.63*10^{-5}$	$1.58*10^{-1}$
2.0000	0.00000000	$3.20*10^{-1}$

CONCLUSION

In this research, modified boundary value method for solving two-dimensional hyperbolic partial differential equations has been developed, analyzed and implemented. The proposed method has been tested on two numerical examples to test the accuracy and efficiency of the method. The method is implemented without the need for the development of neither predictors nor requiring any other method to generate starting values. The method has higher order of accuracy and low error constants.

APPENDIX

Where $\alpha_0 = -s - 1$

$\alpha_1 = 2 + s$

$\beta_0(s)=\frac{1}{272160} h^2(2+s)(s+1)(385s^7 - 30s^7 - 2000s^5 + 1860s^4 - 194s^3 + 2112s^2 - 5948s + 13620$

$$\beta_1(s) = \frac{1}{10080} h^2 (2+s)(s+1)(35s^7 + 75s^6 - 235s^5 - 285s^4 + 821s^3 - 213s^2 - 1003s + 3435)$$

$$\beta_2(s) = -\frac{1}{10080} h^2 (2+s)(s+1)(35s^7 + 30s^6 - 280s^5 - 60s^4 + 866s^3 - 168s^2 - 1228s - 1020)$$

$\beta_0(s)=\frac{1}{272160} h^2(2+s)(s+1)(385s^7 + 1185s^6 - 785s^5 - 4215s^4 - 1409s^3 + 897s^2 + 127s - 2175)$

$$\Psi_0(s) = \frac{1}{90720} h^3 (2+s)(s+1)(35s^7 - 15s^6 - 145s^5 + 129s^4 + 29s^3 + 75s^2 - 283s + 699)$$

$$\Psi_1(s) = \frac{1}{10080} h^3 (2+s)(s+1)(35s^7 + 30s^6 - 220s^5 + 12s^4 + 404s^3 - 396s^2 + 380s - 348)$$

$$\Psi_2(s) = \frac{1}{10080} h^3 (2+s)(s+1)(35s^7 + 75s^6 - 175s^5 - 297s^4 + 359s^3 + 357s^2 - 109s - 387)$$

$$\Psi_3(s) = \frac{1}{90720} h^3 (2+s)(s+1)(35s^7 + 120s^6 - 10s^5 - 294s^4 - 106s^3 + 66s^2 + 14s - 174)$$

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