Transactions of the Nigerian Association of Mathematical Physics
***Volume 14, (January - March, 2021 Issue*), pp159 –164**
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IMPROVED SOLOW GROWTH MODEL ON TIME SCALE

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Abstract

In this paper, we further analyze the Solow model on time scales. The simple growth model can only provide an adequate approximation for an initial period, some conditions were revised to incorporate numerical upper bound on the growth size. We establish an existence and uniqueness of solution on time scale. Then, under the more realistic assumption that the labour force growth rate is a monotonically decreasing function, we discuss stability and monotonicity of the solutions of the Solow model. The economic meanings are also indicated in concluding part.

AMS Subject classification: 81S25, 34A37, 34D10
Keyword: time scale, Solow Growth Model, monotonicity, \\asymptotic stability, Δ differentiation.

Introduction

The neoclassical growth model, developed by Solow [1] had a great impact on how the economists think about economic growth. Since then, it has stimulated an enormous amount of work [2, 3]. Since differential equation systems are usually more easily handled than difference systems from the analytical point of view, some of the economic models have used continuous timing [4, 5, 6, 7] while others are given in difference models because some people think economic data are collected at discrete intervals and transformation of capital into investments depends on the length of time lag, etc. [8, 9]. Hence, in economic modeling, either continuous timing or discrete timing is present, and there is not a common view among economists on which representation of time is better for economic models [10]. Meanwhile, many results concerning differential equations may carry over quite easily to corresponding results for difference equations, while other results seems to be completely different in nature from their continuous counterparts [11].

The blanket assumption that economic processes are either solely continuous or solely discrete, while convenient for traditional mathematical approaches may sometimes be inappropriate, because in reality many economic phenomena do feature both continuous and discrete elements. In biology, a familiar example is a 'seasonal breeding population in which generations do not overlap' [11]. A similar typical example in economics is the 'seasonally changing investment and revenue in which seasons play an important effect on this kind of economic activity'. In addition, option pricing and stock dynamics in finance [12] and the frequency and duration of market trading in economics also contain these hybrid continuous-discrete processes. Therefore, there is a great need to find a more flexible mathematical framework to accurately model the dynamical blend of such systems, so that they are precisely described and better understood. To meet this requirement, an emerging, progressive and modern area of mathematics, known as 'dynamic equations on time scales', has been introduced. This calculus has the capacity to act as the framework to effectively describe the above phenomena and to make advances in their associate fields, see e.g., [13, 14].

This theory was introduced by Stefan Hilger in 1988 in his Ph.D. thesis [15] in order to unify continuous and discrete analysis, and has been developed by many mathematicians. A time scale $\mathbb{T}$ is defined as any nonempty closed subset of $\mathbb{R}$. In the time scales setting, once a result is established, special cases include the result for the differential equation when the time scale is the set of all real numbers R and the result for the difference equation when the time scale is the set of all integers $\mathbb{z}$. The induction principle and rules of Δ differentiation plays an important role in the proofs of some of our results, so we give it here.

Definitions

(1) A time scale $\mathbb{T}$ is an arbitrary non empty closed subset of a real number.

Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator

$\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) : \inf[s \in \mathbb{T}: s > t]$

while backward jump operator is

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ρ : $\mathbb{T} \rightarrow \mathbb{T}$ by ρ(t) : sup[s $\in \mathbb{T}$: s < t]

(2) Classification of points

(i) t is right-dense if σ(t) = t

(ii) t is left-dense if ρ (t) = t

(iii) t is right-scattered if σ(t) > t

(iv) t is left-scattered if ρ(t) < t

(v) t is said to be isolated if σ(t) < t < σ(t) and

(vi) t it dense if σ(t) = t = ρ (t)

(3) The graininess function ν is defined by : $\mathbb{T} \rightarrow [0,\infty)$ is defined by μ(t) = σ(t) - t. we define $\mathbb{T}^k$ as follows. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - m$. Else $\mathbb{T}^k = \mathbb{T}$

(4) A function $f : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be right-dense (rd) continuous if g defined by $g(t) = f(t, k(t))$ is right-dense continuous for any continuous function $K : \mathbb{T} \rightarrow \mathbb{R}$

(5) f is regressive at $t \epsilon \mathbb{T}^k$ if the mapping $\left(id + \mu f(t,.)\right): K \rightarrow \mathbb{R}_+$ is invertible (where id is the identity function) and f is said to be regressive on $\mathbb{T}^k$ if f is regressive at each point $t \epsilon \mathbb{T}^k$

Theorem 1(Induction Principle)

Let $t_0 \in \mathbb{T}$ and assume that $[\, s(t) : t \in [t_0, \infty)\,]$ is a family of statements satisfying

(i) The statement $S(t_0)$ is true

(ii) If $t \in [t_0, \infty)$ is right-scattered and $S(t)$ is true, then $S(\sigma(t))$ is also true.

(iii) If $t \in [t_0, \infty)$ is right-dense and $S(t)$ is true, then there exist a neighbourhood U of t such that $S(s)$ is true for all $s \in U \cap t \in [t, \infty)$.

(iv) If $t \in [t_0, \infty)$ is left-dense and $S(s)$ is true for all, $t \in [t_0, \infty)$, then $S(t)$ is true

Proof

Let $S^* = [t \in [t_0, \infty): S(t)$ is not true], we want to show that $S^* = [\phi]$. To achieve a contradiction, we assume that $S^* \neq [\phi]$. Since S^* is nonempty and $\mathbb{T}$ is closed, we have

$$\inf S^* \in \mathbb{T}$$

we claim that $S(t)^*$ is true. If $t^* - t_0$, then $S(t)^*$ is true from (i).
If $t^* \neq t_0$ and $\rho(t^*) = t^*$, then t^* is true from (iv). Finally, if $\rho(t^*) < t^*$ then $S(t^*)$ is true from (ii). Hence in any case $S(t^*) \neq S^*$. Thus t^* can not be right-scattered, and $t^* \neq \max \mathbb{T}$ either. Hence t^* is right-dense. But now (iii) leads to contradiction. ■

Improved Solow Model on Time Scales

The simple Solow growth model assumed that the labour force L grows at a constant rate n on the time scale, i.e.

$$L^{\Delta}(t) n L(t) \quad (2.1)$$

which implies that the labour force grows exponentially, that is,

$$L(t) = L_0\, e_n(t, t_0)$$

where L_0 is the initial labour level at $t_0 \in \mathbb{T}$. With the properties of the exponential function on time scales and the fact that $n > 0$, we have $\lim_{t\to\infty} L(t)$. This means the labour force approaches ∞ when t goes to ∞, which is unrealistic, because in reality the environment has a carrying capacity. So the simple growth model of labour in equation (2.1) can provide an adequate approximation to such growth only for an initial period, but does not accommodate growth reductions due to competition for environmental resources such as food, habitat and the policy factor etc. [4]. Since the 1950s, developing countries have recognized that the high
population growth rate has seriously hampered the economic growth and adopted the population control policy. As a result, the population growth rates of many countries decreased fast in the last 40 years, such as in China. Also due to the ageing of the population and, consequently, a dramatic increase in the number of deaths, the population growth rate decreased below zero in some developed countries, and is projected to decrease to zero during the next few decades in the developing countries[4].

So to incorporate the numerical upper bound on the growth size, on the reference of [9], we have this :

The labour force L satisfies the following properties:

(a) The population is strictly increasing and bounded, i.e.,

$$L > 0, \quad L^{\Delta} > 0 \ on \ \mathbb{T}.^{+}_{t_0} \ \lim_{t\to\infty} L(t) = \infty,$$

(b) The population growth rate is decreasing to 0, i.e., If

$n = \frac{L^{\Delta}}{L}$, *then* $\lim_{t \to \infty} L(t) = 0$ *and* $n^{\Delta} < 0$ *on* $\mathbb{T}_{t_0}^{+}$

Hence, we have [4]

$$K^{\Delta}(t) = \frac{s}{1+\mu(t)n} f(k(t)) - \frac{\delta + n}{\mu(t)n} K(t) \qquad (2.2)$$

Note that this is a non-autonomous dynamic equation on a time scale. Next we give the theorem of existence and uniqueness for solutions of initial value problems for (2.2).

Theorem 2

Assume that

(i) $f(0) = 0$

(ii) $f'(k) > 0$ *and* $f''(k) < 0$ *for all* $k \in \mathbb{R}_+$

(iii) $\lim_{k \to 0^+} f'(k) = \infty$ *and* $\lim_{k \to 0^+} f''(k) = 0$

For any $t_0 \in \mathbb{T}$ *and* $k_0 \in \mathbb{R}_+$

Then, the initial value problem

$$\begin{cases} k^{\Delta}(t) = & \frac{s}{1+\mu(t)n} f(k(t)) - \frac{\delta + n}{1+\mu(t)n} k(t) \\ k(t_0) = & k_0 \end{cases} \qquad (2.4)$$

has a unique solution on $\mathbb{T}_{t_0}^{+} = \{t \in \mathbb{T} : t \neq t_0$

Theorem 3

Assume conditions in theorem 2 hold, let $\delta > 0$ be such that $-\delta \in \mathbb{R}_+$

Let k_1, k_2 be solutions of equation (2.2) on $\mathbb{T}_{t_0}^{+}$ with initial conditions

$k_1(t_0) - k_{0_1}$ *and* $k_2(t_0) - k_{0_2}$ respectively.

If $0 < k_{0_1} < k_{0_2}$ then $k_1 < k_2$

Theorem 4

Assume conditions in theorem 2 hold, let $\delta > 0$ be such that $-\delta \in \mathbb{R}_+$.

Let k_1, k_2 be solutions of dynamic equation on the same time scale

$$k^{\Delta}(t) = \frac{s}{1+\mu(t)n_1} f(k(t)) - \frac{\delta + n_1}{1+\mu(t)n} k(t) = u(k(t), t) \qquad (2.5)$$

and

$$k^{\Delta}(t) = \frac{s}{1+\mu(t)n_2} f(k(t)) - \frac{\delta + n_2}{1+\mu(t)n} k(t) = v(k(t), t) \qquad (2.6)$$

respectively, with the same initial condition$k_1(t_0) = k_2(t_0)$. If $n_1 < n_2$ on $\mathbb{T}_{t_0}^{+}$ then

$k_1 \geq k_2$ on $\mathbb{T}_{t_0}^{+}$

Proof

From $n_1(t) < n_2(t)$ we have $u(k(t), t) > v(k(t), t)$ for all $t \in \mathbb{T}_{t_0}^{+}$.

Let $= k_1 - k_2$. Obviously, we have $z(t_0) = k_1(t_0) - k_2(t_0) = 0$

and

$z^{\Delta}(t_0) = k^{\Delta_1}(t_0) - k^{\Delta_2}(t_0) = u(k((t_0), t) - v(k((t_0), t) > 0$

so, z is right-increasing at t_0 i.e if t_0 is right-scattered, then we have $z(\sigma(t_0)) > z(t_0) - 0$; if t_0 is right-dense, then there exists a non-empty neighbourhood $U^+(t_0) \cap \mathbb{T}$ of t_0 such that $z(t) > 0$ for any $t \in U^+(t_0) \cap \mathbb{T}$. We now show that $z \geq 0$ holds on $\mathbb{T}_{t_0}^{+}$. If this is not the case, then there must be a point $t_1 > t_0$, $t_1 \in \mathbb{T}$, such that $z(t_1) < 0$, $z(t) \geq 0$, when $t \in (t_0, t_1) \cap \mathbb{T}$. If t_1 is left-dense, then continuity of z gives that $z(t_1) > 0$ which contradicts the assumption. Hence t_1 is left-scattered. Let $\rho(t_1) = t_2$ then $z(t_2) > 0$

i.e $k_1(t_2) \geq k_2(t_1)$. Let k_2' be solution of equation (2.6) satisfying the initial condition $k'(t_2) = k_1(t_2)$. From the discussion in the beginning of this proof, we obtain that

$k_1 - k_2'$ is also right-increasing at t_2 which implies

$k_1(t) > k_2'$ *for* $t \in \dot{U}^+(t_2) \cap \mathbb{T}$ ……….(2.7)

where $\dot{U}^+(t_2) \cap \mathbb{T}$ is a nonempty right neighbourhood of t_2 (at least including 1). Taking into account that $k_2(t_2) < k_2'(t_2)$, theorem 2 gives

$k_2(t) \leq k_2'(t)$ *for all* $t \in \mathbb{T}_{t_2}^{+}$ …………..(2.8)

from (2.7) and (2.8), we have

$k_1(t) > k_2(t)$ *for* $t \in \dot{U}^+(t_2) \cap \mathbb{T}$

and thus $k_1(t_1) > k_2(t_1)$, which contradicts the fact $z(t_1) < 0$ \$z(t_1) < 0 ■

Theorem 5
Assume conditions in theorem 2 hold, let $\delta > 0$ be such that $-\delta \epsilon \mathbb{R}_+$.
If k solves equation (2.2), then $\lim_{t\to\infty} k(t) = \widetilde{k_0}$

Proof
We need to show that for any $\varepsilon > 0, there\ exist\ \ T > 0\ such\ that\ t > T, t\ \epsilon \mathbb{T}$.
we have $\left|k(t) - \widetilde{k_0}\right| < \varepsilon$. Now let $\varepsilon > 0$. Since $\lim_{n\to 0^+} \widetilde{k_n} = \widetilde{k_0}$
we know that there exists $\bar{n} > 0$ such that $\widetilde{k_n} - \widetilde{k_0} < \frac{\varepsilon}{3}\ \ for\ all\ n\ \epsilon\ (0, \bar{n})$
Let $t_1 \in \mathbb{T}._{t_0}^{+}$ such that $n_{t_1} = n(t_1) < \bar{n}$ and let $k_{n_{t_1}}$ and k_0 be the solutions of
$k^{\Delta}(t) = \frac{s}{1+\mu(t)n_{t_1}} f\big(k(t)\big) - \frac{\delta + n_{t_1}}{1+\mu(t)n_{t_1}} k(t)$
and
$k^{\Delta}(t) = sf\big(k(t)\big) - \delta k(t)$
respectively, with the initial conditions
$k_{n_{t_1}}(t_1) = k_0\ (t_1) - k(t_1)$
Then Theorem 3 implies that
$k_{n_{t_1}}(t_1) \le k(t) \le k_0(t)\ \ for\ \ all\ \ t \in \mathbb{T}._{t_1}^{+}$
Since $\lim_{t\to\infty} k_0\ (t) = \widetilde{k_0}\ \ there\ exist\ \ T_1 > 0$ such that
$\left|k(t) - \widetilde{k_0}\right| < \frac{\varepsilon}{3}\ \ for\ all\ t \ge T_1$
Moreover, since $\lim_{t\to\infty} k_{n_{t_1}}(t) = \widetilde{k_{n_{t_1}}}\ \ there\ exist\ \ T_2 > 0$ such that
$\left|k_{n_{t_1}}(t) - \widetilde{k_{n_{t_1}}}\right| < \frac{\varepsilon}{3}\ \ for\ all\ t \ge T_2$.
Hence for $t > T : \max T_1, T_2, t_1$, we have
$\widetilde{k_0} - \frac{2}{3}\varepsilon < \widetilde{k_{n_{t_1}}} - \frac{\varepsilon}{3} < k_{n_{t_1}} \le k_0(t) < \widetilde{k_0} + \frac{\varepsilon}{3}$
which implies that $\left|k(t) - \widetilde{k_0}\right| < \varepsilon\ \ for\ any\ t \in \mathbb{T}._{T}^{+}$. ■ $|

Theorem 6
Assume conditions in theorem 2 hold, let $\delta > 0$ be such that $-\delta \epsilon \mathbb{R}_+$. Then, the solution k solves equation (2.2) with $k(t_0) = k_0$ is asymptotically stable.

Proof
To prove the Lyapunov stability of k in equation (2.2) with initial condition $k(t_0) = k_0$, we have to show that for any $\varepsilon > 0\ there\ exist\ \eta > 0$, such that for any solution q of equation (2.2) with initial condition $q(t_0) = q_0$ and such that $|\ k(t_0) - q\ (t_0)| < \eta$
we have
$|k(t) - q(t)| < \varepsilon\ \ for\ any\ \ t \in \mathbb{T}._{t_0}^{+}$
Let $\varphi_1\ \ and\ \ \varphi_2$ be the solutions of equation (2.2) with initial conditions
$\varphi_1(t_0) = \frac{3}{2}\ k(t_0)\ and\ \ \varphi_2(t_0) = \frac{1}{2}\ k(t_0)$
respectively. From Theorem 5, we have
$\lim_{t\to\infty} \varphi_1(t) = \lim_{t\to\infty} \varphi_2(t) = \widetilde{k_0} = \lim_{t\to\infty} k(t)$
Thus, for any $\varepsilon > 0\ there\ exist\ \ t_1 > t_0$, $t_1\ \ t \in \mathbb{T}._{t_1}^{+}$, such that
$|\varphi_1(t) - k(t)| < \frac{\varepsilon}{2}\ \ for\ all\ t \in \mathbb{T}._{t_1}^{+}$
Let q solve (2.2) with the initial condition $q_0\ \epsilon\ \left(\frac{1}{2}\ k(t_0), \frac{3}{2}\ \ k(t_0)\right)$.
From theorem 3, we have
$\varphi_1(t) < q(t) < \varphi_2(t)\ \ for\ all\ t \in \mathbb{T}._{t_1}^{+}$
Thus
$|q(t) - k(t)| < \frac{\varepsilon}{2}\ \ for\ any\ \ t \in \mathbb{T}._{t_1}^{+}$
Next we choose η such that for any solution q with initial value q_0, $|q_0 - k_0\ | < \eta$ implies
$|k - q| < \varepsilon\ on\ [t_0\ ,\ t_1] \cap \mathbb{T}$

Following the proof of the theorem of continuous dependence on initial conditions, making use of the finite covering theorem, we can obtain that for any $\varepsilon > 0, there\ exist\ \eta < \frac{k_0}{2}$ such that
$|q_0 - k_0| < \varepsilon$ implies $k(t) - q(t) < \varepsilon\ \ for\ all\ \ t \in [t_0, t_1] \cap \mathbb{T}$. From Theorem 5, for any solutions $k\ and\ q$ of equation (2.2), we have that

$$\lim_{t\to\infty} k(t) = \lim_{t\to\infty} q(t) = \widetilde{k_0}$$

and then

$$\lim_{t\to\infty} |q(t) - k(t)| = 0$$

So the solution of equation (2.2) is asymptotically stable. ■
Next we will present the monotonicity of the solutions of (2.2).

Theorem 7
Assume conditions in theorem 2 hold, let $\delta > 0$ be such that $-\delta \epsilon \mathbb{R}_+$.
Let $t_0 \in \mathbb{T}\ and\ k,\ k_{n_{t_0}}, k_0$ be solutions of dynamic equation (2.2),

$$k^{\Delta}(t) = \frac{s}{1 + \mu(t_0)n_{t_0}} f\big(k(t)\big) - \frac{\delta + n_{t_0}}{1 + \mu(t)n_{t_0}} k(t) \ \ldots\ldots\ldots (2.8)$$

and

$$k^{\Delta}(t) = sf\big(k(t)\big) - \delta\, k(t) \ \ldots\ldots\ldots (2.9)$$

respectively, with the initial values

$$k(t_0) = k_{n_{t_0}} = k_0\,(t_0)$$

Then

(i) $k_{n_{t_0}} \le k \le k_0\ \ on\ \ \mathbb{T}._{t_0}^{+}$

(ii) If $k\,(t_0) \le \widetilde{k_{n_0}}$, $then\ k\ \ is\ strictly\ increasing\ on\ \mathbb{T}._{t_0}^{+}$

(iii) If $\widetilde{k_{n_0}} < k(t_0) \le \widetilde{k_0}$ $then\ there\ exist\ \tilde{t} \in \mathbb{T}\ such\ that\ k\ is\ decreasing\ on$ $[t_0, \tilde{t}] \cap \mathbb{T}\ and\ is\ increasing\ \ on\ \ \mathbb{T}._{\tilde{t}}^{+}$

(iv) If $k_0 < k(t_0), then\ \ k\ is\ inreasing\ on\ \mathbb{T}._{t_0}^{+}\ or\ there\ exist$ $\tilde{t} \in \mathbb{T}\ such\ that\ k\ is\ decreasing\ on\ [t_0, \tilde{t}] \cap \mathbb{T}\ and\ is\ increasing\ \ on\ \ \mathbb{T}._{\tilde{t}}^{+}$

Here $\widetilde{k_{n_0}}\ and\ \ k(t_0)$ are the equilibria of (2.8) and (2.9), respectively.

Proof

(i) For $t > t_0,\ t \in \mathbb{T},\ we\ have\ \ n(t_0) > n(t) > 0.$ So from theorem 5, we obtain the result easily.

(ii) we want to show that $S(t)\ \ given\ by\ \ k^{\Delta}(t) > 0$ is true for any $t \in \mathbb{T}._{t_0}^{+}$, using the principle of induction (theorem 1)

(a) Since $k\,(t_0) \le \widetilde{k_{n_0}}$ we have $k^{\Delta}(t_0) > 0\ , so\ S(t) hold\ at\ t = 0$

(b) If t is right-scattered and $k^{\Delta}(t) > 0$, then

$$k\big(\sigma(t)\big) = k(t) + \mu(t)k^{\Delta}(t)$$
$$= k(t) + \mu(t)\frac{sf\big(k(t)\big) - (\delta + n(t)(k(t))}{1 + \mu(t)n}$$
$$= \frac{(1 - \mu(t)\delta)\big(k(t)\big) + s\,\mu(t)f\big(k(t)\big)}{1 + \mu(t)n(t)} < \frac{(1 - \mu(t)\delta)\big(k\sigma(t)\big) + s\,\mu(t)f\big(k\sigma(t)\big)}{1 + \mu(t)n\sigma(t)}$$
$$= \frac{[\big(1 + \mu(t)n\sigma(t)\big)]\big(k\sigma(t)\big)}{1 + \mu(t)n\sigma(t)} + \frac{\big(\mu(t)\big)[sf\big(k\sigma(t)\big) - \big(\delta + n\sigma(t)\big)\big(k\sigma(t)\big)]}{1 + \mu(t)n\sigma(t)}$$
$$k\big(\sigma(t)\big) + \frac{\mu(t)}{1 + \mu(t)n\big(\sigma(t)\big)}\big[1 + \mu(t)n\big(\sigma(t)\big)\big]k^{\Delta}(\sigma(t))$$
$$so,\ \ k^{\Delta}\big(\sigma(t)\big) > 0$$

(c) If t is right-dense and $k^{\Delta}\big(\sigma(t)\big) > 0$ then there exists a neighbourhood $\dot{U}^{+}(t) \cap \mathbb{T}$ such that $k^{\Delta}\big(\sigma(t)\big) > 0$ for any $r \in \dot{U}^{+}(t) \cap \mathbb{T}$. To prove this, we assume that there does not exist such a neighbourhood. Then there must exist a decreasing sequence $\{t_n\} \subset \dot{U}^{+}(t) \cap \mathbb{T}$ such that $\lim_{n\to\infty} t_n = t\ and\ k^{\Delta}\, t_n \le 0$. From the properties of f taking limit on both sides, we obtain $k^{\Delta}\, t_n \le 0$ which is a contradiction.

(d) Assume that t is left-dense and $k^{\Delta}\,(r) > 0$ for any $r \in [t_0, t) \cap \mathbb{T}$. From continuity, we can get $k^{\Delta}\big((t)\big) \le 0\ \ if\ \ k^{\Delta}(t) = 0$, then for any $r \in [t_0, t) \cap \mathbb{T}$, from the chain rule in [6], we have

$$[(1+\mu n)k^{\Delta}](r) = [s(f\circ k) - (\delta+n)k]^{\Delta}(r)$$
$$= sf'\big(k(r)\big)k\Delta(r) - n^{\Delta}(r)k(r) - (\delta + n^{\sigma}(r)k\Delta(r)$$

Taking limit on both sides when $r \to t$, we obtain

$[(1+\mu n)k\Delta)]^{\Delta}(t) = -n\Delta(t)k(t) > 0$

So since t is left-dense and from the continuity, we have

$(1+\mu(t)n(t)k^{\Delta}(t) > \big(1+\mu(r)n(r)\big)k^{\Delta}(r) > 0$

for all $r \in \dot{U^{+}}(t) \cap \mathbb{T}$. Hence $k^{\Delta}(t) > 0$.

(e) If $\widetilde{k_{n_{t_0}}} \le k(t_0) \le \widetilde{k_0}$, then

$$k^{\Delta}(t_0) = \frac{s}{1+\mu(t_0)n_{t_0}} f\big(k(t)\big) - \frac{\delta + n_{t_0}}{1+\mu(t)n_{t_0}} k(t_0)$$
$$= \frac{s}{1+\mu(t_0)n_{t_0}} f\left(k_{n_{t_0}}(t)\right) - \frac{\delta+n_{t_0}}{1+\mu(t)n_{t_0}}\left(k_{n_{t_0}} t_0\right)$$
$$k_{n_{t_0}}{}^{\Delta}(t_0) < 0$$

Hence k is right-decreasing at t_0 i.e., if t_0 is right-scattered, then $k\big(\sigma(t_0)\big) < k(t_0)\,; if\ (t_0)$ is right-dense, then there exists a nonempty neighborhood$\dot{U^{+}}(t) \cap \mathbb{T}$ of (t_0) such that $k(t) < k(t_0)$ for any $t \in \dot{U^{-}}(t) \cap \mathbb{T}$ is true on $\mathbb{T}._{t_0}^{+}$ then k is decreasing on $\mathbb{T}._{t_0}^{+}$. Considering

$\lim_{t\to\infty} k(t) = \widetilde{k_0}$ in theorem 5, we have

$\widetilde{k_0} \le k(t) < k(t_0) \le \widetilde{k_0} \quad for\ t \in \mathbb{T}._{t_0}^{+}$

which is a contradiction. So there must exist $\tilde{t} \in \mathbb{T}._{t_0}^{+}$ $such\,that$ $k^{\Delta}(\tilde{t}) > 0$ and for simplicity we assume $\tilde{t}$ is the first point that verifies the inequality. So it must be proved that $k^{\Delta}(t) > 0$ $for\ all\, t \in \mathbb{T}._{\tilde{t}}^{+}$ which is similar to the proof of Statement 2.

item Following the same proof as in Statement 3, we can obtain the monotonicity.■

Conclusion

Theorem 3 means that if two economies have the same fundamentals, then the one with the bigger initial capital per worker will always have the bigger capital per worker for ever on any time scale. The result in Theorem 3 includes the results in [16] and [9] as special cases.

Theorem 4 implies that, on any economic domain, for two economies with the same initial capital per worker, the economy with the smaller population growth rate will always have the bigger capital per worker on any time scale. The result here also includes the results in [16, 6] and [9] as special cases. Theorem 5 says that for any economic domain $\mathbb{T}$ the population growth rate n(t) has no influence on the level of per worker output in the long run. That is, provided that the economy possesses a population growth rate strictly decreasing to zero, the capital per worker always converges to the positive steady state of the Solow model on a time scale with a population growth rate of zero. Theorem 6 says that under the same fundamentals, if two economies operating on the same time domain have nearly the same initial capital per worker, the following capitals per worker will take on similar behaviour.

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