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ON THEORETICAL STUDY OF *RAYLEIGH*-EXPONENTIATED ODD GENERALIZED-*X* FAMILY OF DISTRIBUTIONS

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Abstract

Over the years, several attempts were made by many researchers, to improve the goodness-of-fit of numerous classical probability distributions by inducing additional parameters aimed at enhancing their flexibility in describing datasets from diverse fields of human endeavour. In this article, a new variant of T-Exponentiated Odd Generalized-X family of distributions introduced in an earlier research is presented. The new variant titled Rayleigh-Exponentiated Odd Generalized-X family becomes what it is, when the variable T follows Rayleigh distribution. Some important functions comprising the cumulative distribution, probability density, survival function and the hazard function of the new sub-family are presented. Other vital derivations include moments, moment generating function, quantile function, entropy and function of order statistics. Furthermore, the proposed sub-family is shown to belong to the Exponentiated-G family of distributions. The method of maximum likelihood is used to derive estimates of the unknown parameters; after which parameter asymptotic confidence bounds were also obtained.

Keywords: T-X family, Exponentiated-*G* family, Parameter induction, Order statistics, Maximum likelihood Estimation

1. Introduction

Many challenging problems abound in diverse fields of human endeavour cannot be adequately handled by well-known conventional probability distributions. In an attempt to enhance the capability of these classical probability distributions and as well improve their goodness-of-fit ability in describing different datasets from different walks of human life; many researchers developed new compound distributions either through induction of one or more parameters or via hybridization of two or more probability models. Some of the most notably older studies in this regard include Pearson [1] who introduced Differential Equation Technique; Burr [2] who proposed another system of generalizing probability distributions based on Pearson’s DE approach; Johnson [3] who pioneered the transformation (translation) method of generating new probability distributions. The quantile function technique was earlier worked upon by Hastings *et al*. [4] and much later by Tukey [5] for the development of Lambda distribution. Azzalini [6] studied skew-distribution techniques; Mudholkar and Srivastava [7] developed exponentiated Weibull family of distribtuions. While Marshall and Olkin [8] visualized the art of generalizing probability distributions from lifetime distribution perspective, Gupta *et al*. [9] worked on exponentiated-*G* family of distributions.

Some fairly recent and notable researches include Alzaatreh *et al*. [10] who introduced the T-X family of distributions; Alzaghal *et al*. [11] who introduced exponentiated T-X family of distributions. On the other hand, Bourguignon *et al*. [12] introduced Weibull-G family; Tahir *et al*. [13] worked on Poisson-X family; Yahaya and Abba [14] presented Odd Generalized Exponential Inverse-Exponential distribution, Ieren and Yahaya [15] worked on two Lomax-based probability distributions, and Falgore and Doguwa [16] studied the properties of New Weibull distribution.

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Recently, Yahaya [17] proposed *T*-Exponentiated Odd Generalized-*X* family of distributions using an exponentiated odd ratio of the distribution function of a (baseline) random variable *X* while utilizing the novel idea introduced in Alzaatreh *et al*. [10]. According to Yahaya [17]; a new CDF $F_{\boldsymbol{\sigma}}\left(x \mid \boldsymbol{\sigma}=(\alpha,\Theta,\boldsymbol{\psi})\right)$ can be obtained as:

$$F_{\boldsymbol{\sigma}}\left(x \mid \boldsymbol{\sigma}=(\alpha,\Theta,\boldsymbol{\psi})\right)=\int_{l}^{M_{\alpha,\boldsymbol{\psi}}(x)} b_{\Theta}(t)\,dt = B_{\Theta}\left[M_{\alpha,\boldsymbol{\psi}}(x)\right]=B_{\Theta}\left[\frac{G_{\boldsymbol{\psi}}^{\alpha}(x)}{\bar{G}_{\boldsymbol{\psi}}^{\alpha}(x)}\right]=B_{\Theta}(T) \tag{1}$$

while the corresponding PDF obtainable from equation (1) is given by:

$$f_{\boldsymbol{\sigma}}\left(x \mid \boldsymbol{\sigma}=(\alpha,\Theta,\boldsymbol{\psi})\right)=\alpha g_{\boldsymbol{\psi}}(x)G_{\boldsymbol{\psi}}^{\alpha-1}\left[\bar{G}_{\boldsymbol{\psi}}^{\alpha}(x)\right]^{-2} b_{\Theta}\left[M_{\alpha,\boldsymbol{\psi}}(x)\right] \quad \forall x \tag{2}$$

Where α is a nonnegative scale parameter $\left(\alpha\in\mathbb{R}^{+}\right)$; $G_{\boldsymbol{\psi}}(x)$ and $g_{\boldsymbol{\psi}}(x)$ are the respective CDF and PDF of the random variable *X* indexed by parameter space, $\boldsymbol{\psi}$, $M_{\alpha,\boldsymbol{\psi}}(x)=\left[\frac{G_{\boldsymbol{\psi}}^{\alpha}(x)}{\bar{G}_{\boldsymbol{\psi}}^{\alpha}(x)}\right]$ and $\bar{G}_{\boldsymbol{\psi}}^{\alpha}(x)=\left[1-G_{\boldsymbol{\psi}}^{\alpha}(x)\right]$. While $B_{\Theta}(t)$ and $b_{\Theta}(t)$ are the respective CDF and PDF of the random variable *T* indexed by parameter vector Θ . *l* is the lower limit within the domain of *T*.

Now, assuming *T* is a Rayleigh distributed random variable; then the *T*-Exponentiated Odd Generalized-*X* family of distributions proposed earlier reduces to *Rayleigh*-Exponentiated Odd Generalized-*X* (*Rayleigh*-EOG-*X*) family; and any variable *X* that conforms to this new family can be regarded as *Rayleigh*-EOG-*X* random variable. Henceforth, this newly proposed family, its applications, estimation and simulations remain the focal point of this article.

2. The *Rayleigh*-EOG-*X* Family of Distributions

Let $T\in[0,\infty)$ be a Rayleigh distributed random variable indexed by parameter $\theta>0$ having CDF and PDF respectively given by $R_{\theta}(t)=1-e^{\frac{-\theta}{2}t^{2}}$ and $r_{\theta}(t)=\theta t e^{\frac{-\theta}{2}t^{2}}$. Let *X* be any arbitrary variable with CDF, $G_{\boldsymbol{\psi}}(x)$ indexed by say, *p*-parameter vector $\boldsymbol{\psi}=\left(\psi_{1},\psi_{2},\cdots,\psi_{p}\right)$. Given a link function $H_{\alpha,\boldsymbol{\psi}}(x)=\frac{G_{\boldsymbol{\psi}}^{\alpha}(x)}{\bar{G}_{\boldsymbol{\psi}}^{\alpha}(x)}$ with $(p+1)$ parameters; equation (1) which gives the CDF of the new *Rayleigh*-EOG-*X* family of distributions having $(p+2)$ parameters is given by:

$$F\left(x;\alpha,\theta,\boldsymbol{\psi}\right)=1-\mathrm{e}^{\frac{-\theta}{2}\left\{\frac{G_{\boldsymbol{\psi}}^{\alpha}(x)}{\bar{G}_{\boldsymbol{\psi}}^{\alpha}(x)}\right\}^{2}} \quad \forall\, x,\alpha,\theta>0;\ \boldsymbol{\psi}>\mathbf{0} \tag{3}$$

while the corresponding PDF obtainable from equation (2) is given by:

$$f\left(x;\alpha,\theta,\boldsymbol{\psi}\right)=\frac{\alpha\theta g_{\boldsymbol{\psi}}(x)G_{\boldsymbol{\psi}}^{2\alpha-1}(x)}{\left[\bar{G}_{\boldsymbol{\psi}}^{\alpha}(x)\right]^{3}}e^{\frac{-\theta}{2}\left\{\frac{G_{\boldsymbol{\psi}}^{\alpha}(x)}{\bar{G}_{\boldsymbol{\psi}}^{\alpha}(x)}\right\}^{2}} \quad \forall\, x,\alpha,\theta>0;\ \boldsymbol{\psi}>\mathbf{0} \tag{4}$$

Where $G_{\boldsymbol{\psi}}(x)$ and $g_{\boldsymbol{\psi}}(x)$ represent the CDF and PDF of the baseline random variable *X* indexed by a *p*-parameter vector $\boldsymbol{\psi}=\left(\psi_{1},\psi_{2},\cdots,\psi_{p}\right)$.

The *Rayleigh*-EOG-*X* family can be interpreted as follows: suppose the random variable *T* represents a (lifetime) variable aimed at defining a certain stochastic phenomenon via the CDF $G_{\boldsymbol{\psi}}^{\alpha}(x)$ for some α > 0. Furthermore, if we let $H_{\alpha,\boldsymbol{\psi}}(x)$ to signify the risk (odds ratio) that a certain component ceases to operate or expires at some time, say *x*; then the variability of this odds ratio can be modelled by utilizing a Rayleigh probability model, $R_{\boldsymbol{\Theta}}$ as in: $P(X\le x)=P\left[X\le H_{\alpha,\boldsymbol{\psi}}(x)\right]=R_{\theta}\left[\frac{G_{\boldsymbol{\psi}}^{\alpha}(x)}{\bar{G}_{\boldsymbol{\psi}}^{\alpha}(x)}\right]=F\left(x;\alpha,\theta,\boldsymbol{\psi}\right)$ which gives the same relation as found in equation (4) – the CDF of *Rayleigh*-EOG-*X* family of distributions. The *Rayleigh*-EOG-*X* family can be shown to belong to an exponentiated-*G* family of distributions due to Gupta *et al*. [9] through the following propositions:

Proposition 1: Let *X* be any nonnegative random variable having respective CDF and PDF denoted as $g_{\psi}(x)$ and $G_{\psi}(x)$ indexed by *p*-parameter vector $\psi = \left(\psi_1, \psi_2, \cdots, \psi_p\right)$; then the PDF of *Rayleigh*-EOG-*X* family of distributions can be represented as:

$$f\left(x;\alpha,\theta,\psi\right) = \sum_{i,j=0}^{\infty} \frac{(-1)^i \theta^{i+1} (2i+j+1)!}{2^i i!\, j!(2i+2)!} \alpha\left(2i+j+2\right) g_{\psi}\left(x\right)\left[G_{\psi}\left(x\right)\right]^{\alpha(2i+j+2)-1} \quad (5)$$

Proof:

The PDF of *Rayleigh*-EOG-*X* family of distributions given in equation (4) can be rewritten as:

$$f\left(x;\alpha,\theta,\psi\right) = \alpha\theta g_{\psi}\left(x\right) G_{\psi}^{2\alpha-1}\left(x\right)\left[1-G_{\psi}^{\alpha}\left(x\right)\right]^{-3} e^{\frac{-\theta}{2} G_{\psi}^{2\alpha}(x)\left[1-G_{\psi}^{\alpha}(x)\right]^{-2}} \quad (6)$$

By utilizing power series expansions, one can express $e^{\frac{-\theta}{2} G_{\psi}^{2\alpha}(x)\left[1-G_{\psi}^{\alpha}(x)\right]^{-2}}$ as:

$$e^{\frac{-\theta}{2} G_{\psi}^{2\alpha}(x)\left[1-G_{\psi}^{\alpha}(x)\right]^{-2}} = \sum_{i=0}^{\infty} \frac{(-1)^i \theta^i}{2^i i!} G_{\psi}^{2\alpha i}\left(x\right)\left[1-G_{\psi}^{\alpha}\left(x\right)\right]^{-2i} \quad (7)$$

Thus, equation (6) becomes:

$$f\left(x;\alpha,\theta,\psi\right) = \sum_{i=0}^{\infty} \frac{(-1)^i \theta^{i+1}}{2^i i!} \alpha g_{\psi}\left(x\right) G_{\psi}^{2\alpha(i+1)-1}\left(x\right)\left[1-G_{\psi}^{\alpha}\left(x\right)\right]^{-(2i+3)} \quad (8)$$

By utilizing binomial expansions, one can write $\left[1-G_{\psi}^{\alpha}\left(x\right)\right]^{-(2i+3)}$ as:

$$\left[1-G_{\psi}^{\alpha}\left(x\right)\right]^{-(2i+3)} = \sum_{j=0}^{\infty} (-1)^j \binom{-(2i+3)}{j} \left[G_{\psi}\left(x\right)\right]^{\alpha j} \quad (9)$$

By substituting equation (9) into (8); then simplifying the resultant expression, equation (6) becomes:

$$f\left(x;\alpha,\theta,\psi\right) = \sum_{i,j=0}^{\infty} u_{i,j}\, \delta_{\alpha(2i+j+2)}\left(x\right) \quad (10)$$

Where $u_{i,j} = \frac{(-1)^i \theta^{i+1} (2i+j+1)!}{2^i i!\, j!(2i+2)!}$ and $\delta_c\left(x\right) = c g_{\psi}\left(x\right)\left[G_{\psi}\left(x\right)\right]^{c-1}$ represents the PDF of *exponentiated-G* distribution with power parameter *c*.

Proposition 2: Let *X* be any nonnegative random variable having respective CDF and PDF denoted as $g_{\psi}(x)$ and $G_{\psi}(x)$ indexed by *p*-parameter vector $\psi = \left(\psi_1, \psi_2, \cdots, \psi_p\right)$; then the CDF of *Rayleigh*-EOG-*X* family of distributions can be represented as:

$$F\left(x;\alpha,\theta,\psi\right) = \sum_{i,j=0}^{\infty} \frac{(-1)^i \theta^{i+1} (2i+j+1)!}{2^i i!\, j!(2i+2)!} \left[G_{\psi}\left(x\right)\right]^{\alpha(2i+j+2)} \quad (11)$$

Proof:

Going by the relation that $F\left(x\right) = \int_{-\infty}^{\infty} f\left(x\right) dx$; one can then write equation (3) as:

$$F\left(x;\alpha,\theta,\psi\right) = \int_0^{\infty} f\left(x;\alpha,\theta,\psi\right) dx = \sum_{i,j=0}^{\infty} u_{i,j} \int_0^{\infty} \delta_{\alpha(2i+j+2)}\left(x\right) dx \quad (12)$$

But one can write $\int_0^{\infty} \delta_{\alpha(2i+j+2)}\left(x\right) dx$ as:

$$\int_0^{\infty} \delta_k\left(x\right) dx = k\int_0^{\infty} g_{\psi}\left(x\right)\left[G_{\psi}\left(x\right)\right]^{k-1} dx \Bigg|_{k=\alpha(2i+j+2)} \quad (13)$$

By using integration by parts, equation (13) simplifies to:

$$k\int_0^{\infty} g_{\psi}\left(x\right)\left[G_{\psi}\left(x\right)\right]^{k-1} dx = k\left\{\left[G_{\psi}\left(x\right)\right]^{k} - \int_0^{\infty}\left(k-1\right) g_{\psi}\left(x\right)\left[G_{\psi}\left(x\right)\right]^{k-1} dx\right\} \quad (14)$$

After further simplifications and rearrangements of like terms, equation (14) becomes:

$$k\int_0^\infty g_\psi(x)\left[G_\psi(x)\right]^{k-1}dx=\left[G_\psi(x)\right]^k \tag{15}$$

Hence, equation (12) can, finally, be written as:

$$F(x;\alpha,\theta,\psi)=\sum_{i,j=0}^\infty u_{i,j}\,\Delta_{\alpha(2i+j+2)}(x) \tag{16}$$

Where $u_{i,j}=\frac{(-1)^i\theta^{i+1}(2i+j+1)!}{2^i i!\,j!(2i+2)!}$ and $\Delta_c(x)=\left[G_\psi(x)\right]^c$ represents the CDF of *exponentiated-G* distribution with power parameter *c*.

2.1 Checking the validity *Rayleigh*-EOG-*X* family of distributions

It is important to ascertain whether the PDF of *Rayleigh*-EOG-*X* family of distributions as given in equation (4) constitute a valid probability density; and this can be achieved by making sure that its integral over the domain of *X* equates to unity. That is to say $\int_{-\infty}^{\infty} f_{\boldsymbol{\omega}}\left(x_{Ray-EOGX}\mid\boldsymbol{\omega}=(\alpha,\theta,\psi)\right)dx=1$.

Thus,

$$\int_0^\infty f_{\boldsymbol{\omega}}\left(x_{Ray-EOGX}\mid\boldsymbol{\omega}=(\alpha,\theta,\psi)\right)dx=\alpha\theta\int_0^\infty g_\psi(x)G_\psi^{2\alpha-1}(x)\left[\bar{G}_\psi^\alpha(x)\right]^{-3}e^{\frac{-\theta}{2}\left\{\frac{G_\psi^\alpha(x)}{\bar{G}_\psi^\alpha(x)}\right\}^2}dx \tag{17}$$

Let $y=e^{\frac{\theta}{2}\left\{\frac{G_\psi^\alpha(x)}{\bar{G}_\psi^\alpha(x)}\right\}^2}$; then $dx=\frac{dy\left[\bar{G}_\psi^\alpha(x)\right]^3}{\alpha\theta g_\psi(x)G_\psi^{2\alpha-1}(x)}$. Furthermore, as $x\to 0;\ y\to 0$ and as $x\to\infty;\ y\to\infty$.

Thus, equation (17) can be rewritten as:

$$\int_0^\infty f_{\boldsymbol{\omega}}\left(x_{Ray-EOGX}\mid\boldsymbol{\omega}=(\alpha,\theta,\psi)\right)dx=\alpha\theta\int_0^\infty g_\psi(x)G_\psi^{2\alpha-1}(x)\left[\bar{G}_\psi^\alpha(x)\right]^{-3}e^{-y}\frac{\left[\bar{G}_\psi^\alpha(x)\right]^3}{\alpha\theta g_\psi(x)G_\psi^{2\alpha-1}(x)}dy=\int_0^\infty e^{-y}dy=1$$

Hence, the PDF of *Rayleigh*-EOG-*X* family of distributions is a valid PDF as required.

2.2 Survival and Hazard rate functions of *Rayleigh*-EOG-*X* Family of Distributions

The survival function of *Rayleigh*-EOG-*X* family of distributions is given by:

$$S(x;\alpha,\theta,\psi)=1-F(x;\alpha,\theta,\psi)=\mathrm{e}^{\frac{-\theta}{2}\left\{\frac{G_\psi^\alpha(x)}{\bar{G}_\psi^\alpha(x)}\right\}^2} \tag{18}$$

The hazard rate function of *Rayleigh*-EOG-*X* family of distributions is given by:

$$h(x;\alpha,\theta,\psi)=\frac{f(x;\alpha,\theta,\psi)}{S(x;\alpha,\theta,\psi)}=\alpha\theta g_\psi(x)G_\psi^{2\alpha-1}(x)\left[\bar{G}_\psi^\alpha(x)\right]^{-3} \tag{19}$$

3. Properties of *Rayleigh*-EOG-*X* Family of Distributions

In this section, some of the properties of *Rayleigh*-EOG-*X* family of distributions will be derived. Some of these properties comprise moments, moment generating function, quantile function, Entropy measures, and distributions of Order Statistics.

3.1 Moments

The r^{th} moment, say $E(x^r)$ of a random variable *X* that follows *Rayleigh*-EOG-*X* family of distributions can be obtained through the relation:

$$E(x^r)=\mu_r'=\sum_{i,j=0}^\infty u_{i,j}E\left[A_\xi^r(x)\right];\qquad r=1,2,\cdots \tag{20}$$

Where $\xi=\alpha(2i+j+2)$, $E\left[A_\gamma^c(x)\right]=\int_{-\infty}^{\infty}\gamma x^c g_\psi(x)\left[G_\psi(x)\right]^{\gamma-1}dx$ and $u_{i,j}=\frac{(-1)^i\theta^{i+1}(2i+j+1)!}{2^i i!\,j!(2i+2)!}$.

3.2 Moments Generating Function

The moment generating function, say $M_X(t)$ of a random variable *X* that follows *Rayleigh*-EOG-*X* family of distributions can be obtained through the relation:

$$M_X(t) = E\left(e^{tx}\right) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu'_r; \tag{21}$$

Where μ'_r is the r^{th} moment as defined earlier in equation (20)

3.3 Quantile Function

Proposition 3: The quantile function of *Rayleigh*-EOG-*X* family, denoted as $Q_{X_{Rayleigh-EOG-X}}(p)$ is given by:

$$Q_{X_{Rayleigh-EOG-X}}(p) = Q_X\left\{\left(\frac{\left[2\log_e\left(\frac{1}{1-p}\right)\right]^{\frac{1}{2}}}{\theta^{\frac{1}{2}} + \left[2\log_e\left(\frac{1}{1-p}\right)\right]^{\frac{1}{2}}}\right)^{\frac{1}{\alpha}}\right\} \tag{22}$$

Proof:

Consider a random variable *Z* having CDF *F*(*Z*), the quantile function, $\left[Q_Z(p) \mid 0 < p < 1\right]$ can be obtained via the relation:

$$Q_Z(p) = F^{-1}(p) \tag{23}$$

Hence from equation (23), in relation to our variable *X*; while utilizing equation (3), one can write:

$$p = F\left[Q_{X_{Rayleigh-EOG-X}}(p); \alpha, \theta, \psi\right] = 1 - e^{\frac{-\theta}{2}\left\{\frac{G_\psi^\alpha\left[Q_X(p)\right]}{\bar{G}_\psi^\alpha\left[Q_X(p)\right]}\right\}^2} \tag{24}$$

Upon simplifying equation (24), it can be deduced that:

$$G_\psi^\alpha\left[Q_{X_{Rayleigh-EOG-X}}(p)\right] = \left\{\frac{\left[2\log_e\left(\frac{1}{1-p}\right)\right]^{\frac{1}{2}}}{\theta^{\frac{1}{2}} + \left[2\log_e\left(\frac{1}{1-p}\right)\right]^{\frac{1}{2}}}\right\} \tag{25}$$

Thus

$$Q_{X_{Rayleigh-EOG-X}}(p) = G_\psi^{-1}\left\{\left(\frac{\left[2\log_e\left(\frac{1}{1-p}\right)\right]^{\frac{1}{2}}}{\theta^{\frac{1}{2}} + \left[2\log_e\left(\frac{1}{1-p}\right)\right]^{\frac{1}{2}}}\right)^{\frac{1}{\alpha}}\right\} = Q_X\left\{\left(\frac{\left[2\log_e\left(\frac{1}{1-p}\right)\right]^{\frac{1}{2}}}{\theta^{\frac{1}{2}} + \left[2\log_e\left(\frac{1}{1-p}\right)\right]^{\frac{1}{2}}}\right)^{\frac{1}{\alpha}}\right\} \tag{26}$$

Where $Q_X(.)$ represents the quantile function of the random variables *X*.

Consequently, by setting $p = \frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$; we obtain the 1st quartile, median and 3rd quartile of *Rayleigh*-EOG-*X* sub-family respectively denoted by $\tilde{X}^1_{Ray-EOGX}$, $\tilde{X}_{Ray-EOGX}$, and $\tilde{X}^3_{Ray-EOGX}$ as:

$$\tilde{X}^1_{Rayleigh-EOG-X} = Q_{X_{Rayleigh-EOG-X}}\left(\tfrac{1}{4}\right)$$

$$= Q_X\left\{\left(\frac{\left[2\left(\log_e(4) - \log_e(3)\right)\right]^{\frac{1}{2}}}{\theta^{\frac{1}{2}} + \left[2\left(\log_e(4) - \log_e(3)\right)\right]^{\frac{1}{2}}}\right)^{\frac{1}{\alpha}}\right\} = Q_X\left\{\left[\frac{(0.7585)}{\left(\theta^{\frac{1}{2}} + 0.7585\right)}\right]^{\frac{1}{\alpha}}\right\} \tag{27}$$

$$\tilde{X}_{Rayleigh-EOG-X} = Q_{X_{Rayleigh-EOG-X}}\left(\tfrac{1}{2}\right)$$
$$= Q_X\left\{\left(\frac{\left[2\log_e(2)\right]^{\frac{1}{2}}}{\theta^{\frac{1}{2}}+\left[2\log_e(2)\right]^{\frac{1}{2}}}\right)^{\frac{1}{\alpha}}\right\} = Q_X\left\{\left[\frac{(1.1774)}{\left(\theta^{\frac{1}{2}}+1.1774\right)}\right]^{\frac{1}{\alpha}}\right\} \quad (28)$$

and

$$\tilde{X}^{3}_{Rayleigh-EOG-X} = Q_{X_{Rayleigh-EOG-X}}\left(\tfrac{3}{4}\right)$$
$$= Q_X\left\{\left(\frac{\left[2\left(\log_e(4)\right)\right]^{\frac{1}{2}}}{\theta^{\frac{1}{2}}+\left[2\left(\log_e(4)\right)\right]^{\frac{1}{2}}}\right)^{\frac{1}{\alpha}}\right\} = Q_X\left\{\left[\frac{(1.6651)}{\left(\theta^{\frac{1}{2}}+1.6651\right)}\right]^{\frac{1}{\alpha}}\right\} \quad (29)$$

3.4 Shannon Entropy

Proposition 4: The Shannon entropy of *Rayleigh*-EOG-*X* family, denoted as $\eta_{X_{Rayleigh-EOG-X}}$ is given by:

$$\eta_{X_{Rayleigh-EOG-X}} = 1+\eta_X - \log_e(\alpha) - \frac{1}{2\alpha}\log_e(\theta) - \left(\frac{2\alpha-1}{2\alpha}\right)(0.6931-\gamma) - \left(\frac{\alpha+1}{2\alpha\theta}\right)\frac{\pi\, erfi\left(\frac{\theta}{2}\right) - Ei\left(\frac{\theta}{2}\right)}{e^{\frac{\theta}{2}}} \quad (30)$$

Where

Euler's constant $=\gamma = 0.577215664\ldots$, *Exponential Integral function of x* = $Ei(x) = -\int_{-x}^{\infty}\frac{e^{-t}}{t}dt = \int_{-\infty}^{x}\frac{e^{-t}}{t}dt$, *imaginary error function of x* = $erfi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$

Proof:

The Shannon entropy of any given PDF, say *f*(*Z*) of a random variable, *Z* is given by: $\eta_Z = E\left[-log\ f(Z)\right]$. Thus, for a random variable, *X* ~ *Rayleigh*-EOG-*X*; the Shannon entropy is:

$$\eta_{X_{Rayleigh-EOG-X}} = E\left[-\log\ f(x;\alpha,\theta,\psi)\right] \quad (31)$$

Thus, by using equation (4), one can write equation (31) as:

$$\eta_{X_{Rayleigh-EOG-X}} = E\left[-\log\left\{\alpha\theta g_\psi(x)G_\psi^{2\alpha-1}(x)\left[1-G_\psi^\alpha(x)\right]^{-3} e^{\frac{-\theta}{2}\left[\frac{G_\psi^\alpha(x)}{1-G_\psi^\alpha(x)}\right]^{2}}\right\}\right] \quad (32)$$

Thus equation (32) becomes:

$$\eta_{X_{Rayleigh-EOG-X}} = -\log(\alpha) - \log(\theta) + E\left[-\log g_\psi(x)\right] - (2\alpha-1)E\left[\log G_\psi(x)\right] + 3E\left\{\log\left[1-G_\psi^\alpha(x)\right]\right\} + \frac{\theta}{2}E\left\{\left[\frac{G_\psi^\alpha(x)}{1-G_\psi^\alpha(x)}\right]^2\right\} \quad (33)$$

Now, based on equation (1), it can be inferred that $H_{\alpha,\psi}(x) = \frac{G_\psi^\alpha(x)}{\bar{G}_\psi^\alpha(x)} = T$, where *T* is Rayleigh distributed variable. Hence,

$G_\psi^\alpha(x) = \left(\frac{T}{1+T}\right)$; $\left[1-G_\psi^\alpha(x)\right] = \left(\frac{1}{1+T}\right)$ and $G_\psi(x) = \left(\frac{T}{1+T}\right)^{\frac{1}{\alpha}}$. Thus, equation (33) can be written as:

$$\eta_{X_{Rayleigh-EOG-X}} = -\log(\alpha) - \log(\theta) + E\left[-\log g_\psi(x)\right] - (2\alpha-1)E\left\{\log\left[\left(\frac{T}{1+T}\right)^{\frac{1}{\alpha}}\right]\right\} + 3E\left\{\log\left[\left(\frac{1}{1+T}\right)\right]\right\} + \frac{\theta}{2}E\left(T^2\right) \quad (34)$$

After some algebraic simplifications, equation (34) becomes:

$$\eta_{X_{Rayleigh-EOG-X}} = \eta_x - \log(\alpha) - \log(\theta) - \frac{(2\alpha-1)}{\alpha} E\left[\log(T)\right] - \frac{(\alpha+1)}{\alpha} E\left[\log(1+T)\right] + \frac{\theta}{2} E\left(T^2\right) \quad (35)$$

Recall for a Rayleigh distributed random variable T with parameter θ ; $E(T) = \sqrt{\frac{\pi}{2\theta}}$, $E(T^2) = \frac{2}{\theta}$, $Var(T) = \frac{(4-\pi)}{2\theta}$. Furthermore,

$$E(\log T) = \int_{-\infty}^{\infty} (\log T) r_\theta(T)\,dT = \int_0^{\infty} \theta T (\log T) e^{\frac{-\theta T^2}{2}}\,dT = \frac{1}{2}(\log 2 - \log\theta - \gamma)$$

and $E(\log(1+T)) = \int_{-\infty}^{\infty} (\log(1+T)) r_\lambda(T)\,dT = \int_0^{\infty} \theta T \log(1+T) e^{-\frac{\theta}{2}T^2}\,dT = \frac{\pi\, erfi\left(\frac{\theta}{2}\right) - Ei\left(\frac{\theta}{2}\right)}{2\theta e^{\frac{\theta}{2}}}$

Hence, equation (35) can be, finally, written as:

$$\eta_{X_{Rayleigh-EOG-X}} = 1 + \eta_X - \log_e(\alpha) - \frac{1}{2\alpha}\log_e(\theta) - \left(\frac{2\alpha-1}{2\alpha}\right)(0.6931-\gamma) - \left(\frac{\alpha+1}{2\alpha\theta}\right)\frac{\pi\, erfi\left(\frac{\theta}{2}\right) - Ei\left(\frac{\theta}{2}\right)}{e^{\frac{\theta}{2}}} \quad (36)$$

Where

Euler's constant $=\gamma = 0.577215664...$, *Exponential Integral function of* x = $Ei(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t}\,dt = \int_{-\infty}^{x} \frac{e^{-t}}{t}\,dt$, *imaginary error function of* x = $erfi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$

3.5 Order Statistics

Given $X_1, X_2, \cdots, X_n$ as n independent and identically distributed random variables with PDF and CDF denoted respectively by $a(x)$ and $A(x)$. Let the corresponding order statistics of the given random sample be denoted by $X_{(1:n)}, X_{(2:n)}, \cdots, X_{(i:n)}, \cdots, X_{(n:n)}$. Then the PDF of a certain random variable, say $X_{(i:n)}$ (known as the *i*th order statistic) can be obtained, according to Zwillinger [18] via:

$$a_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} a(x)\left[A(x)\right]^{i-1}\left[1-A(x)\right]^{n-i} \quad (37)$$

Thus, based on the foregoing, the PDF of the *i*th order statistic, $X_{(i:n)}$ of *Rayleigh*-EOG-*X*, can be captured in the following proposition:

Proposition 5: The PDF of the *i*th order statistic $X_{(i:n)}$ of *Rayleigh*-EOG-*X* family is given by:

$$f_{i:n}(x;\alpha,\theta,\boldsymbol{\psi}) = \sum_{j=0}^{(n-i)} \sum_{k=0}^{(i+j-1)} \sum_{m,r=0}^{\infty} \nu_{jkmr}\, \delta_{\alpha(2m+r+2)}(x) \quad (38)$$

where

$$\nu_{jkmr} = \frac{n!(-1)^{j+k+m}\theta^{m+1}(i+j-1)!(k+1)^m(2m+r+3)(2m+r+1)!}{2^m(i-1)!(n-i-j)!(i+j-k-1)!(2m+2)!\,j!\,k!\,m!\,r!}\ ;\ \delta_\xi(x) = \xi\, g_{\boldsymbol{\psi}}(x) G_{\boldsymbol{\psi}}^{\xi-1}(x)$$

Proof:

Based on equation (37), the PDF of the *i*th order statistic $X_{(i:n)}$ of *Rayleigh*-EOG-*X* family can be written as:

$$f_{i:n}(x;\alpha,\theta,\boldsymbol{\psi}) = \frac{n!}{(i-1)!(n-i)!} f(x;\alpha,\theta,\boldsymbol{\psi})\left[F(x;\alpha,\theta,\boldsymbol{\psi})\right]^{i-1}\left[1-F(x;\alpha,\theta,\boldsymbol{\psi})\right]^{n-i} \quad (39)$$

By utilizing power series expansion equations (39) becomes:

$$f_{i:n}(x;\alpha,\theta,\boldsymbol{\psi}) = \sum_{j=0}^{(n-i)} \frac{(-1)^j n!}{(i-1)!(n-i-j)!\,j!} f(x;\alpha,\theta,\boldsymbol{\psi})\left[F(x;\alpha,\theta,\boldsymbol{\psi})\right]^{i+j-1} \quad (40)$$

By utilizing equations (3) and (4), (40) becomes:

$$f_{i:n}(x) = \sum_{j=0}^{(n-i)} \frac{\alpha\theta(-1)^j n!}{(i-1)!(n-i-j)!\,j!} g_{\boldsymbol{\psi}}(x) G_{\boldsymbol{\psi}}^{2\alpha-1}(x)\left[1-G_{\boldsymbol{\psi}}^{\alpha}(x)\right]^{-3} e^{\frac{-\theta}{2}\left[\frac{G_{\boldsymbol{\psi}}^{\alpha}(x)}{\bar{G}_{\boldsymbol{\psi}}^{\alpha}(x)}\right]^2} \left\{1 - e^{\frac{-\theta}{2}\left[\frac{G_{\boldsymbol{\psi}}^{\alpha}(x)}{\bar{G}_{\boldsymbol{\psi}}^{\alpha}(x)}\right]^2}\right\}^{i+j-1} \quad (41)$$

By utilizing Binomial & power series expansions: $(1-x)^n = \sum_{j=0}^{n}(-1)^j\binom{n}{j}x^j$; $(1-x)^{-a} = \sum_{j=0}^{\infty}\frac{\Gamma(a+j)}{j!\Gamma(a)}x^j$ and employing additional algebraic simplifications, equation (41) reduces to:

$$f_{i:n}(x) = \sum_{j=0}^{(n-i)}\sum_{k=0}^{(i+j-1)}\sum_{m,r=0}^{\infty}\nu_{jkmr}\,\delta_{\alpha(2m+r+2)}(x) \quad (42)$$

Where

$$\nu_{jkmr} = \frac{n!(-1)^{j+k+m}\theta^{m+1}(i+j-1)!(k+1)^m(2m+r+3)(2m+r+1)!}{2^m(i-1)!(n-i-j)!(i+j-k-1)!(2m+2)!\,j!k!m!r!};\ \delta_{\xi}(x) = \xi\, g_{\psi}(x)G_{\psi}^{\xi-1}(x)$$

Corollary 5.1: The PDF of the 1st (minimum) order statistic $X_{(1:n)}$ of *Rayleigh*-EOG-*X* family is given by:

$$f_{1:n}(x) = \sum_{j=0}^{(n-1)}\sum_{k=0}^{j}\sum_{m,r=0}^{\infty}u_{jkmr}\,\delta_{\alpha(2m+r+2)}(x) \quad (43)$$

Proof: The proof follows from Proposition 5 for $i = 1$ and $u_{jkmr} = \frac{n!(-1)^{j+k+m}\theta^{m+1}(k+1)^m(2m+r+3)(2m+r+1)!}{2^m(n-j-1)!(j-k)!(2m+2)!k!m!r!}$

Corollary 5.2: The PDF of the n^{th} (maximum) order statistic $X_{(n:n)}$ of *Rayleigh* -EOG-*X* family is given by:

$$f_{n:n}(x) = \sum_{k=0}^{(n-1)}\sum_{m,r=0}^{\infty}w_{kmr}\,\delta_{\alpha(2m+r+2)}(x) \quad (44)$$

Proof: The proof follows from Proposition 6 for $i = n$ and $w_{kmr} = \frac{n!(-1)^{k+m}\theta^{m+1}(k+1)^m(2m+r+3)(2m+r+1)!}{2^m(n-k-1)!(2m+2)!k!m!r!}$

3.6 Parameter Estimation and ACBs for *Rayleigh*-EOG-*X* Family of Distributions

This section discusses the MLE for the parameters of the proposed family as well as ACBs for such parameters that can be utilized in interval estimation and tests of statistical hypotheses.

3.6.1 MLE for *Rayleigh*-EOG-*X* Family

Suppose $X_1, X_2, \cdots, X_n$ is a random sample from *Rayleigh*-Exponentiated Odd Generalized *X* family of distributions indexed by a $(r+2)$-parameter vector, $\boldsymbol{\omega} = (\alpha, \theta, \boldsymbol{\psi})^T$, where $\boldsymbol{\psi}$ is a $(r\times 1)$ parameter vector of the transformer density function, $g_{\psi}(x)$, while θ is a parameter of the Rayleigh density function, $r_{\theta}(t)$.

Then the likelihood function, $L(\boldsymbol{\omega})$ can be obtained from equation (4) as:

$$L(\boldsymbol{\omega}) = \prod_{i=1}^{n}\left[f(x_i;\alpha,\theta,\boldsymbol{\psi})\right] = \alpha^n\theta^n\prod_{i=1}^{n}\left\{g_{\psi}(x_i)G_{\psi}^{2\alpha-1}(x_i)\left[1-G_{\psi}^{\alpha}(x_i)\right]^{-3}e^{\frac{-\theta}{2}H_{\alpha,\psi}^2(x_i)}\right\}\Bigg|_{H_{\alpha,\psi}(x_i)=G_{\psi}^{\alpha}(x_i)/\left[1-G_{\psi}^{\alpha}(x_i)\right]} \quad (45)$$

The corresponding log-likelihood function, $l(\boldsymbol{\omega})$ can be obtained from equation (45) as:

$$l(\boldsymbol{\omega}) = n\log_e\alpha + n\log_e\theta + \sum_{i=1}^{n}\log_e g_{\psi}(x_i) - (2\alpha-1)\sum_{i=1}^{n}\log_e G_{\psi}(x_i) - 3\sum_{i=1}^{n}\log_e\left[1-G_{\psi}^{\alpha}(x_i)\right] - \frac{\theta}{2}\sum_{i=1}^{n}H_{\alpha,\psi}^2(x_i) \quad (46)$$

The ML estimators of the parameters that maximize the likelihood function (equation 45), can be obtained by differentiating the log-likelihood function (equation 46) with respect to the unknown $(r+2)$ parameters, $\boldsymbol{\omega} = (\alpha,\theta,\boldsymbol{\psi})^T$. By so doing, we get the following nonlinear system of equations:

$$\frac{\partial l(\boldsymbol{\omega})}{\partial\alpha} = \frac{n}{\alpha} - 2\sum_{i=1}^{n}\log_e G_{\psi}(x_i) + 3\sum_{i=1}^{n}H_{\alpha,\psi}(x_i)\log_e G_{\psi}(x_i) - \theta\sum_{i=1}^{n}H_{\alpha,\psi}(x_i)\left(\frac{\partial H_{\alpha,\psi}(x_i)}{\partial\alpha}\right) \quad (47)$$

$$\frac{\partial l(\boldsymbol{\omega})}{\partial\theta} = \frac{n}{\theta} - \frac{1}{2}\sum_{i=1}^{n}H_{\alpha,\psi}^2(x_i) \quad (48)$$

For the parameters of the baseline distribution, the associated gradients can be obtained through:

$$\frac{\partial l(\boldsymbol{\omega})}{\partial\psi_j}\Bigg|_{j=1,2,\cdots,r} = \sum_{i=1}^{n}g_{\psi}^{-1}(x_i)\left(\frac{\partial g_{\psi}(x_i)}{\partial\psi_j}\right) - (2\alpha-1)\sum_{i=1}^{n}G_{\psi}^{-1}(x_i)\left(\frac{\partial G_{\psi}(x_i)}{\partial\psi_j}\right) + 3\alpha\sum_{i=1}^{n}\frac{H_{\alpha,\psi}(x_i)}{G_{\psi}(x_i)}\left(\frac{\partial G_{\psi}(x_i)}{\partial\psi_j}\right) - \theta\sum_{i=1}^{n}H_{\alpha,\psi}(x_i)\left(\frac{\partial H_{\alpha,\psi}(x_i)}{\partial\psi_j}\right) \quad (49)$$

On setting the nonlinear equations (47), (48) and (49) to zero and solving them simultaneously, one obtains the estimates $\hat{\boldsymbol{\omega}} = \left(\hat{\alpha}, \hat{\boldsymbol{\Theta}}, \hat{\boldsymbol{\psi}}\right)^T$ of the unknown parameters:

$$\frac{\partial l(\boldsymbol{\omega})}{\partial \alpha} = 0$$
$$\Rightarrow \frac{n}{\alpha} - 2\sum_{i=1}^{n} \log_e G_{\boldsymbol{\psi}}(x_i) + 3\sum_{i=1}^{n} H_{\alpha,\boldsymbol{\psi}}(x_i)\log_e G_{\boldsymbol{\psi}}(x_i) - \theta\sum_{i=1}^{n} H_{\alpha,\boldsymbol{\psi}}(x_i)\left(\frac{\partial H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial \alpha}\right) = 0 \tag{50}$$

$$\frac{\partial l(\boldsymbol{\omega})}{\partial \theta} = 0$$
$$\Rightarrow \frac{n}{\theta} - \frac{1}{2}\sum_{i=1}^{n} H^2_{\alpha,\boldsymbol{\psi}}(x_i) = 0 \quad \Rightarrow \hat{\theta} = \frac{2n}{\sum_{i=1}^{n} H^2_{\alpha,\boldsymbol{\psi}}(x_i)} \tag{51}$$

$$\left.\frac{\partial l(\boldsymbol{\omega})}{\partial \psi_j}\right|_{j=1,2,\cdots,r} = 0$$
$$\Rightarrow \sum_{i=1}^{n} g^{-1}_{\boldsymbol{\psi}}(x_i)\left(\frac{\partial g_{\boldsymbol{\psi}}(x_i)}{\partial \psi_j}\right) - (2\alpha-1)\sum_{i=1}^{n} G^{-1}_{\boldsymbol{\psi}}(x_i)\left(\frac{\partial G_{\boldsymbol{\psi}}(x_i)}{\partial \psi_j}\right) + 3\alpha\sum_{i=1}^{n}\frac{H_{\alpha,\boldsymbol{\psi}}(x_i)}{G_{\boldsymbol{\psi}}(x_i)}\left(\frac{\partial G_{\boldsymbol{\psi}}(x_i)}{\partial \psi_j}\right) - \theta\sum_{i=1}^{n} H_{\alpha,\boldsymbol{\psi}}(x_i)\left(\frac{\partial H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial \psi_j}\right) = 0 \tag{52}$$

Due to the complexity inherent in the above nonlinear equations, closed-form analytical solutions cannot be obtained; however, some iterative optimization techniques such as Newton-Raphson, Particle Swarm Optimization, Simulated Annealing and Genetic Algorithms can be used to obtain numerical solutions of the parameter estimates that will be found to maximize the likelihood function based on some given datasets.

3.6.2 ACB for parameters of *Rayleigh*-EOG-*X* Family

The ACB for the $(r+2)$ unknown parameters $(\alpha, \theta \text{ and } \boldsymbol{\psi})$ of the *Rayleigh*-Exponentiated Odd Generalized *X* sub-family of distributions can be utilized and would be found very useful in interval estimation and statistical test of hypotheses. In order to obtain these bounds; we've to (first) determine the asymptotic variance-covariance matrix $\Sigma(\boldsymbol{\omega})$ of the estimators {which is square of order $\left[(r+2)\times(r+2)\right]$} from the inverse of the Fisher Information matrix $\left[I^{-1}(\boldsymbol{\omega})\right]$, given by:

Table 1: Fisher Information Matrix for *Rayleigh*-EOG-*X* family

$$\Sigma(\boldsymbol{\omega}) = \begin{pmatrix} -\frac{\partial^2 l(\boldsymbol{\omega})}{\partial \alpha^2} & -\frac{\partial^2 l(\boldsymbol{\omega})}{\partial \alpha \partial \theta} & -\frac{\partial^2 l(\boldsymbol{\omega})}{\partial \alpha \partial \psi_1} & \cdots & -\frac{\partial^2 l(\boldsymbol{\omega})}{\partial \alpha \partial \psi_j} & \cdots & -\frac{\partial^2 l(\boldsymbol{\omega})}{\partial \alpha \partial \psi_r} \\ & -\frac{\partial^2 l(\boldsymbol{\omega})}{\partial \theta^2} & -\frac{\partial^2 l(\boldsymbol{\omega})}{\partial \theta \partial \psi_1} & \cdots & -\frac{\partial^2 l(\boldsymbol{\omega})}{\partial \theta \partial \psi_j} & \cdots & -\frac{\partial^2 l(\boldsymbol{\omega})}{\partial \theta \partial \psi_r} \\ & & -\frac{\partial^2 l(\boldsymbol{\omega})}{\partial \psi_1^2} & \cdots & -\frac{\partial^2 l(\boldsymbol{\omega})}{\partial \psi_1 \partial \psi_j} & \cdots & -\frac{\partial^2 l(\boldsymbol{\omega})}{\partial \psi_1 \partial \psi_r} \\ & & & \ddots & \vdots & \vdots & \vdots \\ & & & & -\frac{\partial^2 l(\boldsymbol{\omega})}{\partial \psi_j^2} & \cdots & -\frac{\partial^2 l(\boldsymbol{\omega})}{\partial \psi_j \partial \psi_r} \\ & & & & & \ddots & \vdots \\ & & & & & & -\frac{\partial^2 l(\boldsymbol{\omega})}{\partial \psi_r^2} \end{pmatrix}^{-1} = \begin{pmatrix} Var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\theta}) & cov(\hat{\alpha},\hat{\psi}_1) & \cdots & cov(\hat{\alpha},\hat{\psi}_j) & \cdots & cov(\hat{\alpha},\hat{\psi}_r) \\ & Var(\hat{\theta}) & cov(\hat{\theta},\hat{\psi}_1) & \cdots & cov(\hat{\theta},\hat{\psi}_j) & \cdots & cov(\hat{\theta},\hat{\psi}_r) \\ & & Var(\hat{\psi}_1) & \cdots & cov(\hat{\psi}_1,\hat{\psi}_j) & \cdots & cov(\hat{\psi}_1,\hat{\psi}_r) \\ & & & \ddots & \vdots & \vdots & \vdots \\ & & & & Var(\hat{\psi}_j) & \cdots & cov(\hat{\psi}_j,\hat{\psi}_r) \\ & & & & & \ddots & \vdots \\ & & & & & & Var(\hat{\psi}_r) \end{pmatrix}$$

From the foregoing Fisher Information matrix in Table (1), it can be observed that, its elements constitute the negatives of the second derivatives of the log-likelihood function with respect to the unknown parameters. These elements are obtainable:

$$\frac{\partial^2 l(\boldsymbol{\omega})}{\partial \alpha^2} = -\frac{n}{\alpha^2} + 3\sum_{i=1}^{n} \log_e G_{\boldsymbol{\psi}}(x_i)\left(\frac{\partial H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial \alpha}\right) - \theta\sum_{i=1}^{n} H_{\alpha,\boldsymbol{\psi}}(x_i)\left(\frac{\partial^2 H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial \alpha^2}\right) - \theta\sum_{i=1}^{n}\left(\frac{\partial H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial \alpha}\right)^2 \tag{53}$$

$$\frac{\partial^2 l(\boldsymbol{\omega})}{\partial\alpha\partial\theta} = -\sum_{i=1}^{n} H_{\alpha,\boldsymbol{\psi}}(x_i)\left(\frac{\partial H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial\alpha}\right) \tag{54}$$

$$\left.\frac{\partial l(\boldsymbol{\omega})}{\partial\alpha\partial\psi_j}\right|_{j=1,\cdots,r} = -2\sum_{i=1}^{n} G_{\boldsymbol{\psi}}^{-1}(x_i)\left(\frac{\partial G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right) + 3\sum_{i=1}^{n}\frac{H_{\alpha,\boldsymbol{\psi}}(x_i)}{G_{\boldsymbol{\psi}}(x_i)}\left(\frac{\partial G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right) + 3\sum_{i=1}^{n}\log_e G_{\boldsymbol{\psi}}(x_i)\left(\frac{\partial H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right)$$
$$-\theta\sum_{i=1}^{n} H_{\alpha,\boldsymbol{\psi}}(x_i)\left(\frac{\partial^2 H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial\alpha\partial\psi_j}\right) - \theta\sum_{i=1}^{n}\left(\frac{\partial H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial\alpha}\right)\left(\frac{\partial H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right) \tag{55}$$

$$\frac{\partial^2 l(\boldsymbol{\omega})}{\partial\theta^2} = \frac{-n}{\theta^2} \tag{56}$$

$$\left.\frac{\partial^2 l(\boldsymbol{\omega})}{\partial\theta\partial\psi_j}\right|_{j=1,2,\cdots,r} = -\sum_{i=1}^{n} H_{\alpha,\boldsymbol{\psi}}(x_i)\left(\frac{\partial H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right) \tag{57}$$

$$\left.\frac{\partial^2 l(\boldsymbol{\omega})}{\partial\psi_j^2}\right|_{j=1,2,\cdots,r} = \sum_{i=1}^{n} g_{\boldsymbol{\psi}}^{-1}(x_i)\left(\frac{\partial^2 g_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j^2}\right) - \sum_{i=1}^{n} g_{\boldsymbol{\psi}}^{-2}(x_i)\left(\frac{\partial g_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right)^2 - (2\alpha-1)\sum_{i=1}^{n} G_{\boldsymbol{\psi}}^{-1}(x_i)\left(\frac{\partial^2 G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j^2}\right)$$
$$+(2\alpha-1)\sum_{i=1}^{n} G_{\boldsymbol{\psi}}^{-2}(x_i)\left(\frac{\partial G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right)^2 + 3\alpha\sum_{i=1}^{n}\frac{H_{\alpha,\boldsymbol{\psi}}(x_i)}{G_{\boldsymbol{\psi}}(x_i)}\left(\frac{\partial^2 G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j^2}\right) - 3\alpha\sum_{i=1}^{n}\frac{H_{\alpha,\boldsymbol{\psi}}(x_i)}{G_{\boldsymbol{\psi}}^2(x_i)}\left(\frac{\partial G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right)^2 \tag{58}$$
$$+3\alpha\sum_{i=1}^{n} G_{\boldsymbol{\psi}}^{-1}(x_i)\left(\frac{\partial G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right)\left(\frac{\partial H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right) - \theta\sum_{i=1}^{n} H_{\alpha,\boldsymbol{\psi}}(x_i)\left(\frac{\partial^2 H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial\psi_j^2}\right) - \theta\sum_{i=1}^{n}\left(\frac{\partial H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right)^2$$

$$\left.\frac{\partial^2 l(\boldsymbol{\omega})}{\partial\psi_j\partial\psi_k}\right|_{j<k=1,2,\cdots,r} = \sum_{i=1}^{n} g_{\boldsymbol{\psi}}^{-1}(x_i)\left(\frac{\partial^2 g_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j\partial\psi_k}\right) - \sum_{i=1}^{n} g_{\boldsymbol{\psi}}^{-2}(x_i)\left(\frac{\partial g_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right)\left(\frac{\partial g_{\boldsymbol{\psi}}(x_i)}{\partial\psi_k}\right)$$
$$-(2\alpha-1)\sum_{i=1}^{n} G_{\boldsymbol{\psi}}^{-1}(x_i)\left(\frac{\partial^2 G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j\partial\psi_k}\right) + (2\alpha-1)\sum_{i=1}^{n} G_{\boldsymbol{\psi}}^{-2}(x_i)\left(\frac{\partial G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right)\left(\frac{\partial G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_k}\right)$$
$$+3\alpha\sum_{i=1}^{n}\frac{H_{\alpha,\boldsymbol{\psi}}(x_i)}{G_{\boldsymbol{\psi}}(x_i)}\left(\frac{\partial^2 G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j\partial\psi_k}\right) - 3\alpha\sum_{i=1}^{n}\frac{H_{\alpha,\boldsymbol{\psi}}(x_i)}{G_{\boldsymbol{\psi}}^2(x_i)}\left(\frac{\partial G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right)\left(\frac{\partial G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_k}\right) \tag{59}$$
$$+3\alpha\sum_{i=1}^{n} G_{\boldsymbol{\psi}}^{-1}(x_i)\left(\frac{\partial G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right)\left(\frac{\partial H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial\psi_k}\right) - \theta\sum_{i=1}^{n} H_{\alpha,\boldsymbol{\psi}}(x_i)\left(\frac{\partial^2 H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial\psi_j\partial\psi_k}\right)$$
$$-\theta\sum_{i=1}^{n}\left(\frac{\partial H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right)\left(\frac{\partial H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial\psi_k}\right)$$

Where $H_{\alpha,\boldsymbol{\psi}}(x_i) = G_{\boldsymbol{\psi}}^{\alpha}(x_i)\big/\left[1-G_{\boldsymbol{\psi}}^{\alpha}(x_i)\right]$; $\left(\frac{\partial H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial\alpha}\right) = G_{\boldsymbol{\psi}}^{\alpha}(x_i)\left[1-G_{\boldsymbol{\psi}}^{\alpha}(x_i)\right]^{-2}\log G_{\boldsymbol{\psi}}(x_i)$;

$$\left(\frac{\partial H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right) = \alpha G_{\boldsymbol{\psi}}^{\alpha-1}(x_i)\left[1-G_{\boldsymbol{\psi}}^{\alpha}(x_i)\right]^{-2}\left(\frac{\partial G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right);\left(\frac{\partial^2 H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial\alpha^2}\right) = G_{\boldsymbol{\psi}}^{\alpha}(x_i)\left[1-G_{\boldsymbol{\psi}}^{\alpha}(x_i)\right]^{-3}\left[1+G_{\boldsymbol{\psi}}^{\alpha}(x_i)\right]\log^2 G_{\boldsymbol{\psi}}(x_i);$$

$$\left(\frac{\partial^2 H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial\alpha\partial\psi_j}\right) = G_{\boldsymbol{\psi}}^{\alpha-1}(x_i)\left[1-G_{\boldsymbol{\psi}}^{\alpha}(x_i)\right]^{-3}\left(1-G_{\boldsymbol{\psi}}^{\alpha}(x_i)+\alpha\log G_{\boldsymbol{\psi}}(x_i)+\alpha G_{\boldsymbol{\psi}}^{\alpha}(x_i)\log G_{\boldsymbol{\psi}}(x_i)\right)\left(\frac{\partial G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right);$$

$$\left(\frac{\partial^2 H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial\psi_j^2}\right) = \alpha G_{\boldsymbol{\psi}}^{\alpha-2}(x_i)\left[1-G_{\boldsymbol{\psi}}^{\alpha}(x_i)\right]^{-3}\left(G_{\boldsymbol{\psi}}(x_i)\left[1-G_{\boldsymbol{\psi}}^{\alpha}(x_i)\right]\left(\frac{\partial^2 G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j^2}\right)+\left(\alpha G_{\boldsymbol{\psi}}^{\alpha}(x_i)+G_{\boldsymbol{\psi}}^{\alpha}(x_i)+\alpha-1\right)\left(\frac{\partial G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right)^2\right);$$

$$\left(\frac{\partial^2 H_{\alpha,\boldsymbol{\psi}}(x_i)}{\partial\psi_j\partial\psi_k}\right) = \alpha G_{\boldsymbol{\psi}}^{\alpha-2}(x_i)\left[1-G_{\boldsymbol{\psi}}^{\alpha}(x_i)\right]^{-3}\left(G_{\boldsymbol{\psi}}(x_i)\left[1-G_{\boldsymbol{\psi}}^{\alpha}(x_i)\right]\left(\frac{\partial^2 G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j\partial\psi_k}\right)+\left(\alpha G_{\boldsymbol{\psi}}^{\alpha}(x_i)+G_{\boldsymbol{\psi}}^{\alpha}(x_i)+\alpha-1\right)\left(\frac{\partial G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_j}\right)\left(\frac{\partial G_{\boldsymbol{\psi}}(x_i)}{\partial\psi_k}\right)\right)$$

It is noteworthy to state that, when the sample size gets larger $(i.e.\ n\to\infty)$, the distribution of the vector of parameter estimates, $\hat{\boldsymbol{\omega}} = \left(\hat{\alpha},\hat{\theta},\hat{\boldsymbol{\psi}}\right)^T$, is approximated by a multivariate normal distribution with zero mean vector [of order $(r+2)\times 1$] and a variance-covariance matrix given by the inverse of the Fisher information matrix.

Based on the foregoing, one can derive approximate confidence intervals for the unknown parameters of the new distribution. For instance, one can construct $(1-\zeta)100\%$ approximate confidence intervals for the unknown parameters $\boldsymbol{\omega} = (\alpha,\theta,\boldsymbol{\psi})^T$ using the information matrix as follows:

$\hat{\alpha} \pm Z_{\left(1-\frac{\zeta}{2}\right)} \sqrt{var\left(\hat{\alpha}\right)}$; $\hat{\theta} \pm Z_{\left(1-\frac{\zeta}{2}\right)} \sqrt{var\left(\hat{\theta}\right)}$; $\hat{\psi}_j \pm Z_{\left(1-\frac{\zeta}{2}\right)} \sqrt{var\left(\hat{\psi}_j\right)}$ $j = 1,\cdots,r$

Where $Z_{\left(1-\frac{\zeta}{2}\right)}$ is the upper $\left(\frac{\zeta}{2}\right)^{th}$ percentile of the standard normal distribution.

4 Conclusion

In this article, a new probability distribution generator titled *Rayleigh*-Exponentiated Odd Generalized-*X* family of distributions was proposed. All important functions of the proposed family such as distribution function, probability density function, survival function, and hazard rate function were derived. It was further established that, the proposed family belongs to the Exponentiated–*G* class of distributions introduced in earlier research. Some important properties of the new family including moments, moment generating function, quantile function, entropy as well as functions of order statistics were obtained. The method of maximum likelihood was used to estimate the parameters of the new family and the functions for obtaining the Asymptotic Confidence Bounds of the parameters were provided. It was further demonstrated how the asymptotic confidence bounds can be used to obtain interval estimates of the parameters of the proposed family.

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