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A NEW WEIBULL- ODD FRECHET G FAMILY OF DISTRIBUTIONS

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Abstract

Generalization of distributions plays an important role inmodeling lifetime data of different shape for different purpose, especially in engineering, physical and life science and others.We propose a New Weibull-Odd Frèchet family of distributions and derive useful expansions and some structural properties of this family of distributions. Asub-model of this family called the New Weibull Odd Frechet Inverse Exponential distribution isproposed. The distribution is used to fit to two data sets using the Maximum Likelihood Estimation procedure. The robustness of the parameters of this distributionis tested through Monte Carlo Simulation,using bias and mean squared error. The results of fitting the new distribution to two different data sets suggest that the proposed distribution outperformed its competitors.

Keywords: New Weibull- G family, odd Frèchet G family, Renyi Entropy, Moments, New Weibull odd Frechet Inverse Exponential distribution, Monte Carlo simulation.

1.0 Introduction

Generalization of distributions play an important role in both theory and application. Everyday, several well-known distributions are generalized using different methods in order to provide greater flexibility. These generalizations provide more information in modeling data in practice and make the generalized distributions tractable for easier use. Also, generalization of distributions provides varieties of compound distributions capable of modeling lifetime dataset of different shapes that have not been adequately captured by the existence distributions.

This article introduced a New Weibull-odd Frechet G family of distributions. The new family combines the new Weibull - G family proposed by Tahir et al. [1] and odd Frechet –G family of distributions proposed by Ulhaq et al. [2]. The combination of these two families is expected to provide greater flexibility and enhanced performance than each of the existing families in modeling lifetime dataset. This method of combining two families was used by: Oluyede et al. [3] that proposed the Gamma Weibull-G family; Beta Weibull- Generalizedfamily by Makubate et al. [4]; Burr X - Exponential-G family proposed by Sanusi et al. [5]; Topp Leone Exponentiated- G family by Ibrahim et. al. [6] and New Generalized Weibull-odd Frechet family of distributions by Usman, et. al.[7].

Several other generators proposed using different methods are the Weibull-G family by Bourguignon [8]; new generalized Weibull by Cordeiro, et al. [9]; Burr X genetator by Yousof,et al. [10]; Weibull-X familyof distributions by Alzaatreh Indranil [11];the beta extended Weibull-G by Cordeiro, et al. [12]; the Exponentiated Weibull family by Mudholkar et al. [13]; the families of beta and generalized gamma-generated distributions by Zagrafos and Balakrishnan [14];the Exponentiated -G class of distributions by Cordeiro, et al.[15];beta-Generalized by Eugene, et al.[16]; the Lomax Generator by Cordeiro, et al.[17]; Kumaraswamy transmuted-Generalized by Afify, et al. [18]; Kumaraswamy Marshall-Olkin by Alizadeh et al. [19] and Exponentiated T-X family by Alzaghal, et al. [20].

Weibull distribution by Weibull [21] is the mostwidely used lifetime distribution and has been identified as a life testing model in reliability and engineering. In reliability analysis, the Weibull distribution can be used to find the percentage of items that are expected to fail during the burn-in period.

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If the random variable M is from a weibull distribution with parameters a and b, then itscumulative distribution function (cdf) and probability density function (pdf) are:

$$F(m;a,\ b) = 1 - exp\left[-am^{b}\right], \qquad m \geq 0;\ a > 0,\ b > 0 \tag{1}$$

$$f(m;a,b) = abm^{b-1}exp\left[-am^{b}\right] \tag{2}$$

Consider the pdf of the Weibull distribution given in equation (2). Then Tahir et al. [1] proposed the cdf of the new Weibull-G Family by replacing y by $-\log\left(G(w;\Lambda)\right)$ as follows

$$F(w;a,b,\Lambda) = 1 - a\,b\int_{0}^{-\log(G(w;\Lambda))} m^{b-1}\exp(-am^{b})dm = exp\left[-a\left(-\log\left[G(w;\Lambda)\right]\right)^{b}\right], a,b > 0 \tag{3}$$

and pdf

$$f(w;\ a,\ b,\Lambda) = \frac{a\,bg\,(w;\Lambda)}{G\,(w;\Lambda)}\left\{-\log\left(G\,(\,w;\Lambda)\right)\right\}^{b-1} exp\left[-a\left(-\log\left(G(w;\Lambda)\right)\right)^{b}\right] \tag{4}$$

Frèchet distribution was introduced by Maurice Frèchet [22]and its application can be found in Harlow [23].If the random variable W is from Frèchet distribution with one shape parameter d, then its cdf and pdf are given as:

$$F(w;\ d) = .exp\left[-\left(\frac{1}{w^{'}}\right)^{d}\right], \qquad w \geq 0,\ \ d > 0 \tag{5}$$

and

$$f(w;d) = \frac{d}{w^{d+1}}exp\left[-\left(\frac{1}{w}\right)^{d}\right] \tag{6}$$

Consider the Frèchet pdf given in equation (6) with positive parameter $d > 0$, then the cdf of the Odd Frèchet-G family by Ulhaq et al. [2] is defined by replacing the argument of t by $G(y;\Lambda)\,/\,\bar{G}(y;\Lambda)$ as follows:

$$F_{OFG}(m;d,\Lambda) = d\int_{0}^{\left(\frac{G(m;\Lambda)}{1-G(m;\Lambda)}\right)} \frac{1}{w^{d+1}}exp\left[-\left(\frac{1}{w}\right)^{d}\right]dw = \exp\left[-\left(\frac{1-G(m;\Lambda)}{G(m;\Lambda)}\right)^{d}\right]; \qquad d > 0 \tag{7}$$

with pdf

$$f_{OFG}(.m;d,\Lambda) = \frac{d\ g[m;\Lambda]}{[1-G(m;\Lambda)]^{d-1}\left[G(.m;\Lambda)\right]^{d+1}}\exp\left[-\left(\frac{1-G(m;\Lambda)}{.G(m;\Lambda)}\right)^{d}\right]; \qquad d > 0 \tag{8}$$

The rest of the article is structured as follows. Section two proposes the New Weibull-Odd Frechet family of distributions. While useful expansions of the family are presented in Section three, its structural properties are discussed in Section four. Parameter estimation is discussed in Section five, Monte Carlo Simulation study to test the robustness of the estimated parameters is conducted in Section six. Section seven fits the proposed New Weibull odd Frechet Inverse Exponential distribution and other comparator distributions to the two real datasets, while the last Section concludes the article.

2.0 The new weibull-odd frechet G (NWoFrG) famıly

We defined the proposed New Weibull- odd Frèchet G (NW-oFrG) family via the cdf and pdf given by:

$$F_{NW-OFG}(y;a,b,d,\Lambda) = 1 - a\,b\int_{0}^{H(y;d,\Lambda)} m^{b-1}\,\mathrm{e\,xp}(-a\,m^{b})d\ m = \exp\left\{-a[\left(-\log\left([F_{OFG}(y;d,\Lambda)]\right)\right)]^{b}\right\} \tag{9}$$

and

$$f_{NW-OFG}(y;a,b,d,\Lambda) = \frac{a\,b\ f_{OFG}(y;d,\Lambda)}{[F_{OFG}(y;d,\Lambda)]}\left(-\log\left[F_{OFG}(y;d,\Lambda)\right]\right)^{b-1}\exp\left\{-a[\left(-\log\left([F_{OFG}(y;d,\Lambda)]\right)\right)]^{b}\right\} \tag{10}$$

where

$H(y;d,\Lambda) = \left(-\log\left([F_{OFG}(y;d,\Lambda)]\right)\right)$

substitution of equation (7) into equations (9) and (10), yields the cdf and pdf of the NW-OFG family of distributions as:

$$F_{NW-OFG}(y;a,b,d,\Lambda) = \exp\left\{-a\left(\frac{1-G(y;\Lambda)}{G(y;\Lambda)}\right)^{bd}\right\} \quad (11)$$

and

$$f_{NW-OFG}(y;a,b,d,\Lambda) = a\,b\,d\,\frac{g[y;\Lambda][1-G(y;\Lambda)]^{b\,d-1}}{G[y;\Lambda]^{1+b\,d}}\exp\left\{-a\left(\frac{1-G(y;\Lambda)}{.G(y;\Lambda)}\right)^{b\,d}\right\} \quad (12)$$

where $g(y;\Lambda)$ *and* $G(y;\,\Lambda)$ are the cdf and pdf of the baseline distribution respectively.

2.1 Survival Function

The survival functionR(y) of NwoFrGis :

$$R(y) = 1-\exp\left\{-a\left(\frac{1-G(y;\Lambda)}{G(y;\Lambda)}\right)^{d\,b}\right\} \quad (13)$$

2.2 Hazard Function

The hazard function H(y) of NWoFrG is :

$$H(y) = \frac{a\,bdg[y;\Lambda][1-G(y;\Lambda)]^{bd-1}\exp\left\{-a\left(\frac{1-G(y;\Lambda)}{G(y;\Lambda)}\right)^{d\,b}\right\}}{[G(y;\Lambda)]^{1+bd}\left(1-\exp\left\{-a\left(\frac{1-G(y;\Lambda)}{G(y;\Lambda)}\right)^{bd}\right\}\right)} \quad (14)$$

2.3 New Weibull- Odd Frèchet– Inverse Exponential (NWoFr-IE) Distribution

If the we take the baseline distribution as the Inverse Exponential distribution with pdf $g(y;\mathrm{e})$ and cdf $G(y;\,\mathrm{e})$ with parameter e, given by:

$$g\,(y;\,e) = \frac{e}{y^{\cdot 2}}.exp\left[-\left(\frac{e}{y}\right)\right] \quad \text{for } y > 0$$

and

$$G\,(y;e) = .exp\left[-\left(\frac{e}{y}\right)\right]$$

then the New Weibull Odd Frèchet- Inverse Exponential (NwoF-IE) distribution, which isa sub-modelof the NwoFG family has the cdf and pdf given in equations (15) and (16), respectively.

$$F_{NWOF-IE}(y;a,b,d,e) = \exp\left\{-a\left(exp\left[\left(\frac{e}{y}\right)\right]-1\right)^{b\,d}\right\} \quad (15)$$

And

$$f_{NWOF-IE}(y;a,b,d,e) = a\,b\,c\,\frac{e}{y^2}\left(.exp\left[\left(\frac{e}{y}\right)\right]-1\right)^{b\,d}\left(1-.exp\left[-\left(\frac{e}{y}\right)\right]\right)^{-1}.\exp\left[-a\left(.exp\left[\left(\frac{e}{y}\right)\right]-1\right)^{b\,d}\right] \quad (16)$$

The survival R(y) and hazard h(y) functions of New Weibull odd Frèchet-Inverse Exponential distribution are provided in equations (17) and (18), respectively,

$$R(y) = 1-.\exp\left[-a\left(.exp\left(\frac{e}{y}\right)-1\right)^{d\,b}\right] \quad (17)$$

$$h(y)=\frac{a\,b\,c\,\frac{e}{y^{2}}\left(1-.exp\left[-\left(\frac{e}{y}\right)\right]\right)^{-1}\left(.exp\left[\left(\frac{e}{y}\right)\right]-1\right)^{d\,b}.\exp\left[-a\left(.exp\left[\left(\frac{e}{y}\right)\right]-1\right)^{d\,b}\right]}{\left(1-.\exp\left[-a\left(.exp\left[\left(\frac{e}{y}\right)\right]-1\right)^{d\,b}\right]\right)} \quad (18)$$

The Quantile function of the New Weibull odd Frèchet-Inverse Exponential distributionis:

$$Q(u)=e\left(\log\left(1+\left[(a)^{-1}(-\log u)\right]^{\frac{1}{b\,d}}\right)\right)^{-1} \quad (19)$$

where u is uniformly distributed between 0 and 1.

Figures 1and 2 present the plots of probability density function and hazard functionof New Weibull- odd Frèchet- Inverse Exponential Distribution. These plots depict varying shapes of the two functions depending on the choice of the four parameter values.

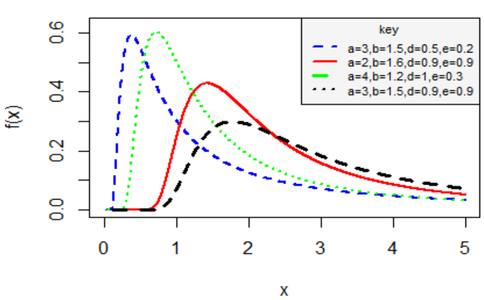

Figure 1.*Pdf of NWoFr-IE distribution with different paramter values*

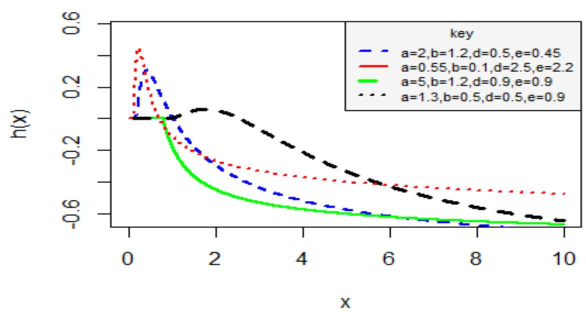

Figure 2. *The HF of NWoFr-IE distribution with different parameter values*

2.4 useful expansions of the proposed family.

This section provides a very useful expansion for the proposed New Weibull- odd Frèchet G Family. Given the cdf of the family:

$$F_{NW-OFG}(y;a,b,d,\Lambda)=\exp\left\{-a\left(\frac{1-G(y;\Lambda)}{G(y;\Lambda)}\right)^{d\,b}\right\}$$

And using power series expansion, we have

$$\exp\left\{-a\left(\frac{1-G(y;\Lambda)}{G(y;\Lambda)}\right)^{d\,b}\right\}=\sum_{i=0}^{\infty}\frac{a^{i}(-1)^{i}}{i!}\frac{(1-G(y;\Lambda))^{i\,b\,d}}{(G(y;\Lambda))^{i\,b\,d}} \quad (20)$$

Using Binomial expansion to the term $[1-G(y;\Lambda)]^{i\,b\,d}=\sum_{j=0}^{\infty}(-1)^{j}\binom{i\,b\,d}{j}G(y;\Lambda)^{j}$

And the cdf becomes,

$$F_{NW-OFG}(y;\,a,\,b,d,\Lambda)=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{a^{i}(-1)^{j+i}}{i!}\binom{i\,b\,d}{j}(G(y;\Lambda))^{j-i\,b\,d} \quad (21)$$

Equation (21) can be rewritten as

$$F_{NW-OFG}(y;a,\,b,d,\Lambda)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{j,i}(G(y;\Lambda))^{j-i\,b\,d} \quad (22)$$

where

$$\Omega_{j,i}=\frac{(-1)^{i+j}a^{i}}{i!}\binom{i\,b\,d}{j} \quad (23)$$

and for the pdf,

$$f_{NW-OFG}(y;a,\,b,\,d,\Lambda)=\frac{a\,b\,dg\,[y;\Lambda][\bar{G}(y;\Lambda)]^{b\,d-1}}{G(y;\Lambda)^{1+b\,d}}.\exp\left\{-a\left(\frac{1-G(y;\Lambda)}{G(y;\Lambda)}\right)^{d\,b}\right\} \tag{24}$$

Using the power series expansion, we have

$$\exp\left\{-a\left(\frac{1-G(y;\Lambda)}{G(y;\Lambda)}\right)^{b\,d}\right\}=\sum_{i=0}^{\infty}\frac{(-1)^i a^i}{i!}\left(\frac{1-G(y;\Lambda)}{G(y;\Lambda)}\right)^{i\,d\,b}$$

Equation (24) can be rewritten as

$$f_{NW-OFG}(y;a,b,d,\Lambda)=a\,b\,d\sum_{i=0}^{\infty}\frac{a^i(-1)^i}{i!}\frac{g(y;\Lambda)[1-G(y;\Lambda)]^{b\,d\,(i+1)-1}}{\left[G(y;\Lambda)\right]^{b\,d\,(i+1)+1}} \tag{25}$$

Using Binomial expansion to the Last term of the numerator in equation (25), the equation becomes

$$f_{NW-OFG}(y;a,\,b,\,d,\Lambda)=a\,b\,d\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{a^i(-1)^{j+i}}{i!}\binom{b\,di+b\,d-1}{j}g(y;\Lambda)G(y;\Lambda)^{j-b\,d\,(i+1)+1} \tag{26}$$

Equation (26) can be express as

$$f_{NW-OFG}(y;a,b,d,\Lambda)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Phi_{i,j}h_{j-b\,d\,(1+i)+2}(y;\Lambda) \tag{27}$$

where

$$h_{j-b\,d\,(1+i)+2}(x)=(j-d\,b\,(i+1)+2)\,g(y;\Lambda)G(y;\Lambda)^{j-b\,d\,(1+i).+1}$$ and

$$\Phi_{i,j}=a\,b\,d\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{a^i(-1)^{j+i}}{i!(j-b\,d\,(i+1)+2)}\binom{b\,d\,(1+i).-1}{j} \tag{28}$$

4.0 structural propertıes of the proposed famıly

4.1. Moments of the new family

The r^{th} moments of the proposed familyusing equation (27) is given by:

$$E(y^r)=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\Phi_{i,j}E(Z_{j,i}^{\,r}) \tag{29}$$

where $Z_{j,i}$ is the (exponential-G $(j-b\,d\,(i+1)+1)$). The incomplete moments of Y isgiven as

$$I_Y(x)=\int_{-\infty}^{y}y^r f(y;a,b,d,\Lambda)dy=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Phi_{i,j}I_{j,i}(x)$$

where

$$I_{j,i}(x)=\int_{-\infty}^{x}y^r h_{j-b\,d\,(i+1)+2}(y;\Lambda)dy$$

4.2. Moment Generating Function (MGF)

The MGF of the proposed family is given as;

$$M_x(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Phi_{i,j}E(\mathrm{e}^{tz_{i,j}}) \tag{30}$$

where

$$E(\mathrm{e}^{tz_{i,j}})=\int_{0}^{\infty}\mathrm{e}^{tz_{i,j}}\,h_{(j-b\,d\,(i+1)+2)}(z;\Lambda)dz$$

4.3. Quantile function

The quantile function also called inverse cumulative distribution function is associated with probability distribution used for simulation study.

$$Q(u) = G(y;\Lambda)^{-1}\left(1+\left[\frac{1}{a}\left(-\log(u)\right)\right]^{\frac{1}{bd}}\right)^{-1} \tag{31}$$

where the inverse of the cdf is the quantile function of the baseline distribution.

4.4. Entropy

The Renyi entropy is defined as:

$$I_c(y) = \frac{1}{1-c}\log\int_{-\infty}^{\infty} f(y)^c\,dy, c>0 \text{ and } \theta \neq 1$$

Using the pdf of our proposed class of distributions given in equation (27),

$$f(x)^c = \left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Phi_{i,j}h_{j-bd(i+1)+2}(y;\Lambda)\right)^c$$

the Renyi entropy of the NW-OFG is:

$$I_c(y) = \frac{1}{1-c}\log\left\{\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Phi_{i,j}\right)^c\int_{-\infty}^{\infty}\left(h_{j-bd(i+1)+2}(y;\Lambda)\right)^c dy\right\}; c>0 \text{ and } c \neq 1 \tag{32}$$

5.0 parameter estımatıon

In the spirit of other researchers, theparameters of the proposed family of distributions are estimatedusing the ML method. Let $\vartheta = (a, b, d, \Lambda)^T$, then from equation (12) the log likelihood function of the family is obtained as:

$$l(\vartheta) = n\log(a+b+d) + \sum_{\tau=1}^{n}\log g(y_\tau;\Lambda) - (bd+1)\sum_{\tau=1}^{n}\log G(y_\tau;\Lambda) +$$

$$(bd-1)\sum_{\tau=1}^{n}\log \bar{G}(y_\tau;\Lambda) - a\sum_{\tau=1}^{n}\left(H_\tau(y_\tau;\Lambda)\right)^{bd}$$

Where

$$H_\tau(y_\tau;\Lambda) = \frac{\bar{G}(y_\tau;\Lambda)}{G(y_\tau;\Lambda)}.$$

We can express all the partial derivatives for the parameters as:

$$Q(\vartheta) = \frac{\partial l}{\partial \vartheta} = \left(\frac{\partial l}{\partial a},\frac{\partial l}{\partial b},\frac{\partial l}{\partial d},\frac{\partial l}{\partial \Lambda}\right)^T = \left(Q_a, Q_b, Q_d, Q_{\Lambda_r}\right)^T,$$

where

$$Q_a = \frac{n}{a} - \sum_{\tau=1}^{n}\left(H_\tau(y_\tau;\Lambda)\right)^{bd}$$

$$Q_b = \frac{n}{b} + d\sum_{\tau=1}^{n}\log(1-G(y_\tau;\Lambda)) - d\sum_{\tau=1}^{n}\log G(y_\tau;\Lambda) - ab\sum_{\tau=1}^{n}\left(H_\tau(y_\tau;\Lambda)\right)^{bd-1}$$

$$Q_d = \frac{n}{d} + bd\sum_{\tau=1}^{n}\log(1-G(y_\tau;\Lambda)) - b\sum_{\tau=1}^{n}\log G(y_\tau;\Lambda) - ad\sum_{\tau=1}^{n}\left(H_\tau(y_\tau;\Lambda)\right)^{bd-1}$$

$$Q_{\Lambda_r} = \sum_{\tau=1}^{n}\frac{g(y_\tau;\Lambda)'}{g(y_\tau;\Lambda)} - (bd+1)\sum_{\tau=1}^{n}\frac{G(y_\tau;\Lambda)'}{G(y_\tau;\Lambda)}$$

$$-(bd-1)\sum_{\tau=1}^{n}\frac{g(y_\tau;\Lambda)}{(1-G(y_\tau;\Lambda))} + abd\sum_{\tau=1}^{n}\frac{g(y_\tau;\Lambda)}{(G(y_\tau;\Lambda))^2}\left(H_\tau(y_\tau;\Lambda)\right)^{bd-1}$$

Then setting these equations to zero, we can solve the system of equations numerically using iterative methods.

6.0 Sımulatıon Study

In order to assess the robustness and consistency of the parameter estimates of the NWOF-IE distribution, the simulation study was conducted to estimate the mean, bias, variance and mean squared error of the estimated parameters from the maximum likelihood estimates. The simulated data is generated using the quantile function of the distribution defined in equation (19) with true parameter values: a=5, b=1.1, d=1.4, e=1.1, for selected sample sizes n=25, 50, 100, and 225. Each sample is replicated 1000 times.

Table 1.*Results of Monte Carlo Simulation with true parameter values (a=5,b=1.1,d=1.4, e=1.1)*

Sample sizes (n)	Parameters	Estimates	Bias	Variance	MSE
	A	5.0739080	0.0739085	0.0001011	0.0055636
	b	2.2371010	1.1371010	0.4006936	1.6936930
25	d	2.9799290	1.5799290	0.4150030	2.9111780
	e	0.0521640	-1.0478360	0.0010085	1.0989690
	A	5.0678090	0.0678095	0.0000262	0.0046243
	b	2.1450810	1.0450810	0.0674728	1.1596660
50	d	2.8498780	1.4498780	0.0738948	2.1760420
	e	0.0616802	-1.0383200	0.0011222	1.0792300
	A	5.0584870	0.0584869	0.0000127	0.0034334
	b	2.0252640	0.9252643	0.0070287	0.8631426
100	d	2.6700250	1.2700250	0.0087326	1.6216960
	e	0.1095937	-0.9904063	0.0011081	0.9827104
	A	5.0499720	0.0499717	0.0000113	0.0025083
	b	1.9317390	0.8317391	0.0043911	0.6961810
225	d	2.5204630	1.1204630	0.0067763	1.2622140
	e	0.1465148	-0.9534852	0.0005775	0.9149090

Table 1 shows the results obtained from the Monte Carlo Simulation study. These results indicate that the bias, variance and mean square error of all the estimated parameters decreases as sample size increases. The simulation results suggest the robustness and consistency of the estimates.

7. applıcatıons

Here, we demonstrate the performance of the proposed New Weibull- odd Frèchet – Inverse Exponential Distribution using two real datasets. The maximum likelihood estimates, as well as goodness-of-fit measures, are computed and compared with these comparator models:

(a)Weibull -Inverse Exponential distribution (WIED) defined by Chandrakant et al. [24] with cdf

$$F_{WIE}(w;a,b,e)=1-\exp\left[-a\left(exp\left(\frac{e}{w}\right)-1\right)^{-b}\right],\quad where \quad a>0,\ b>0\ \ and\ \ e>0.$$

and the pdf

$$f_{WIE}(w;a,b,e)=\frac{a\,b\,e}{w^2}\left(exp\left(\frac{e}{w}\right)-1\right)^{-b}\left(1-exp\left[-\left(\frac{e}{w}\right)\right]\right)^{-1}\exp\left[-a\left(exp\left(\frac{e}{w}\right)-1\right)^{-b}\right]$$

(b) Odd Frechet-Inverse Exponential distribution (OFIED) defined by Sharifah et al. [25] with cdf

$$F_{OFIE}(w,d,e)=\exp\left[-\left(exp\left(\frac{e}{w}\right)-1\right)^{d}\right];\qquad ,\quad where\quad w>0,\ d>0\ \ and\ \ e>0,$$

and pdf

$$f_{OFIE}(w;d,e)=\frac{d\,e}{w^2}\left(1-exp\left[-\left(\frac{e}{w}\right)\right]\right)^{-1}\left(exp\left(\frac{e}{w}\right)-1\right)^{d}\exp\left[-\left(exp\left(\frac{e}{w}\right)-1\right)^{d}\right]$$

(c) Frechet distribution (FrD) defined by Maurice Frechet [22] with cdf

$$F(w;d,e)=exp\left[-\left(\frac{e}{w}\right)^{d}\right],\qquad where\ w>0,\ d>0\ \ and\ \ e>0,$$

and pdf

$$f(w;d,e) = d\, e^{d}\ w^{-(d+1)} exp\left[-\left(\frac{e}{w} \right)^{d} \right]$$

7.1 First Data Set

The given data represents intervals of times of an event and is sourced from Jørgensen [26]. The observations are as follows:

2.50, 2.60, 2.60, 2.70, 2.80, 2.80, 2.90, 3.00, 3.00, 3.10, 3.20, 3.40, 3.70, 3.90, 3.90, 3.90, 4.60, 4.70, 5.00, 5.60, 5.70, 6.00, 6.00, 6.10, 6.60, 6.90, 6.90, 7.30, 7.60, 7.90, 8.00, 8.30, 8.80, 8.80, 9.30, 9.40, 9.50, 10.1, 11.0, 11.3, 11.9, 11.9, 12.3, 12.9, 12.9, 13.0, 13.8, 14.5, 14.9, 15.3, 15.4, 15.9, 16.2, 17.6, 20.1, 20.3, 20.6, 21.4, 22.8, 23.7, 23.7, 24.7, 29.7, 30.6, 31.0, 34.1, 34.7, 36.8, 40.1, 40.2, 41.3, 42.0, 44.8, 49.8, 51.7, 55.7, 56.5, 58.1, 70.5, 72.6, 87.1, 88.6, 91.7, 119.8

Table 2*.ResultsofParameters Estimates and Goodness of fit measures for the First Data Set*

Models	*Parameters Estimates*				*Goodness of Fit Measures*		
	$\hat{a}$	$\hat{b}$	$\hat{d}$	$\hat{e}$	*AIC*	*BIC*	*HQIC*
NWOF-IED	1.907465	0.952376	0.845190	1.981255	695.331	705.054	699.234
W-IED	0.083776	0.934989	-	1.434425	685.151	692.443	688.082
OFr-IED	-	-	0.667550	1.992155	743.218	748.079	745.172
FrD	-	-	1.944688	0.679438	770.295	775.156	772.249

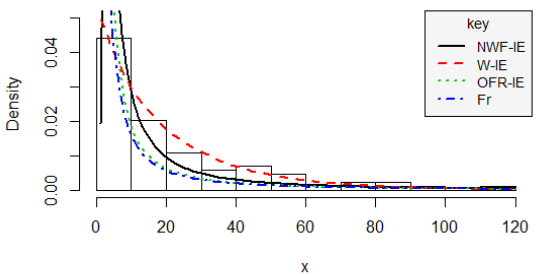

Figure 3. *Estimated Density for the Data Set 1.*

From Table 2 and Figure 3 it is clear that except for the Weibull - Inverse Exponential, the new Weibull- odd Frèchet-Inverse Exponential distribution (NWoF-IED) outperformed its competitors.

7.2 Second Data Set

The given data set represents the strength of carbon fibres at gauge lengths of 10 mm and is sourced fromBi and Gui [27]. The observations are:

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020

Table 3*.Results of Parameters Estimates and Goodness of fit measures for the second Data Set*

Models	*Parameters Estimates*				*Goodness of Fit*		
	$\hat{a}$	$\hat{b}$	$\hat{d}$	$\hat{e}$	*AIC*	*BIC*	*HQIC*
NWoFr-IED	1.477774	1.868562	1.830836	1.773950	130.761	139.334	134.133
W-IED	0.693926	1.967006	-	1.884405	164.275	170.704	166.804
OFr-IED	-	-	1.920383	1.855323	165.1739	169.4602	166.860
FrD	-	-	1.848149	1.772918	235.1510	239.4373	236.837

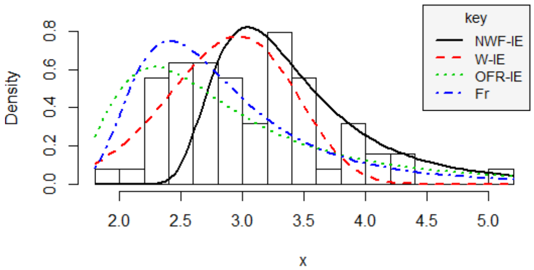

Figure 4. *Estimated Density for the Data Set 2.*

From Table 3 and Figure 4 it is clear that the New Weibull- odd Frèchet- Inverse Exponential distribution (NWOF-IED) performs better than its competitors.

8. conclusıon

Generalizing well-known distributions is very important for modeling lifetime data, especially data from Engineering, Sciences and social Sciences. This paper proposed a New Weibull-odd Frechet G (NWoFG) family capable of generalizing any continuous distribution. We derived some of its properties using the simplest form of cdf and pdf. The model parameters were estimated and Monte Carlo simulation is used to test the robustness of the estimated parameters. The New Weibull Odd Frechet-Inverse Exponential distribution of the family was fitted to two different datasets to illustrate its performance. The results of fits to the two different data sets suggested that the proposed distribution outperformed its competitors.

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