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A SHORT NOTE ON THE TWO-PERSON ZERO-SUM GAME

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Abstract

This paper presents the strategic form of a two-person zero-sum game and showcases solution techniques for both a pure strategy game and a mixed strategy game. Illustrative examples are provided to aid understanding of the solution techniques.

Keywords**:** dominant strategy, game, strategic game, two-person game, zero-sum game.

1. Introduction

The term 'game' refers to a situation of conflict and competition in which two or more competitors (or participants) are involved in the decision making process in anticipation of certain outcomes [1]. For example, in entry and exit decisions, where the manager of a firm is considering the possibility of entering a new market, which has only one other firm operating, the manager's decision will be based on certain quantitative measures (e.g., the profitability of the market) and this depends on how the incumbent firm will react to the entry. The incumbent firm could be accommodating and let the entrant take his share of the market or could respond aggressively with a price war. Such a price war typifies a conflict situation and creates competition between the incumbent firm and the entry firm. Other examples of a game are: advertisement campaigns for competing products, war, etc. The competitors in a game are referred to as players. A player may be an individual, a group of individuals or an organization. A series of analytical tools have been designed to explain interactive decision making phenomena under conditions of conflict and competition (for example, [1, 2]). These analytical tools are studied in what is referred to as game theory. Game theory entails the construction of mathematical models for game theoretic problems on the assumptions that the decision makers pursue well-defined exogenous objectives (they are rational) and that they take into account the knowledge or expectations of other decision makers (they reason strategically) cf. [2].

A game where the players do not have any information about their competitors' choices while they make their own is called a simultaneous move game or a game in strategic (or normal) form. If the game evolves over several time periods, then it is called a dynamic game; whereas if it takes place in one single period, it is termed a static game. In this paper, we concentrate on static games (or games in strategic form). The interested readers on dynamic game may refer to [3].

Consider a game that describes the strategic interaction between the players, where the outcome for each player depends upon the collective actions of all players involved. To describe this strategic interaction, the following are important [2]:

i. The number of players. Let p denote the number of players. If $p = 2$, the game is a two-person game; otherwise, it is referred to as a $p -$ person game ($p > 2$).
ii. The rules of the game that specify the sequence of moves as well as the possible actions and information available to each player whenever they move.
iii. The outcome of the game for each possible set of actions.

The (expected) payoffs based on the outcome, which is a quantitative measure of satisfaction that a player gets at

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iv. the end of the play in terms of gains or losses when players select their particular strategies.

A strategy is a complete action plan for all possible ways a game can proceed. The outcome resulting from a particular pair of strategies is assumed known to the players in advance and is expressed in terms of numerical values (e.g., money, percentage market share or utility). The expected outcome per play, when players follow their optimal strategy, is called the value of the game. The vector of all actions of players is called a strategy profile. Two types of strategies are used by players, viz. pure strategy and mixed strategy. A pure strategy is a decision rule that is always used by the player to select only one particular strategy in a well-known strategy profile, regardless of the other player's strategy. In a mixed strategy situation, a particular strategy is selected with a certain fixed probability. More specifically, a mixed strategy is a probability distribution over a set of pure strategies. A strategy s_i is called a dominant strategy for player i, if no matter what the other players choose, playing s_i maximises the player i's payoff. A strategy profile (i.e., a vector of actions by the different players) such that no player wants to alter his own action unilaterally, given that all other players play according to this strategy profile, is termed Nash equilibrium (see [2]).

This paper focuses on a game with only two players wherein one player's gain is equal to the loss of the other player in a way that the total sum of gains and losses is zero. This kind of game is well-known in the literature as the two-person zero-sum game (cf. [1, 4]).

2. Materials and Method

The strategic form (or normal form) of a two-person zero-sum game is given by a triplet (X, Y, A), where $X \neq \phi$ is the set of strategies of player I, $Y \neq \phi$ is the set of strategies of player II and A is a real-valued function defined on $X \times Y$. To be more precise, A is the payoff (or game) matrix, which is written for a specific player, usually player I. This is because in a two-person zero-sum game, the payoff function of player II is the negative of the payoff of player I. In the matrix A, player I chooses a row, player II chooses a column and player II pays player I the entry in the chosen row and column.

An example of a two-person zero-sum game is penalty kick in football [4, 5]. A penalty kick is a simultaneous-move strategic game that involves two players (a kicker and a goalkeeper) and the actions of the players are governed by a precisely defined set of rules. The outcome, which is a goal or no goal, is decided immediately after the kick. Thus, penalty kicks may be said to be a one-shot two-person zero-sum game between the kicker and the goalkeeper since the ball is kicked once and no second chance is given to the kicker when no goal is scored [4, 6]. The action of the kicker results in four possible directions. The ball can be shot wide or hit the goalpost or crossbar (denoted by the symbol O), at the middle (M), at the right hand side (R) or at the left hand side of the goalpost (L).The goalkeeper may jump towards the right or left, R or L, or maintain his position, M. From the foregoing, $G = (X, Y, A)$ is a game modelling penalty kicks, where $X = \{O, M, R, L\}$, $Y = X/\{O\}$ and $A = (a_{ij})$ is a real-valued function defined on $X \times Y$ with

$$a_{ij} = \begin{cases} -1 & \text{for } i = O \\ (1 - 2p_j) & \text{for } i = j \neq O,\ i \in X,\ j \in Y \\ 1 & \text{otherwise} \end{cases}$$

and p_j is the probability that the goalkeeper successfully defends the ball in direction j [4].

Several solution techniques abound in the literature for the two-person zero-sum game $G = (X, Y, A)$. We discuss a few of them. These include: the principle of dominated strategy, the minimax-maximin principle, the algebraic method and the linear programming method. The first two methods are more appropriate for a pure strategy game, while the last two are suitable for a mixed strategy game. Solutions to pure strategy games are straightforward.

Let $A = (a_{ij})$. If $a_{ij} \geq a_{kj}$ for all j, then the $i-$th row of A is said to dominate the $k-$th row. The $i-$th row of A strictly dominates the $k-$th row if $a_{ij} > a_{kj}$ for all j. Similarly, the $j-$th column of A dominates (strictly dominates) the $k-$th column if $a_{ij} \leq a_{ik}$ ($a_{ij} < a_{ik}$) for all i. The common practice is to remove the dominated row or column whenever the relations stated above are observed. The removal of a dominated row or column does not alter the value of the game. For instance, in the 3×3 game

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 4 & 1 & 2 \end{pmatrix},$$

the last column is dominated by the middle column. Thus, the last column may be discarded so that the problem becomes that of a 3×2 game.

Pure strategy games can be solved using the minimax-maximin principle, which states that the best of the worst payoffs is selected for each player. If an entry $\tilde{a}_{ij}$ of the matrix $A = (a_{ij})$ has the property that: $\tilde{a}_{ij}$ is the minimum of the $i-$th row and the maximum of the $j-$th column, then $\tilde{a}_{ij}$ is a saddle point. This $\tilde{a}_{ij}$ is also the value of the game. Thus, player I can win at least $\tilde{a}_{ij}$ by choosing row i and player II can keep her loss to at most $\tilde{a}_{ij}$ by choosing column j. For example, it can be deduced using the minimax-maximin principle that the payoff matrix

$$A = \begin{pmatrix} 8 & -2 & 9 & -3 \\ 6 & 5 & 6 & 8 \\ -2 & 4 & -9 & 5 \end{pmatrix}$$

is a game of pure strategy that has a saddle point of $\tilde{a}_{22} = 5$. However, the game is not fair as it is favourable to player I. A game is said to be fair if the value of the game is zero.

The minimax-maximin principle also provides information on the interval for the value of the game in a mixed strategy situation. The game matrix

$$A = \begin{pmatrix} 1 & 9 & 6 & 0 \\ 2 & 3 & 8 & 4 \\ -5 & -2 & 10 & -3 \\ 7 & 4 & -2 & -5 \end{pmatrix}$$

does not have a pure strategy solution. Thus, both players are most likely to use a random mixes of their respective strategies. Herein, the optimal value of the game will occur somewhere between the maximin and the minimax values of the game, that is, maximin (lower) value <value of the game<minimax (upper) value.

Consider the payoff matrix with $\{M, R, L\}$ as the strategy space given as [5]:

$$A = \begin{pmatrix} p_L & \pi_L & \pi_L \\ \mu & 0 & \mu \\ \pi_R & \pi_R & p_R \end{pmatrix}.$$

Suppose that $1 > \pi_i > p_i$, $\pi_i > p_j$ for $i \in \{R, L\}$, $i \neq j$ and $\pi_i > \mu$. Then, by the minimax-maximin principle, the value of the game, v, should lie in the interval

$$\max\ (p_L, p_R) < v < \min\ (\pi_R, \pi_L).$$

Let $G = (X, Y, A)$ be a game of mixed strategies with probabilities $x_1, x_2, \cdots, x_m$ for player I and $y_1, y_2, \cdots, y_n$ for player II. Then the maximin problem for player I is a non-linear problem of the form [7]

$$\max_{x_i} \left\{ \min \left(\sum_{i=1}^{m} a_{i1} x_i, \sum_{i=1}^{m} a_{i2} x_i, \cdots, \sum_{i=1}^{m} a_{in} x_i \right) \right\}$$

subject to

$$x_1 + x_2 + \cdots + x_m = 1,\ x_i \geq 0,\ i = 1, 2, \cdots, m.$$

If either $m = 2$ or $n = 2$, then the game can be solved using the graphical method [1, 7]. Suppose that $m = 2$. Then the mixed strategies of player I have probabilities x_1 and $x_2 = 1 - x_1$, $0 \leq x_1 \leq 1$. In this case, the expected payoff of player I corresponding to player II's $j-$th pure strategy is given as

$$E_j = a_{1j} x_1 + a_{2j} x_2 = (a_{1j} - a_{2j}) x_1 + a_{2j},\ j = 1, 2, \cdots, n.$$

Thus, the maximin problem for player I is

$$\max_{x_1} \min_{j} \{(a_{1j} - a_{2j})x_1 + a_{2j}\}.$$

The best of the worst situation for player I is obtained by finding the point of intersection of the lower envelope of the straight lines defining player II's pure strategies.

Suppose that $m > 2$. We let

$$v = \min \left(\sum_{i=1}^{m} a_{i1} x_i , \sum_{i=1}^{m} a_{i2} x_i , \cdots , \sum_{i=1}^{m} a_{in} x_i \right).$$

This equation implies that $\sum_{i=1}^{m} a_{ij} x_i \geq v$, $j = 1, 2, \cdots, n$. After some algebra, the maximin problem of player I is obtained as a linear programming problem (LPP) of the form

$\max \ z = v$

subject to

$$v - \sum_{i=1}^{m} a_{ij} x_i \leq 0,$$

$$x_1 + x_2 + \cdots + x_m = 1,$$

$x_i \geq 0$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$ and v is unrestricted in sign.

This maximin problem in LPP form may be solved using the (two-phase) simplex method. The dual of the above LPP is the minimax problem for player II.

3. Illustrative Examples

In this section, examples are provided from [7] to illustrate how to implement the mixed strategy solution techniques.

Example 1 The 2×4 game

$$A = \begin{pmatrix} 2 & 2 & 3 & -1 \\ 4 & 3 & 2 & 6 \end{pmatrix}$$

is a mixed strategy game since maximin $\neq$ minimax. Let the mixed strategies of player I have probabilities x_1 and $x_2 = 1 - x_1$, $0 \leq x_1 \leq 1$. His expected payoffs corresponding to player II's $j-$th ($j = 1, 2, 3, 4$) pure strategy are: $E_1 = -2x_1 + 4$, $E_2 = -x_1 + 3$, $E_3 = x_1 + 2$, $E_4 = -7x_1 + 6$. The values for x_1 at the point of intersection of the straight lines defining player II's pure strategies are obtained by equating and solving any two of the E_j's ($j = 1, 2, 3, 4$) for x_1. The results are presented in Table 1 as well as the expected values for player I.

Table 1. Values for x_1 at the point of intersection and the expected values for player I

x_1	1	2/3	2/5	1/2
E_1	2	8/3	16/5	3
E_2	2	7/3	13/5	5/2
E_3	3	8/3	12/5	5/2
E_4	-1	4/3	16/5	5/2
Minimum values	-1	4/3	12/5	5/2

As the problem of player I is a maximin problem, then the value of x_1 is the one corresponding to $v = \max_{x_1} \{-1, 4/3, 12/5, 5/2\} = 5/2$. Thus, $x_1 = 1/2$. Since $v > 0$, it follows that the game favours player I.

Example 2 Consider the game matrix

$$A = \begin{pmatrix} 3 & -1 & -3 \\ -2 & 4 & -1 \\ -5 & -6 & 2 \end{pmatrix}.$$

This game is a mixed strategy game and the value, v , lies in the interval $[-2, 2]$. Taking x_1, x_2, x_3 as the probabilities of mixed strategy for player I, the maximin problem at hand in the LPP form is

$\max \ z = v$

subject to

$v - 3x_1 + 2x_2 + 5x_3 \leq 0$,

$v + x_1 - 4x_2 + 6x_3 \leq 0$,

$v + 3x_1 + x_2 - 2x_3 \leq 0$,

$x_1 + x_2 + x_3 = 1$,

$x_i \geq 0$, $i = 1, 2, 3$, and v is unrestricted in sign.

To deal with the unrestricted sign for v , we set

$$v = v^+ - v^- = \begin{cases} > 0 & \text{for} \quad v^+ > v^- \\ = 0 & \text{for} \quad v^+ = v^- \\ < 0 & \text{for} \quad v^+ < v^- \end{cases}.$$

We solve the LP problem using the two-phase simplex method. This method is computationally tedious. The two-phase simplex algorithm which has been programmed in MATLAB (see [8]) was debugged and used to solve the LPP. The LLP is executed in the MATLAB environment using the code

```
clc
type='max';
c=[1 -1 0 0 0];
A=[1 -1 -3 2 5; 1 -1 1 -4 6; 1 -1 3 1 -2; 0 0 1 1 1];
rel= '<<<=';
b=[0; 0; 0; 1];
simplex2p(type,c,A,rel,b)
```

After seven iterations, the following results are obtained: $v = -0.9083$, $x_1 = 0.3945$, $x_2 = 0.3119$, $x_3 = 0.2936$. Since $v < 0$, this game does not really favour player I. However, player I can minimise the gains of player II by mixing his strategies with the probabilities $x_1 = 0.3945$, $x_2 = 0.3119$, $x_3 = 0.2936$. As $x_i > 0$ for each $i = 1, 2, 3$, none of the strategies is dominated by another.

4. Conclusion

This study had provided analytical tools to aid the evaluation of competition between two players (agents or firms) who interact while making their decisions. In such interaction, the payoffs of one player depend on the profile of strategies chosen by the other player in such fashion that one player's gain is equal to the loss of the other player and the total sum of gains and losses is zero. This study employed the notion of strategic game to analyse this situation. The readers had been familiarised with a pure strategy game and a mixed strategy game. By formulating the non-linear maximin problem as a linear programming problem, the mixed strategy game was solved by the two-phase simplex method. The results indicate that the game may favour either player. We believe that game theory could aid managers in their decisions and improve the managers' understanding of the dynamics in business interactions. It could also assist government in warfare as in the fight against insurgency. This warlike situation would require identifying the strategy profile of the insurgents vis-à-vis that of the government's and then quantifying the payoffs. Such a problem may not necessarily be a game in static form. It is most likely to be a dynamic game. Further investigation on dynamic games is worthwhile as the present study did not consider such games.

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