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OPTIMAL DEBT RATIO AND CONSUMPTION PLAN FOR AN INVESTOR UNDER INFLATION RISK AND CAPITAL GAINS TAX.

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Abstract

In this paper, we derive the optimal debt ratio and optimal consumption strategy for an investor whose asset price follows a diffusion process and capital gains are taxed. We put into consideration a market that is exposed to three background risks which include inflation, investment and income growth rate risks. Tax is levied on the payoff of the wealth process. Using dynamic programming principle and the CRRA utility function, we derive the optimal debt ratio and optimal consumption rate. We find the following: (i) optimal debt ratio is higher with taxation (ii) that optimal debt ratio is negatively impacted by increased volatility of both inflation and asset price risk (iii) that terminal consumption is positively related to the tax rate (iv) that a risk averse investor will have a decreasing optimal debt ratio.

JEL Classification: G11, G12, C02, C22, C61

Keywords: optimal debt ratio; corporation tax; inflation risk, jump risk

1. Introduction

Optimal debt management has become a vital part of most organization because of the role played by debt financing in increasing the financial leverage of firms. Therefore, having a good understanding of the factors that impacts debt is necessary for a firm that wants to survive in the current complex financial world. The debt ratio gives an insight into the financial health of a firm and represent an important factor that may influence the investment and financing decisions of firms. Also, the performance of a firm's investment is affected by debt exposure especially in periods of a financial crisis and how well the firm is able to manage this exposure is critical for its survival. A high debt ratio indicates that the firm has high leverage and might be highly exposed with increased volatility in asset prices. The higher the debt ratio, the greater the financial risk exposure of the firm's asset to increased movement in asset prices, government policies, inflation index, natural disaster etc. The business activities of firms in Nigeria are exposed to several risks which include: economic risk, political risk, policy risk, inflation risk, regulatory risk, operational risk, currency risk, environmental risk, corruption, COVID-19 pandemic etc. In this paper, we consider inflation, investment and income growth risks and address the problem of optimal debt ratio and consumption strategy of the investor.

Most firms now consider inflation risk when making investment decisions. This is because increasing prices have made it imperative for investors to know their real rather than nominal wealth, so they can make the right investment decision. The inflation index in [1] was given as a stochastic process and assumed to follow the Ornstein–Uhlenbeck process. In [2] and [3] a one factor stochastic inflation dynamics was considered. A three-factor inflation index dynamics with jumps was considered in [4]. However, in [5], a four-factor stochastic inflation index for an investor was used in discounting nominal wealth. Apart from inflation, the amount of tax paid can also impact on the debt ratio. Hence, understanding the dynamic impact of taxes on investment and debt ratio is critical for an investor decision making. In [6], an optimal investment problem in the presence of jump risk and capital gains tax was considered and [7] investigated optimal intertemporal asset allocation and location decisions for investors making taxable and tax-deferred investments. Further, [8] considered asset allocation and optimal tax-timing when tax rebates on capital losses are limited and [9] proposed a continuous-time model to investigate the impact of asymmetric tax structure and limited tax return on the behaviour of investors.

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The literature is quite rich with works on optimal debt and consumption plan. The optimal debt ratio and consumption plan for an investor during financial crisis in a stochastic setting was studied in [10]. The impact of labor market condition was also studied. He assumed that the production rate function of the investor is stochastic and being influenced by the government policy, employment and unanticipated risks. A stochastic optimal control model and optimal debt ratio management strategies of an investor in a financial crisis was also considered in [11]. They considered productivity of capital, asset return, interest rate and market regime switches of an investor. The utility of terminal wealth was optimized under debt ratio.

In [5], backup security as a buffer for loans in a continuous setting. He derived the optimal portfolio, debt ratio and consumption rate for an investor during a financial crisis and [12] applied the mathematical model of economic system to the development of a financial crisis. The surplus process of an insurer which satisfies a diffusion process was considered in [13] and obtained an optimal policy for debt and dividend payment for an insurer. Further, while [14] considered the optimal debt ratio, investment and dividend payment strategies for an insurance company in a finite time horizon by maximizing the total expected discounted utility of dividend payment, [15] considered the optimal liability management strategies and dividend payment for an insurance company that experiences catastrophic risks. The optimal investment strategies and collaterized optimal debt in the presence of intangible asset within a jump-diffusion framework was considered in [16].

In this paper, the investor we consider operates within a diffusion framework. The asset price, inflation index and income growth rate dynamics are assumed to follow a diffusion process. The wealth process of the investor is described as the difference between asset value and debt. To account for the impact of inflation risk, we derived the real wealth process of the investor. We considered two state variables in our value function which include the real wealth process and income growth rate. The goal of the investor is to maximize the discounted expected utility of consumption over an infinite horizon. The dynamics of real wealth process was solved using stochastic dynamic programming techniques and the resulting Hamilton-Jacobi-Bellman (HJB) equation was derived with transversality condition. We obtained the optimal debt ratio and optimal consumption strategy by assuming a CRRA utility function.

The remainder of the paper is structured as follows. In section 2, we characterize the inflation index, asset dynamics, debt process, income growth rate dynamics and the real wealth process. In section 3, we present the optimal controls and value function, optimal debt ratio and optimal consumption plan. The numerical results for our models are presented in section 4. We conclude the paper in section 5.

2. The Model

In this section, we consider our probability space of interest, inflation rate dynamics, underlying asset dynamics and the real wealth process. First, we present the probability space.

2.1 The probability space

Let $(\Omega, f_t, F(f_t), P)$ be a filtered probability space, where $F(f_t)$ is the $\sigma - algebra$ generated by $\{W_I(t), W_B(t), W_\eta(t) : 0 \le s \le t\}$ such that $W(t) = (W_I(t), W_B(t), W_\eta(t))'$ is a standard Brownian motion that captures the risks from inflation index, asset price and income growth rate respectively at time t, f_t is a filtration, P is the real world probability and the sign (') represent transpose.

2.2 The Dynamics of Inflation Rate

The consumer price index is a measure that helps us appreciate the real impact of price changes on goods and service. The prices of goods and services in Nigeria for example have experienced increase in recent times and hence it is necessary to consider the impact of inflation on the investment portfolio. In this paper, the inflation rate dynamics is assumed to be correlated with inflation risks, asset price risks and income growth rate risks and evolves according to the following diffusion process.

$$\frac{dI(t)}{I(t)} = \mu_I(t)dt + \sigma_I(t)dW(t), \quad I(0) = I_0 > 0. \tag{1}$$

where $\mu_I(t) = r(t) - \overline{r}(t) + \sigma_I(t)\theta_I(t)$ is the expected inflation rate at time t, $r(t)$ is the nominal interest rate at time t, $\overline{r}(t)$ is the real rate of interest at time t, $\sigma_I(t) = (\sigma_{1I}(t), \sigma_{2I}(t), \sigma_{3I}(t))$ is the price index volatility at time t such that $\sigma_{1I}(t) \in \mathbb{R}$ is the volatility of price index with respect to sources of inflation risks, $\sigma_{2I}(t) \in \mathbb{R}$ is the volatility of price index with respect to sources of asset price risks, $\sigma_{3I}(t) \in \mathbb{R}$ is the volatility of price index with respect to sources of income growth rate risks, $\theta(t) = (\theta_I(t), \theta_B(t), \theta_\eta(t))'$ is the market price of risk at time t such that $\theta_I(t)$ is the market price of inflation risk at time t, $\theta_B(t)$ is the market price of asset price risk at time t, $\theta_\eta(t)$ is the market price of income growth rate risk at time t.

2.3 The Asset Dynamics

We consider a firm whose asset value $A(t)$ at time t is given as the product of the asset price $B(t)$ and the quantity of the asset $K(t)$ such that $A(t) = B(t)K(t)$. We assume the asset price follows the following diffusion process

$$\frac{dB(t)}{B(t)} = \mu_B(t)dt + \sigma_B(t)dW(t), \qquad (2)$$

where $\mu_B(t)$ is the expected growth rate of asset price at time t, $\sigma_B(t) = (\sigma_{1B}(t), \sigma_{2B}(t), \sigma_{3B}(t))$ is the volatility tensor of asset price at time t such that $\sigma_{1B}(t) \in \mathbb{R}$ is the volatility of the asset price arising from inflation sources of risks, $\sigma_{2B}(t) \in \mathbb{R}$ is the volatility of the asset price arising from asset price sources of risks and $\sigma_{3B}(t) \in \mathbb{R}$ is the volatility of the asset price arising from income growth rate sources of risks.

The change in asset value of the firm is given as

$$\begin{aligned} dA(t) &= (B(t)K(t)) \\ &= B(t)dK(t) + dB(t)K(t) \\ &= B(t)dK(t) + A(t)\frac{dB(t)}{B(t)} \\ &= B(t)dK(t) + A(t)(\mu(t)dt + \sigma(t)dW(t)). \end{aligned} \qquad (3)$$

We note that the change in asset value of the firm is caused by a change in the price of the asset given by $A(t)(\mu(t)dt + \sigma(t)dW(t))$ and a change in the asset quantity i.e. $B(t)dK(t)$. The change in asset quantity is considered an investment by the firm and thus included as part of the firm expenditure.

We describe the wealth process of the firm as the difference between the asset value and debt at time t. Let $X(t)$ and $L(t)$ be the wealth and debt respectively of the investor at time t. The debt $L(t)$ at time t is expenditure less income. The expenditure component of $L(t)$ includes interest rate paid on debt $r_L(t)$ at time t and also the amount consumed at time t given by $dC(t) = c(t)X(t)dt,$ where $c(t)$ is the consumption rate. The income component of $L(t)$ comes from the product of income growth rate and asset value. Let $Y(t)$ be the income generated by the asset of the investor at time t. We assume that $\eta(t)$ is the income growth rate at time t, so that we have

$$dY(t) = \eta(t)A(t)dt, \qquad (4)$$

The income growth rate $\eta(t)$ is assumed to follow a diffusion process and affected by economic, scientific, human, environmental and political factors. We assume in this study that the productive capacity of the firm is a major factor that impacts the income growth rate at time t. When the productive capacity ω of the firm is high, the firm is considered to be expanding and if ω is low, the firm is said not to be expanding. The income growth rate evolves according to the following diffusion process

$$\begin{aligned} d\eta(t) &= \left[\beta(\eta(t)) + \eta(t)\phi(\omega)\right]dt + \sigma_\eta(t)dW(t), \\ \eta(0) &= \eta_0. \end{aligned} \qquad (5)$$

where $\beta : \mathbb{R} \times [0,T] \to \mathbb{R}$ is the expected growth of $\eta(t), \phi(\omega) : \mathbb{R} \to \mathbb{R}$ represents the impact of the productive capacity of the firm on its asset value. Following [10], if $\phi(\omega) > 0$, it implies that the asset value of the firm is expanding, if $\phi(\omega) < 0$, it implies that the asset value of the firm is contracting and if $\phi(\omega) = 0$, it implies that the asset value of the firm is in a critical position. The tensor $\sigma_\eta(t) = (\sigma_{1\eta}, \sigma_{2\eta}, \sigma_{3\eta})$ is the volatility of income growth rate at time t such that $\sigma_{1\eta} \in [\mathbb{R} \times [0,T]], \sigma_{2\eta} \in [\mathbb{R} \times [0,T]], \sigma_{3\eta} \in [\mathbb{R} \times [0,T]].$

We now have the net change in debt as

$$dL(t) = r_L(t)L(t)dt + B(t)dK(t) + c(t)X(t)dt - \eta(t)A(t)dt \tag{6}$$

Definition 1.

The wealth process $X(t)$ *of the firm at time* t *is defined as*

$$X(t) = A(t) - L(t) \tag{7}$$

We now assume that the government imposes capital gains tax based on accrual on the investor's wealth. This implies that the tax base for this study is represented by the change in the value of the investor's wealth. Let $\tau \in [0,1]$ be the tax rate levied on the payoff of the wealth process. The dynamics of the wealth process is summarized in the following proposition.

Proposition 1.

The wealth process of the investor at time t *is*

$$\frac{dX(t)}{X(t)} = (1-\tau)\left[\left(\mu(t)+\eta(t)\right)\left(1+\varphi(t)\right) - r_L(t)\varphi(t) - c(t)\right]dt + (1-\tau)\left(1+\varphi(t)\right)\sigma(t)dW(t), \tag{8}$$

$$X(0) = x_0.$$

Proof:

We commence by noting that

$$dX(t) = (1-\tau)d\left(A(t) - L(t)\right) \tag{9}$$

Using (3) and (6), we have that

$$dX(t) = (1-\tau)\left[A(t)\left[\mu(t)dt + \sigma(t)dW(t)\right] - r_L(t)L(t)dt - c(t)X(t)dt + \eta(t)A(t)dt\right], \tag{10}$$

but, $A(t) = X(t) + L(t)$, which implies that

$$dX(t) = (1-\tau)\begin{bmatrix}\left(X(t)+L(t)\right)\left[\mu(t)dt + \sigma(t)dW(t)\right] - r_L(t)L(t)dt - c(t)X(t)dt + \\ \eta(t)\left(X(t)+L(t)\right)dt\end{bmatrix} \tag{11}$$

Hence, we have

$$\frac{dX(t)}{X(t)} = (1-\tau)\left[\left(\mu(t)+\eta(t)\right)\left(1+\varphi(t)\right) - r_L(t)\varphi(t) - c(t)\right]dt + \tag{12}$$

$$(1-\tau)\left(1+\varphi(t)\right)\sigma(t)dW(t), \quad X(0) = x_0.$$

where $\varphi(t) = \frac{L(t)}{X(t)}$ is the debt ratio of the firm.

Definition 2.

The real wealth of the investor $\bar{X}(t)$ *is defined as the ratio of the wealth process to the inflation index. It is mathematically given as*

$$\bar{X}(t) = \frac{X(t)}{I(t)} \tag{13}$$

Proposition 2.

$$d\bar{X}(t) = d\left(\frac{X(t)}{I(t)}\right) = \bar{X}\begin{bmatrix}(1-\tau)\left[\left(\mu(t)+\eta(t)\right)\left(1+\varphi(t)\right) - r_L(t)\varphi(t) - c(t)\right] \\ -\mu_I(t) + \sigma_I^2(t) - (1-\tau)(1+\varphi)\sigma_B\sigma_I\end{bmatrix}dt \tag{14}$$

$$+\bar{X}\left[(1-\tau)(1+\varphi)\sigma_B - \sigma_I\right]dW, \quad \bar{X}(0) = \bar{x}_0 > 0.$$

Proof:

Taking the differential of both sides of (13), we have

$$d\bar{X}(t) = d\left(\frac{X(t)}{I(t)}\right) \tag{15}$$

Applying the Ito quotient rule on (15), where X and I satisfies (12) and (1) respectively, the result follows immediately.

In what follows, we shall no longer indicate the variable t when no confusion is made. We give the following definition.

3. The Optimal Control Problem

In this section, we present the admissible strategies, optimal controls and the value function of our problem.

3.1 The Admissible Strategies

Definition 3.

The strategy $u(.) = \{\varphi, c : t \geq 0\}$ *that is progressively measurable with respect to* $\{W : 0 \leq s \leq t\}$ *is called an admissible strategy.*

For admissible debt ratio strategy φ, we assume that for all $T \in (0, \infty)$,

$$E\int_0^T \varphi^2 dt < \infty, \qquad (16)$$

For admissible consumption rate c, we assume that for all $T \in (0, \infty)$,

$$E\int_0^T c^2 dt < \infty. \qquad (17)$$

Let A be the collection of all admissible strategies, then the collection of admissible controls A is defined as

$$A = \left\{ u = (\varphi, c : t \geq 0) \in \square \times \square : E\int_0^\infty c^2(t)\,dt < \infty, E\int_0^\infty \varphi^2(t)\,dt < \infty. \right\} \qquad (18)$$

3.2 The Optimal Controls and Value Function

The desire of the investor is to choose the debt ratio and consumption plan, that will optimize the expected discounted utility of consumption in an infinite horizon. For an arbitrary admissible strategy $u(.) = \{\varphi, c : t \geq 0\}$, we define the value function as

$$G\left(t, \bar{X}(t), \eta(t)\right) := \sup_{u \in A}\left[J(\bar{x}, \eta; u) \mid \bar{X}(t) = \bar{x}, \eta(t) = \eta \right]. \qquad (19)$$

where

$$J(\bar{x}, \eta; u) = E_{\bar{x},\eta}\int_t^\infty e^{-\delta s} H\left(c(s)\right) ds, \qquad (20)$$

$H\left(c(s)\right)$ is utility function with respect to consumption and $0 \leq \delta < 1$ is the discount rate of consumption.

The investor we consider chooses the power utility function and by adopting the stochastic dynamic programming approach and Itó Lemma for semi-martingale processes, our Hamilton–Jacobi–Bellman equation which characterizes the optimal solutions to the problem of the firm becomes

$$\begin{aligned} 0 = \; & G_t + \left[(1-\tau)\left((\mu+\eta)(1+\varphi) - r_L\varphi - c\right) - \mu_I + \sigma_I^2 - (1-\tau)(1+\varphi)\sigma_B\sigma_I^{'}\right]\bar{x}G_{\bar{x}} + \\ & \left(\beta(\eta) + \eta\phi(\omega)\right)G_\eta + e^{-\delta}H(c) + \frac{1}{2}\bar{x}^2\left[(1-\tau)(1+\varphi)\sigma_B - \sigma_I\right]^2 G_{\bar{x}\bar{x}} + \\ & \frac{1}{2}\sigma_\eta^2 G_{\eta\eta} + \bar{x}\left[(1-\tau)(1+\varphi)\sigma_B - \sigma_I\right]\sigma_\eta G_{\bar{x}\eta}. \end{aligned} \qquad (21)$$

with transversality condition $\lim_{t\to\infty} E\left[G(t, \bar{x}, \eta)\right] = 0$, $G_t = \frac{\partial G}{\partial t}$, $G_{\bar{x}} = \frac{\partial G}{\partial \bar{x}}$, $G_\eta = \frac{\partial G}{\partial \eta}$, $G_{\bar{x}\bar{x}} = \frac{\partial G}{\partial \bar{x}^2}$, $G_{\eta\eta} = \frac{\partial G}{\partial \eta^2}$, $G_{\bar{x}\eta} = \frac{\partial G}{\partial \bar{x}\eta}$.

By the standard time-homogeneity argument for infinite-horizon problems, we have

$$\begin{aligned} e^{\delta t}G\left(t, \bar{X}, \eta\right) &= \sup_{\{\varphi(s), c(s)\}\, t \leq s \leq \infty} E_t\int_t^\infty \left[e^{-\delta(s-t)} H\left(c(s)\right)\right] ds \\ &= \sup_{\{\varphi(t+u), c(t+u)\}\, 0 \leq u \leq \infty} E_t\int_0^\infty \left[e^{-\delta(u)} H\left(c(t+u)\right)\right] du \\ &= \sup_{\{\varphi(u),\, c(u)\}\, 0 \leq u \leq \infty} E_0\int_0^\infty \left[e^{-\delta(u)} H\left(c(u)\right)\right] du \\ &\equiv V\left(t, \bar{X}, \eta\right) \end{aligned} \qquad (22)$$

which is independent of time. The third equality in this argument makes use of the fact that the optimal control is Markov.

Hence, $G(t,\bar{x},\eta)=e^{-\delta t}V(t,\bar{x},\eta)$ and (21) reduces to the following time-homogeneous value function V :

$$0 = V_t+\left[(1-\tau)\left((\mu+\eta)(1+\varphi)-r_L\varphi-c\right)-\mu_I+\sigma_I^2-(1-\tau)(1+\varphi)\sigma_B\sigma_I^{'}\right]\bar{x}V_{\bar{x}}+ \tag{23}$$
$$\left(\beta(\eta)+\eta\phi(\omega)\right)V_\eta+\delta V(c)+\frac{1}{2}\bar{x}^2\left[(1-\tau)(1+\varphi)\sigma_B-\sigma_I\right]^2V_{\bar{x}\bar{x}}+$$
$$\frac{1}{2}\sigma_\eta^2V_{\eta\eta}+H(c)+\bar{x}\left[(1-\tau)(1+\varphi)\sigma_B-\sigma_I\right]\sigma_\eta V_{\bar{x}\eta}.$$

with transversality condition $\lim_{t\to\infty}E\left[V(t,\bar{x},\eta)\right]=0$. For more details see [17].

3.3 Power Utility

Our investor has the following power utility function $H(c)=\dfrac{c^{1-\gamma}}{1-\gamma}$ with respect to consumption for $\gamma>0$, such that $\gamma\in(0,1)\cup(1,\infty)$, where γ is the constant relative risk aversion parameter with respect to consumption.

We assume a solution to (23) that is of the form

$$V(t;\bar{x},\eta) = \frac{\bar{x}^{1-\gamma}}{1-\gamma}e^{h(t,\eta)}. \tag{24}$$

From (24), we have the following

$$V_{\bar{x}} = \bar{x}^{-1}(1-\gamma)V(t;\bar{x},\eta),\quad V_{\bar{x}\bar{x}}=-\bar{x}^{-2}(1-\gamma)V(t;\bar{x},\eta)$$
$$V_\eta = h_\eta V(t;\bar{x},\eta),\ V_{\eta\eta}=\left(h_\eta^2+h_{\eta\eta}\right)V(t;\bar{x},\eta) \tag{25}$$
$$V_{\bar{x}\eta}=h_\eta(1-\gamma)\bar{x}^{-1}V(t;\bar{x},\eta),\ V_t=h_tV(t;\bar{x},\eta)$$

Substituting (25) into (23) and dividing through by $-(1-\gamma)V(t;\bar{x},\eta)$, we have

$$0 = -\frac{h_t}{1-\gamma}-\left[(1-\tau)\left((\mu(t)+\eta(t))(1+\varphi(t))-r_L(t)\varphi(t)-c(t)\right)-\mu_I(t)+\sigma_I^2(t)-(1-\tau)(1+\varphi)\sigma_B\sigma_I\right]$$
$$+\left(\beta(\eta)+\eta\phi(\omega)\right)\frac{h_\eta}{1-\gamma}+\frac{\delta}{1-\gamma}+\frac{1}{2}\gamma\left[(1-\tau)(1+\varphi)\sigma_B-\sigma_I\right]^2+\frac{1}{2(1-\gamma)}\sigma_\eta^2\left(h_\eta^2+h_{\eta\eta}\right)$$
$$-\left[(1-\tau)(1+\varphi)\sigma_B-\sigma_I\right]\sigma_\eta h_\eta-\frac{c^{1-\gamma}}{x^{1-\gamma}}e^{-h(t,\eta)}. \tag{26}$$

The optimal debt ratio and optimal consumption strategy of the investor can be deduced from equation (26). In what follows, we obtain the optimal debt ratio.

3.4 Optimal Debt Ratio

Definition 4.

The optimal debt ratio φ^* for the firm at time t is defined as

$$\varphi^*(t)=arg\min_{\varphi}f(\varphi),$$

where the function

$$f(\varphi)=-(1-\tau)\left((\mu+\eta)(1+\varphi)-r_L\varphi\right)+(1-\tau)(1+\varphi)\sigma_B\sigma_I+\frac{1}{2}\gamma\left((1-\tau)(1+\varphi)\sigma_B-\sigma_I\right)^2 \tag{27}$$
$$-\left((1-\tau)(1+\varphi)\sigma_B-\sigma_I\right)\sigma_\eta h_\eta$$

is convex.

Proposition 3.

The optimal debt ratio φ^* of the investor is

$$\varphi^* = \frac{(\mu+\eta-r)}{\sigma_B^2\gamma(1-\tau)} - \frac{\sigma_I(1-\gamma)}{\sigma_B\gamma(1-\tau)}+\frac{\sigma_\eta h_\eta}{\sigma_B\gamma(1-\tau)}-1. \tag{28}$$

Proof:

By first order condition, we have from (27) that

$$\frac{\partial f(\varphi)}{\partial\varphi}=-(\mu+\eta-r_L)+\sigma_B\sigma_I+\gamma\sigma_B\left((1-\tau)(1+\varphi)\sigma_B-\sigma_I\right)-\sigma_B\sigma_\eta h_\eta=0 \tag{29}$$

making φ the subject of the formular, we have

$$\varphi^* = \frac{(\mu+\eta-r)}{\sigma_B^2\gamma(1-\tau)} - \frac{\sigma_I(1-\gamma)}{\sigma_B\gamma(1-\tau)}+\frac{\sigma_\eta h_\eta}{\sigma_B\gamma(1-\tau)}-1. \tag{30}$$

Equation (30) gives the optimal debt ratio of the investor. It shows that it is optimal for the firm to have a debt ratio that comprises the following three components:

1. an optimal debt ratio component $\frac{(\mu+\eta-r_L)}{\sigma_B^2\gamma(1-\tau)}$ that is time varying and inversely proportional to asset price risk, tax rate and risk aversion coefficient,

2. an inflation risk hedging debt ratio component $\frac{\sigma_I(1-\gamma)}{\sigma_B\gamma(1-\tau)}$ inversely proportional to the diffusion risk of asset price, tax rate and the CRRA coefficient $\frac{1-\gamma}{\gamma}$,

3. an income growth rate hedging debt ratio component $\frac{\sigma_\eta h_\eta}{\sigma_B\gamma(1-\tau)}$ with respect to the cross derivative of the function h with respect to η and inversely proportional to asset price risk, tax rate and risk aversion coefficient $\frac{1}{\gamma}$.

Corollary 1.
The optimal debt ratio is higher with taxation.
Proof:
Let φ_τ^* and φ^* be the optimal debt ratio with and without taxation respectively. We assume that $M=\varphi_\tau^*-\varphi^*$, where M is a constant. Hence,

$$\left(\frac{1}{1-\tau}-1\right)\left[\frac{(\mu+\eta-r)}{\sigma_B^2\gamma}-\frac{\sigma_I(1-\gamma)}{\sigma_B\gamma}+\frac{\sigma_\eta h_\eta}{\sigma_B\gamma}\right]=M. \qquad (31)$$

This implies that,

$$\left(\frac{1}{1-\tau}-1\right)=\frac{M}{\left[\frac{(\mu+\eta-r)}{\sigma_B^2\gamma}-\frac{\sigma_I(1-\gamma)}{\sigma_B\gamma}+\frac{\sigma_\eta h_\eta}{\sigma_B\gamma}\right]}. \qquad (32)$$

Suppose that $\varphi_\tau^*=\varphi^*$, then $\frac{1}{1-\tau}-1=0.$

Since $\tau\in(0,1)$, we have $\frac{1}{1-\tau}>1$. Hence,

$$M=\frac{1}{1-\tau}-1>0. \qquad (33)$$

This implies that $\varphi_\tau^*-\varphi^*>0$ and it therefore follows that $\varphi_\tau^*>\varphi^*$.

Remark 1.
We note the following:

1. $\tau=1-\frac{1}{\sigma_B^2\gamma}\left[\mu+\eta-r_L-\sigma_B\sigma_I(1-\gamma)+\sigma_B\sigma_\eta h_\eta\right],\quad for\quad \varphi^*=0.$
2. $\tau>1-\frac{1}{\sigma_B^2\gamma}\left[\mu+\eta-r_L-\sigma_B\sigma_I(1-\gamma)+\sigma_B\sigma_\eta h_\eta\right],\quad for\quad \varphi^*<0.$
3. $\tau<1-\frac{1}{\sigma_B^2\gamma}\left[\mu+\eta-r_L-\sigma_B\sigma_I(1-\gamma)+\sigma_B\sigma_\eta h_\eta\right],\quad for\quad \varphi^*>0.$

We observe that the optimal debt ratio φ^* is negatively related to diffusion risk of inflation. In fact,

$$\frac{\partial\varphi^*}{\partial\sigma_I}=-\frac{(1-\gamma)}{\sigma_B(1-\tau)}<0. \qquad (34)$$

(34) shows that there is an inverse relationship between the optimal debt ratio and inflation risk. However, the extent of this inverse relation depends on asset diffusion risk, tax rate and the coefficient of risk aversion.

3.5 Optimal Consumption

Definition 5.

The optimal consumption rate is defined as

$$c^*(t) = \arg\min_c f(c),$$

where the function

$$f(c) = (1-\tau)c - \frac{c^{1-\gamma}}{x^{1-\gamma}} e^{-h(t,\eta)} \quad (35)$$

is convex.

Proposition 4.

The optimal consumption rate c^* *is given as*

$$c^* = \left[\frac{(1-\tau)}{1-\gamma} x^{1-\gamma} e^{h(t,\eta)}\right]^{-1/\gamma}. \quad (36)$$

Proof:

By first order principle, we have

$$\frac{\partial f(c)}{\partial c} = (1-\tau) - \frac{(1-\gamma)c^{-\gamma}}{x^{1-\gamma}} e^{-h(t,\eta)} = 0, \quad (37)$$

this implies that

$$c^* = \left[\frac{(1-\tau)}{1-\gamma} x^{1-\gamma} e^{h(t,\eta)}\right]^{-1/\gamma}. \quad (38)$$

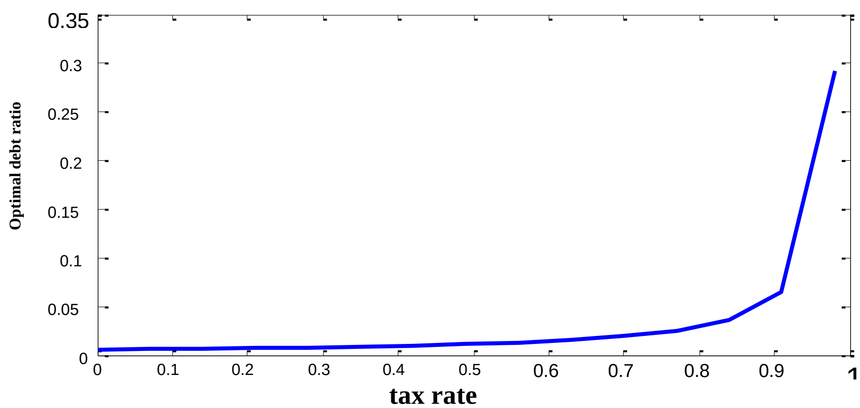

Figure 1: Optimal debt ratio plotted against tax rate

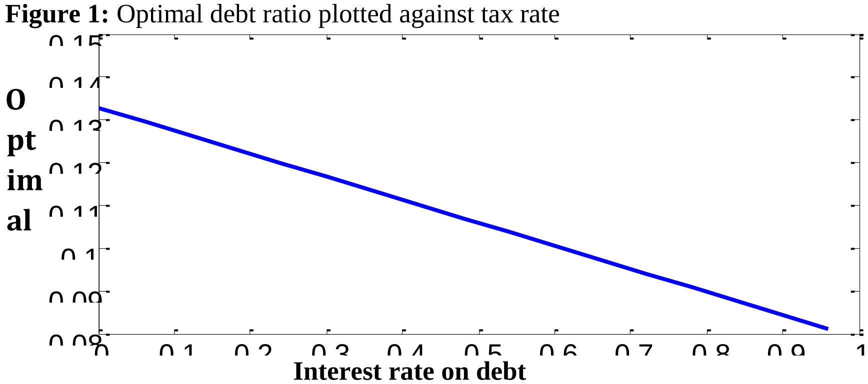

Figure 2: Optimal debt ratio plotted against interest rate on debt

4. Numerical analysis

In this section, we present the numerical analysis of the model developed in this study. We basically consider the effects of model parameters on the optimal debt ratio and optimal consumption rate. The base case values of our parameters are given as follows

Table 1: The Base Case Values of the Parameters

Parameters	values	Parameters	values
Risk aversion coefficient	$\gamma = 0.5$	Income growth rate	$\eta = 0.0228$
Tax rate	$\tau = 0.07$	Asset expected return	$\mu_B = 0.08$
Interest rate on debt	$r_L = 0.06$	Asset volatility	$\sigma_B = 2.0$
Inflation index volatility	$\sigma_I = 0.27$	Income growth rate volatility	$\sigma_\eta = 0.025$

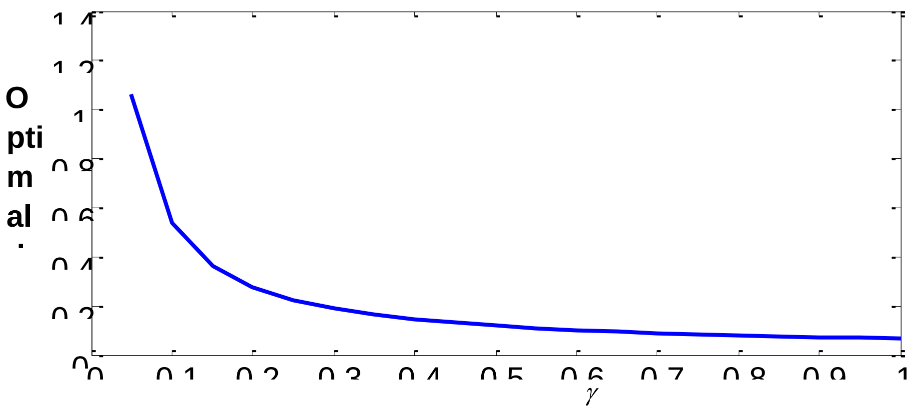

Figure 3: Optimal debt ratio plotted against γ

An increasing tax rate as shown in Figure 1 impacts positively on the debt ratio. Clearly, increasing the tax rate impacts on the debt ratio of the investor in such a way as to make less funds available for portfolio expansion which leads the firm to look for sources of debt financing which ultimately drives up the debt ratio.

Figure 2 shows the impact of interest paid on debt (debt servicing) on the optimal debt ratio. We observe that as the investor increases the amount paid as debt servicing, the optimal debt ratio decreases and vice versa. In other words, debt servicing is negatively related to the optimal debt ratio.

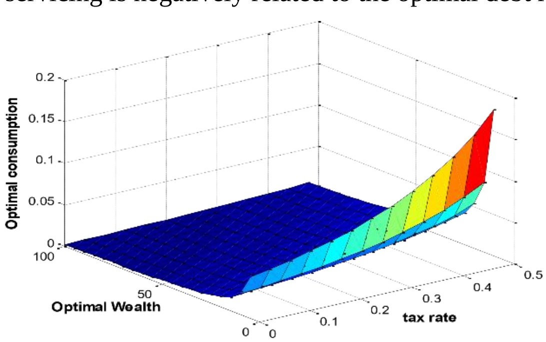

Figure 4: Debt ratio plotted against income growth rate and interest rate on debt for $\tau = 0.30$, $\gamma = 5$, $q = 0.8$, $\sigma_B = 0.2$ and $\mu_B = 0.08$

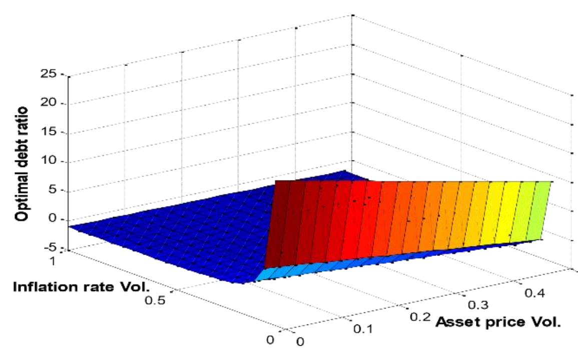

Figure 5: Debt ratio plotted against inflation rate volatility and asset price volatility

Figure 3 demonstrates the impact of the coefficient of risk aversion on the optimal debt ratio of the investor. It is observed that as the investor sentiment for taking risk reduces, the optimal debt reduces as well. In other words, as the risk aversion of the investor is increasing, the debt ratio is decreasing which indicates that a risk averse investor is likely to have a lower debt ratio as they might not be willing to take advantage of available debt financing opportunities to increase portfolio return at the expense of a higher debt ratio.

Figure 4 demonstrates the effects of optimal wealth and tax rate on optimal consumption rate of the investor at the terminal time. We observe that as optimal terminal wealth increases, optimal terminal consumption decreases and vice versa. However, we observe that the tax rate has a positive relationship with optimal terminal consumption. The effects of inflation risk and asset price diffusion risk on the optimal debt ratio is shown in Figure 5. We observe that both inflation risk and asset price risk are negatively related to the optimal debt ratio. It indicates that an investor that is facing increased asset price and inflation rate volatility will likely not contract debt financing that will ultimately increase the debt ratio. In other words, asset price volatility is negatively related to the optimal debt ratio.

5. Conclusion

In this paper, we study the optimal debt ratio and optimal consumption strategy of an investor that operates in a market with diffusion risk. The investor faces three background risks: inflation, asset price and income growth rate risks. The objective of the investor is to maximize the expected discounted utility of consumption in an infinite horizon. Considering the CRRA utility function, we obtained the optimal debt ratio and the optimal consumption strategy. Our results suggest that an investor that faces capital gains tax is likely to have a higher debt ratio than an investor not facing taxation. It was also found that the optimal debt ratio depends negatively on the coefficient of risk aversion, debt servicing, inflation rate and asset price volatility.

In [6], it was found an investor with a high coefficient of risk aversion and capital gains tax will reduce the investor's consumption level. However, the numerical result of our study shows that optimal terminal consumption and the capital gains tax rate is positively related.

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