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ON THE SPECIAL CASES OF THE GENERALIZED MAX-PLUS EIGENVALUE PROBLEM

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Abstract

We study the generalized eigenvalue problem in max algebra $\boldsymbol{A \otimes x = \lambda \otimes B \otimes x}$***, where*** $\boldsymbol{A, B \in \overline{\mathbb{R}}^{m\times n}}$***. We prove that if*** $\boldsymbol{A}$ ***and*** $\boldsymbol{B}$ ***are binary, then*** $\boldsymbol{\lambda}$ ***is finitely generated in a closed set. We give the general solution for the*** $\boldsymbol{2 \times 2}$ ***case for binary entries.***

Keywords: Matrix, Max-plus Algebra, Generalized Eigenvalue, Spectrum

1.0 INTRODUCTION

Max-Algebra can be define as a Semi-ring $(R,\oplus,\otimes)$ as follows:
Let G be a set, define two operations on G as $\otimes$ and the linear order $\leq$, thus
$G = (G,\otimes,\leq)$ is a linearly ordered commutative group. Denote $\bar{G} = G \cup \{\epsilon\}$ where ϵ is an adjoined element such that $\epsilon < a \quad \forall\, a \in G$.
Define $a \oplus b = \max\{a,b\} \forall\, a,b \in \bar{G}$ and extend $\otimes \quad \bar{G}$ by setting $a \otimes \epsilon = \epsilon = \epsilon \otimes a$. Thus it is clear that $(\bar{G},\oplus,\otimes)$ is and idempotent commutative semi-ring. Max-algebra is defined here when G is an additive group of real numbers i.e $(R,\oplus,\otimes)$. In max-algebra we define the operation $a \otimes b = a + b$ *and* $a \oplus b = max\{a,b\}$
We denote $-\infty$ by ϵ (the neutral element with respect to $\oplus$), and for convenience we also denote by the same symbol any vector with all components
$-\infty$; and a matrix with all entries $-\infty$. 0 is denoted by e (the neutral element with respect to $(\otimes)$. If $a \in \mathbb{R}$; then the symbol a^{-1} stands for $-a$.
Recall that in max-algebra, we define the operation $a \otimes b = a + b$ and $a \oplus b = max(a,b)$
Max-plus algebra is derived from the analog of linear algebra developed for the pair of operations $(\oplus,\otimes)$, and it is extended to matrices and vectors as in conventional algebra. It has many applications areas such as machine scheduling, optimization, combinatorics, mathematical physics and algebraic geometry. Nowadays it solves many problems of Parallel processing systems, telecommunication networks, discrete events processes and control theory. For the proofs and more information about max-algebra, the reader is referred to [1, 3, 4 and 12].

2.0 PREREQUISITES

In this section we give the definitions of terms that will be used in the formulation of our results.
A binary operation $*$ is called idempotent on a set $\mathbb{R}$ if $\forall\, x \in \mathbb{R},\ x \times x = x$
A monoid is a closed set under an associative binary operation which has multiplicative identity.
A semi-ring is a commutative monoid which has no additive identity.
Two important aspects of max-algebra are that it does not have additive inverses and it is idempotent. This is why max-algebra is considered a semi-ring and not a ring.
An $n \times n$ matrix is called diagonal, notation $diag(d_1,\ldots,d_n)$, or just $diag(d)$, if its diagonal entries are $d_1,\ldots,d_n \in \mathbb{R}$ and off-diagonal entries are ϵ.
A digraph is an ordered pair $G = (V, E)$ where V is a nonempty finite set (of nodes) and $E \subset V \times V$ (the set of arcs).
A subdigraph of G is any digraph $G' = (V', E')$ such that $V' \subset V$ and $E' \subset E$. If $e = (u,v) \in E$ for some $u,v \in V$ then we say that e is leaving u and entering v. Any arc of the form (u,u) is called a loop.
Let $G = (V, E)$ be a given digraph. A sequence $\pi = (v_1,\ldots,v_p)$ of nodes in G
is called a path (in G) if $p = 1$ or $p > 1$ and $(v_i,v_{i+1}) \in E$ for all $i = 1,\ldots,p-1$. The node v_1 is called the starting node and v_p the endnode of π, respectively.

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The number $p - 1$ is called the length of π and will be denoted by $l(\pi)$. If u is the starting node and v is the endnode of π then we say that π is a $u - v$ path.
If there is a $u - v$ path in G then v is said to be reachable from u, notation $u \rightarrow v$. Thus $u \rightarrow u$ for any $\in V$.
A path $(v_1, \ldots, v_p)$ is called a cycle if $v_1 = v_p$ and $p > 1$ and it is called an
elementary cycle if, moreover, $v_i \neq v_j$ for $i, j = 1, \ldots, p - 1, i \neq j$. If there is no cycle in G then G is called acyclic.
The weight of a path from vertex i to j of length k is given by $||p||_w = a_{i_2 i_1} \oplus \ldots \oplus a_{i_{k+1} i_k}$ *where* $i = i_1, j = i_{k+1}$.
The average weight of a cycle is given by

$$\frac{||p||_w}{||p||_l} = \max \frac{a_{i_2 i_1} + \ldots + a_{i_{k+1} i_k}}{k}$$

A path \pi is called positive if $w(\pi) > 0$. In contrast, a cycle $\delta = (u_1, \ldots, u_p)$ is called a zero cycle if $w(u_k, u_{k+1}) = 0$ for all $k = 1, \ldots, p - 1$. Since w stands for "weight" rather than "length", we will use the word "heaviest path/cycle" instead of "longest path/cycle".

A digraph G is called strongly connected if $u \rightarrow v$ for all nodes $u, v \in G$. A subdigraph G' of G is called a strongly connected component of G if it is a maximal strongly connected subdigraph of G, that is, G' is a strongly connected subdigraph of G and if G' is a subdigraph of a strongly connected subdigraph G'' of G then $G' = G''$. Note that a digraph consisting of one node and no arc is strongly connected and acyclic; however, if a strongly connected digraph has at least two nodes then it obviously cannot be acyclic. Because of this singularity we will have to assume in some statements that $|V| > 1$.
If $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ then the symbol $F_A(Z_A)$ will denote the digraph with the node set N and arc set $E = \{(i, j); a_{ij} > \epsilon\}$ $(E = \{(i, j); a_{ij} = 0\})$. Z_A will be called the zero digraph of the matrix A. If F_A is strongly connected then A is called irreducible otherwise reducible.

3.0 MAX-ALGEBRAIC EIGENVALUE PROBLEM

Given $A \in \overline{\mathbb{R}}^{n \times n}$ find the vectors $\in \overline{\mathbb{R}}^{n \times n}$ $(x \neq \epsilon)$ (Eigenvectors) and scalars $\lambda \in \overline{\mathbb{R}}$ (Eigenvalues) such that:

$$A \otimes x = \lambda \otimes x \qquad (1)$$

is called the (max-algebraic) eigenproblem.
The theory of max-algebraic eigenproblem is well known [9, 10, 15]. In this section we will give an overview of some results which will be useful in the forthcoming sections.
The set of all eigenvalues will be called the spectrum of the pair (A, B).
Given $A \in \overline{\mathbb{R}}^{n \times n}$, $\lambda(A)$ is the maximum cycle mean of A, that

$$\lambda(A) = max\, \mu\, (\sigma\, , A) \qquad (2)$$

Where the maximization is taken over all elementary cycles in G_A, and

$$\mu\, (\sigma\, , A) = \frac{(\sigma\, , A)}{l(\sigma)} \qquad (3)$$

Denotes the mean of a cycle σ. Clearly, $\lambda(A)$ always exists since the number of Elementary cycles is finite. It follows from this definition that G_A is acyclic if and only if $\lambda(A) = \epsilon$.

LEMMA 1 [27]
$\lambda(A)$ remains unchanged if the maximization is taken over all cycles.

THEOREM 2 [12]
If $A \in \overline{\mathbb{R}}^{n \times n}$ then:
a) $\lambda(A)$ is the greatest eigenvalue of A, that is
$\lambda(A) = \max\{\lambda \in \overline{\mathbb{R}}: A \otimes x = \lambda \otimes x, x \in \overline{\mathbb{R}}; x \neq \epsilon\}$
and dually b) We have
$\lambda(A) = \inf\{\lambda \in \overline{\mathbb{R}}: A \otimes x \leq \lambda \otimes x, x \in \overline{\mathbb{R}}^n$

LEMMA 3 [12, 21]
Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ have columns $A_1, A_2, \ldots, A_n$. If $\lambda(A) = \epsilon$ then $\Lambda(A) = \epsilon$,at least one column of A is ϵ and the eigenvectors of A are exactly the vectors $(x_1, \ldots, x_n)^T \in \overline{\mathbb{R}}^n, x \neq \epsilon$ such that $x_j = \epsilon$ whenever $A_j \neq \epsilon$ $(j \in N)$. Hence $V(A, \epsilon) = \{G \otimes z: z \in \overline{\mathbb{R}}^n\}$, where G \in \rbarsq has columns g_1,g_2, . . . and for all $j \in N$

$$g_j = \begin{cases} e^j, & if \;\; A_j = \epsilon \\ \epsilon, & if \;\; A_j \neq \epsilon \end{cases}$$

The maximum cycle mean of a matrix is of fundamental importance in max-algebra because for any square matrix A, it is the greatest eigenvalue of A, and every eigenvalue of A is the maximum cycle mean of some principal submatrix of A.

4.0 GENERALIZED EIGENVALUE PROBLEM

The two-sided eigenvalue problem in max-algebra is described as follows:-

For two matrices $A, B \in \overline{\mathbb{R}}^{n\times m}$ find scalars $\lambda \in \mathbb{R}$ called eigenvalues and vectors $x \in \mathbb{R}^n$ called eigenvectors with atleast one component not equals $-\infty$ such that

$$A \otimes \; x = \lambda \otimes B \otimes x \qquad (4)$$

In conventional notation this reads

$$\max_{i=1}^{n} (a_{ij} + x_j) = \lambda + \max_{i=1}^{n} (b_{ij} + x_j) \; for \;\; j = 1,2,3 \ldots, m$$

Let A and B be finite matrices, such that $A = a_{ij};\; B = b_{ij} \in \overline{\mathbb{R}}^{n\times m}$ are given matrices and we denote $M = \{1, \ldots, m\}$ and, $N = \{1, \ldots, n\}$.

Let

$C = (c_{ij}) = (a_{ij} \otimes b_{ij}^{-1})$

and

$D = (d_{ij}) = (b_{ij} \otimes a_{ij}^{-1})$

Define the set of all real numbers λ such that $A \otimes x = \lambda \otimes B \otimes x$ is satisfied

$V(A, B, \lambda) = \{x \in \overline{\mathbb{R}}^n : A \otimes x = \lambda \otimes B \otimes x, x \neq \epsilon\}$

$\Lambda(A, B) = \{\lambda \in \mathbb{R} : V(A, B, \lambda) \neq \emptyset\}$

The set $\Lambda(A, B)$ is called the spectrum of the pair (A, B)

Define $\underline{\lambda}(C) = max_{i\in M} \; min_{j\in N} c_{ij}$ and $\overline{\lambda}(C) = min_{i\in M} \; max_{j\in N}$

THEOREM 4 [11] If (A, B) is solvable and $\lambda \in \Lambda(A, B)$ then C satisfies

$$max_{i\in M} \; min_{j\in N} c_{ij} \leq \lambda \leq min_{i\in M} \; max_{j\in N} c_{ij} \qquad (5)$$

COROLLARY 4.1 If (A, B) is solvable then C satisfies

$max_{i\in M} min_{j\in N} c_{ij} \leq min_{i\in M} \; max_{j\in N} c_{ij}$

COROLLARY 4.2 If $m = n, (A, B)$ is solvable and $\lambda \in \Lambda(A, B)$ then C satisfies $\lambda'(C) \leq \lambda \leq \lambda(C)$

LEMMA 5 [*CANCELLATION RULE*] [24] Let $v, w, a, b, \in \mathbb{R};\; a < b$: Then for any real x we have $v \oplus a \otimes x = w \oplus b \otimes x$ if and only if $v = w \oplus b \otimes x$

PROPOSITION 6 If $a_{ij} < b_{ij}$ and $\underline{\lambda} < \overline{\lambda}$ then $\Lambda(A, B) = \{\gamma\}$ where γ is a projection onto define as:

$$\gamma = \begin{cases} \underline{\lambda} & if \quad \gamma \leq \underline{\lambda} \\ \gamma & if \quad \gamma \in (\underline{\lambda}, \overline{\lambda}) \\ \overline{\lambda} & if \quad \gamma \geq \overline{\lambda} \end{cases}$$

proof

The proof to this theorem for an arbitrary value of a_{ij}, b_{ij} can be found in [19].

Here we consider a special case where $\forall \; a_{ij}, b_{ij} \in \{0, 1\}$, such that $\underline{\lambda} = 0$ and $\overline{\lambda} = 1$.

Let $S = [\underline{\lambda}, \overline{\lambda}] \cap \Lambda(A, B)$, thenit is sufficient to prove the following statements

i. $S \neq \emptyset \Rightarrow \{\gamma\}$
ii. $\gamma \in (\underline{\lambda}, \overline{\lambda}) \Rightarrow \gamma \in S$
iii. $\gamma \in (\underline{\lambda}, \overline{\lambda}) \Rightarrow \underline{\lambda}, \overline{\lambda} \notin \Lambda(A, B)$
iv. $\gamma \leq \overline{\lambda} \Rightarrow \Lambda(A, B) = \{\overline{\lambda}\}$
v. $\gamma \geq \underline{\lambda} \Rightarrow \Lambda(A, B) = \{\underline{\lambda}\}$

In order to prove (i) suppose $\lambda \in S$ hence we have

$a_{ii} < \lambda \otimes b_{ii}, \;\; and \;\; a_{ij} > \lambda \otimes b_{ij}$

Using cancellation rule of Lemma 4.2 in this case we have:

$a_{ij} \otimes x_j = \lambda \otimes b_{ii}$

$a_{ij} = \lambda \otimes b_{ii} \otimes x_j$

So that $x_2 = \lambda \otimes b_{ii} \otimes a_{ij}^{-1}$ and $x_2 = \lambda^{-1} \otimes a_{ji} \otimes b_{ii}^{-1}$ from which $\lambda = \gamma$ follows.

(ii) Suppose $\gamma \in (\underline{\lambda}, \overline{\lambda})$ and put $\lambda = \gamma$.
by taking $x_2 = \lambda \otimes b_{ii} \otimes a_{ij}^{-1} = \lambda^{-1} \otimes a_{ji} \otimes b_{ii}^{-1}$ we see that $\lambda \in \Lambda(A, B)$
(iii) Suppose that $\gamma \in (\underline{\lambda}, \overline{\lambda})$ and $\lambda = \underline{\lambda} \in \Lambda(A, B)$ if $a_{ij} < b_{ij}$ then
$a_{ij} \otimes x_j = \lambda \otimes b_{ii}$
$a_{ij} \oplus \lambda \otimes b_{jj} \otimes x_j = \lambda \otimes b_{ii} \otimes x_j$
which indicate $\lambda^2 \geq \gamma^2$ a contradiction. A contradiction in similar way is obtained when $a_{ii} > b_{jj}$
(iv) Suppose $\gamma \leq \underline{\lambda}$ due to (i) it is sufficient to show that $\underline{\lambda} \in \Lambda(A, B)$ and $\overline{\lambda} \notin \Lambda(A, B)$. Let $\lambda = \underline{\lambda}$. It is easily verified that

$$x_2 = \begin{cases} \lambda \otimes b_{ii} \otimes a_{ij}^{-1} & if \quad a_{ij} < b_{ij} \\ \lambda^{-1} \otimes a_{ij} \otimes b_{ii}^{-1} & if \quad a_{ij} > b_{ij} \\ any\ value\ in\ [a_{ii} \otimes b_{jj}, a_{ij} \otimes b_{ij}^{-1}] & if \quad a_{ij} = b_{ij} \end{cases}$$

(v) The proof of this part is similar to that of (iv) and it is omitted here.
PROPOSITION 7 $\Lambda(A, B) \subseteq [\underline{\lambda}, \overline{\lambda}]$ holds for every $\mathbb{R}^{m\times n}$ The interval $[\underline{\lambda}, \overline{\lambda}]$ will be called the feasibility interval for the generalized eigen-problem.

5.0 GENERALIZED EIGENVALUE PROBLEM FOR BINARY ENTRIES

Suppose that the entries of the matrices A and B of the generalized eigenvalue problem (GEP) $A \otimes x = \lambda \otimes B \otimes x$,are in the set $\{0, 1\}$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Thus we have:

$$a_{11} \otimes x_1 \oplus \ldots \oplus a_{1n} \otimes x_n = \lambda \otimes (b_{11} \otimes x_1 \oplus \ldots \oplus b_{1n} \otimes x_n)$$
$$a_{21} \otimes x_1 \oplus \ldots \oplus a_{n1} \otimes x_n = \lambda \otimes (b_{21} \otimes x_1 \oplus \ldots \oplus b_{2n} \otimes x_n)$$
$$\vdots$$
$$a_{m1} \otimes x_1 \oplus \ldots \oplus a_{mn} \otimes x_n = \lambda \otimes (b_{m1} \otimes x_1 \oplus \ldots \oplus \otimes b_{mn} \otimes x_n)$$

And therefore

$$\bigotimes_{\substack{i=1\\j=1}}^{m} \bigoplus_{\substack{i=1\\j=1}}^{n} a_{ij}x_j = \bigotimes_{\substack{i=1\\j=1}}^{m} \bigoplus_{\substack{i=1\\j=1}}^{n} \lambda b_{ij}x_j$$

THEOREM 8 If $A = a_{ij}$, $B = b_{ij} \in \mathbb{R}^{m\times n}$ are given matrices of the generalized eigenvalue problem in which the respective value of each a_{ij} and b_{ij} are binary, then

$$\lambda \in \Lambda(A, B) = \begin{cases} a_{ij}b_{ij}^{-1} & if \quad a_{ij} > b_{ij} \\ a_{ij}^{-1}b_{ij} & if \quad a_{ij} < b_{ij} \\ any\ value\ in\ \left[a_{ij}b_{ij}^{-1}, a_{ij}^{-1}b_{ij}\right]\ if\ a_{ij} = b_{ij} \end{cases}$$

λ is finitely generated from the set {-1, 0, 1}

PROOF

Now given $A \otimes x = \lambda \otimes B \otimes x$ where a_{ij} and b_{ij} are binary $\lambda \in \mathbb{R}$ we have that

$$\bigotimes_{\substack{i=1\\j=1}}^{m} \bigoplus_{\substack{i=1\\j=1}}^{n} a_{ij}x_j = \bigotimes_{\substack{i=1\\j=1}}^{m} \bigoplus_{\substack{i=1\\j=1}}^{n} \lambda b_{ij}x_j \qquad (6)$$

From Proposition 7 a_{ij}, b_{ij} satisfy on of the following:

i. $a_{ij} > b_{ij} \Rightarrow \gamma > \underline{\lambda}$
ii. $a_{ij} = b_{ij} \Rightarrow \gamma \in (\underline{\lambda}, \overline{\lambda})$
iii. $a_{ij} < b_{ij} \Rightarrow \gamma < \overline{\lambda}$

(i) If $a_{ij} > b_{ij}$

$$\bigotimes_{\substack{i=1\\j=1}}^{m} a_{ij} \bigoplus_{\substack{i=1\\j=1}}^{n} x_j = \lambda \bigotimes_{\substack{i=1\\j=1}}^{m} b_{ij} \bigoplus_{\substack{i=1\\j=1}}^{n} x_j \qquad (7)$$

And therefore $\underline{\lambda} < \overline{\lambda}$ and by proposition 6 we can obtain the eigenvalue of the system in (7) as:

$$\bigoplus_{\substack{i=1\\j=1}}^{n} a_{ij} = \lambda \bigoplus_{\substack{i=1\\j=1}}^{n} b_{ij}$$

Thus

$$\lambda = \bigotimes_{\substack{i=1\\j=1}}^{m} \bigoplus_{\substack{i=1\\j=1}}^{n} a_{ij}\, b_{ij}^{-1} \tag{8}$$

(ii) if for all (i,j), $a_{ij} = b_{ij}$ then from proposition 7 and the cancellation property of lemma 5, (6) becomes

$$\bigoplus_{\substack{i=1\\j=1}}^{n} x_j = \bigotimes_{\substack{i=1\\j=1}}^{m} \bigoplus_{\substack{i=1\\j=1}}^{n} \lambda x_j \tag{9}$$

The system in (9) can only be feasible if $\lambda = 0$. This implies that if $a_{ij} = b_{ij}$ the system $A \otimes x = \lambda \otimes B \otimes x$ have a trivial solution.

(iii) By similar argument from (i) if $a_{ij} < b_{ij}$ we have that

$$\lambda = \bigotimes_{\substack{i=1\\j=1}}^{m} \bigoplus_{\substack{i=1\\j=1}}^{n} a_{ij}^{-1} b_{ij} \tag{10}$$

5.1 Generalized Eigenvalue Problem For 2 By 2 Matrices With Binary Entries

Here we solve the generalized eigenvalue problem for 2×2 matrices with binary entries.
The system below is considered the GEP for 2×2 matrices.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Simplified as

$$a_{i1} \otimes x_1 \oplus a_{i2} \otimes x_2 = b_{i1} \otimes x_1 \oplus b_{i2} \otimes x_2 \tag{11}$$

We assume by homogeneity that x2 = 0 thus from (11) we have

$$a_{11} \otimes x_1 \oplus a_{12} = b_{11} \otimes x_1 \oplus b_{12}$$
$$a_{21} \otimes x_1 \oplus a_{22} = b_{21} \otimes x_1 \oplus b_{22}$$

Where al $a_{ij}, b_{ij} \in \{0,1\}$

If $a_{i1} = b_{i2}$ for some $i \in M$ then by Theorem 4 and Corollary 4.1 this value is the unique candidate for the generalized eigenvalue.

Otherwise if $a_{i1}b_{i1}^{-1} < a_{i2}b_{i2}^{-1}$ (or similarly $a_{i1}b_{i1}^{-1} > a_{i2}b_{i2}^{-1}$) then the feasibility interval for the $i\,th$ equation is

$$[a_{i1}b_{i1}^{-1}, a_{i2}b_{i1}^{-1}] \tag{12}$$

Thus Equation (11) using cancellation reduces to

$$a_{i1} \otimes x_1 = \lambda \otimes b_{i1} \tag{13}$$

Hence

$$; x_1 = \lambda \otimes b_{i1} \otimes a_{i1}^{-1} \tag{14}$$

Example 1: (*Numerical Example*)

Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \; and \; B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

The generalized eigenvalue problem is expressed as:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \otimes \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Thus,

$$1 \otimes x_1 \oplus 1 \otimes x_2 = \lambda \otimes 1 \otimes x_1 \oplus \lambda \otimes x_2 \tag{15}$$
$$1 \otimes x_1 \oplus x_2 = \lambda \otimes 1 \otimes x_1 \oplus \lambda \otimes 1 \otimes x_2 \tag{16}$$

From (15)

$\lambda = 0$ *or* $\lambda = 1$

This satisfy the feasibility interval for the i^{th} *equation (12) i.e.* $\lambda \in [0, 1]$. *From A and B it is clear that* $\forall i \in M, a_{i1}b_{i1}^{-1} < a_{i2}b_{i2}^{-1}$. *Now by homogeneity suppose without loss of generality that* $x_2 = 0$,

*S*o that

$$1 \otimes x1 \oplus 1 = \lambda \otimes 1 \otimes x1 \oplus \lambda \otimes 0 \tag{17}$$

Using cancellation property

$1 \otimes x_1 = \lambda$

$x_1 = \lambda \otimes 1 - 1$

$x_1 = \lambda$

This satisfy the result of equation (13).

Now since either $\lambda = 0$ *or* $\lambda = 1$ and x_1 cannot be 0 thus $x_1 = 1$

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