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DERIVING THE RIEMMANIAN CURVATURE TENSOR THROUGH HOWUSU METRIC

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Abstract

The Howusu metric tensor was used to derive the Riemmanian Curvature Tensor $R^{\alpha}_{\mu\alpha v}$. Results obtained were compared with the Riemmanian Curvature Tensor $R^{\alpha}_{\mu\alpha v}$ derived from the Schwarzschild metric tensor. It was found that, at $r \to 0$, the Riemannian Curvature Tensors for both metric tensors were different except for R^{1}_{010} where they were the same. Also, as $r \to \infty$, it was found the Riemannian Curvature Tensor were the same for both metric tensors except for R^{1}_{212} and R^{2}_{323} derived from the Howusu metric tensor.

Keywords: Riemannian Curvature Tensor, Howusu Metric Tensor, Schwarzschild Metric Tensor.

INTRODUCTIO

The Schwarzschild metric represents the gravitational field around a symmetrically spherical object without angular momentum. This Schwarzschild metric can be summarized as [1]

$$g_{\mu v} = \begin{pmatrix} -\left(1-\frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\left(1-\frac{2M}{r}\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix} \quad (1)$$

where M is the mass of the object and r is the distance away from the object. In 2009, Howusu Metric Tensor was introduced and was said to be a solution to Einstein field equations that describes the gravitational field for all gravitational fields in nature. The metric is written as follows [1];

$$g_{\mu v} = \begin{pmatrix} -exp\left(\frac{2GM}{c^2 r}\right) & 0 & 0 & 0 \\ 0 & exp\left(-\frac{2GM}{c^2 r}\right) & 0 & 0 \\ 0 & 0 & r^2 exp\left(-\frac{2GM}{c^2 r}\right) & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta\, exp\left(-\frac{2GM}{c^2 r}\right) \end{pmatrix} \quad (2)$$

where C is the speed of light; G is the universal constant of gravitation; M is the mass of the object and r is the distance away from the object. In this paper, we derive the Riemannian Curvature Tensor in the Spherical coordinate based upon the Howusu Metric Tensor, and then compared the results with the well-known Riemannian Metric Tensor in the Spherical Coordinate based on Schwarzschild Metric Tensor.

THEORY

$R^{\alpha}_{\mu\alpha v}$ is the Riemann Curvature Tensor, which is a measure of the basic curvature of geometry; a measure that is not transformed away by changing coordinates[2]. The Riemann Curvature Tensor is given by[3-5]

$$R^{\alpha}_{\mu\alpha v} = \Gamma^{\alpha}_{\mu v,\alpha} - \Gamma^{\alpha}_{\mu\alpha,v} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu v} - \Gamma^{\alpha}_{\beta v}\Gamma^{\beta}_{\mu\alpha} \quad (3)$$

where $\Gamma^{\alpha}_{\mu v}$ is the Christoffel symbol of the second kind pseudo tensor.

Let's look at R^{0}_{101}.

Making α = o, μ = 1 and ν = 1,then substituting into equation (3), we obtained

$$R^{0}_{101} = \Gamma^{0}_{11,0} - \Gamma^{0}_{10,1} + \Gamma^{0}_{\beta 0}\Gamma^{\beta}_{11} - \Gamma^{0}_{\beta 0}\Gamma^{\beta}_{10} \quad (4)$$

Summing over all the values of β, we have

$$R^{0}_{101} = \Gamma^{0}_{11,0} - \Gamma^{0}_{10,1} + \Gamma^{0}_{00}\Gamma^{0}_{11} + \Gamma^{0}_{10}\Gamma^{1}_{11} + \Gamma^{0}_{20}\Gamma^{2}_{11} + \Gamma^{0}_{30}\Gamma^{3}_{11} - \Gamma^{0}_{01}\Gamma^{0}_{10} - \Gamma^{0}_{11}\Gamma^{1}_{10} - \Gamma^{0}_{21}\Gamma^{2}_{10} - \Gamma^{0}_{31}\Gamma^{3}_{10} \quad (5)$$

Substituting the Christoffel symbols in equation (5) and differentiating with respect to the coordinate where necessary, we arrived at

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$$R^0_{101} = 0 - \frac{2GM}{c^2r^3} + 0 + \left[\left(-\frac{2GM}{c^2r^3}\right)\left(-\frac{2GM}{c^2r^2}\right)\right] + 0 + 0 - \left[\left(-\frac{2GM}{c^2r^2}\right)\left(-\frac{2GM}{c^2r^2}\right)\right] - 0 - 0 - 0 \quad (6)$$

$$R^0_{101} = -\frac{2GM}{c^2r^3} + \frac{2GM}{c^4r^4} - \frac{2GM}{c^4r^4} \quad (7)$$

$$R^0_{101} = -\frac{2GM}{c^2r^3} \quad (8)$$

Similarly, we can obtain the mathematical results for the other non-zero terms from the calculation of Riemann Tensor and it should be noted that only half the amount of Riemann Tensor will be listed because of the following property:

$$R^\alpha_{\mu\alpha v} = -R^\alpha_{\mu v\alpha} \quad (9)$$

We have;

$$R^0_{101} = -\frac{2GM}{c^2r^3} \quad (10)$$

$$R^0_{202} = \frac{GM}{c^2r} - \frac{G^2M^2}{c^4r^2} \quad (11)$$

$$R^0_{303} = \frac{GM}{c^2r}sin^2\theta - \frac{G^2M^2}{c^4r^3}sin^2\theta \quad (12)$$

$$R^1_{010} = \frac{2GM}{c^2r^3} \quad (13)$$

$$R^1_{212} = 1 - \frac{GM}{c^2r} \quad (14)$$

$$R^1_{313} = -\frac{GMsin^2\theta}{c^2r} \quad (15)$$

$$R^2_{020} = \frac{GM}{c^4r^3} - \frac{G^2M^2}{c^4r^4} \quad (16)$$

$$R^2_{121} = -\frac{GM}{c^2r^3} \quad (17)$$

$$R^2_{323} = \frac{2GMsin^2\theta}{c^2r} - \frac{G^2M^2sin^2\theta}{c^4r^2} - cos^2\theta \quad (18)$$

$$R^3_{030} = \frac{GM}{c^4r^3} - \frac{G^2M^2}{c^4r^4} \quad (19)$$

$$R^3_{131} = -\frac{GM}{c^2r^3} \quad (20)$$

$$R^3_{232} = -\frac{G^2M^2}{c^4r^2} \quad (21)$$

where c is the speed of light; G is the universal constant of gravitation; M is the mass of object and r is the distance away from the object. Comparing (10) – (21) with the well known Riemannian Curvature Tensor calculated from the Schwarzschild Metric Tensor. We now obtained the following results

RESULTS AND DISCUSSION

The Riemannian curvature tensor $R^0_{101}$of the Howusu Metric Tensor is

$$R^0_{101} = -\frac{2GM}{c^2r^3} \quad (22)$$

It can be seen from the (22) that, as $r \to 0$, $R^0_{101} = -\infty$ and as $r \to \infty$, R^0_{101} tends to zero. In the case of the Schwarzschild Metric Tensor, Riemannian curvature tensor R^0_{101} is

$$R^0_{110} = -R^0_{101} = -\frac{2M}{(2M-r)r^2} \quad (23)$$

Clearly, as $r \to 0$, $R^0_{101} = -\infty$ and as $r \to \infty$, R^0_{101} tends to zero and at

$$r = \frac{2GM}{c^2-G} \quad (24)$$

The value of Riemannian curvature tensor R^0_{101} is equal for both Howusu Metric Tensor and Schwarzschild Metric Tensor.

Consider the Riemannian curvature tensor R^0_{202} of the Howusu Metric:

$$R^0_{202} = \frac{GM}{c^2r} - \frac{G^2M^2}{c^4r^2} \quad (25)$$

It is clear from (26) that, as $r \to 0$, $R^0_{202} = 0$ and as $r \to \infty$, R^0_{202} tends to zero. For Schwarzschild Metric Tensor, Riemannian curvature tensor R^0_{202} is

$$R^0_{220} = -R^0_{202} = -\frac{M}{r} \quad (26)$$

Clearly, as $r \to 0$, $R^0_{202} = -\infty$ and as $r \to \infty$, R^0_{202} tends to zero and at$r = \frac{G^2M}{c^2(c^2+G)}$ (27)

The Riemannian curvature tensor R^0_{202} is equal in the Howusu Metric Tensor and the Schwarzschild Metric Tensor.

Consider the Riemannian curvature tensor R^0_{303} of the Howusu Metric Tensor;

$$R^0_{303} = \frac{GM}{c^2r}sin^2\theta - \frac{G^2M^2}{c^4r^3}sin^2\theta \quad (28)$$

It is clear from the (28) that, as $r \to 0$, $R^0_{303} = 0$ and as $r \to \infty$, R^0_{303} tends to zero. On the other hand, for the Schwarzschild Metric Tensor the Riemannian curvature tensor R^0_{303} is

$$R^0_{330} = -R^0_{303} = -\frac{M}{r} sin^2\theta \quad (29)$$

Clearly, as $r \to 0$, $R^0_{303} = -\infty$ and as $r \to \infty$, R^0_{303} tends to zero.

For the Riemannian curvature tensors of the Howusu Metric Tensor and the Schwarzschild Metric Tensor to be equal then

$$r = \sqrt{\frac{G^2M}{c^2(G-c^2)}} \quad (30)$$

The Riemannian curvature tensor R^1_{010} of the Howusu Metric tensor is

$$R^1_{010} = \frac{2GM}{c^2r^3} \quad (31)$$

It is clear from the above equation that, as $r \to 0$, $R^1_{010} = \infty$ and as $r \to \infty$, R^1_{010} tends to zero. For the Schwarzschild Metric Tensor, the Riemannian curvature tensor R^1_{010} is

$$R^1_{010} = \frac{2M(2M-r)}{r^4} \quad (32)$$

Clearly, as $r \to 0$, $R^1_{010} = \infty$ and as $r \to \infty$, R^1_{010} tends to zero. At

$$r = \frac{2Mc^2}{G+c^2} \quad (33)$$

the Riemannian curvature tensor R^1_{010} result in the Howusu Metric Tensor is equal to the Riemannian curvature tensor in the Schwarzschild Metric Tensor.

The Riemannian curvature tensor R^1_{212} of the Howusu Metric tensor

$$R^1_{212} = 1 - \frac{GM}{c^2r} \quad (34)$$

It is clear from (34) that, as $r \to 0$, $R^1_{212} = -\infty$ and as $r \to \infty$, R^1_{212} tends to zero. In the case the Schwarzschild Metric Tensor Riemannian curvature tensor R^1_{212} is

$$R^1_{221} = -R^1_{212} = -\frac{M}{r} \quad (35)$$

Clearly, as $r \to 0$, $R^1_{212} = -\infty$ and as $r \to \infty$, R^1_{212} tends to zero.

To equate the Riemannian curvature tensor of the Howusu Metric Tensor with the Schwarzschild Metric Tensor, then

$$r = \frac{GM}{c^2} - \mathrm{M} \quad (36)$$

The Riemannian curvature tensor R^1_{313} of the Howusu Metric tensor is

$$R^1_{313} = -\frac{GMsin^2\theta}{c^2r} \quad (37)$$

It is clear from (37) that, as $r \to 0$, $R^1_{313} = -\infty$ and as $r \to \infty$, R^1_{313} tends to zero. In the case of the Schwarzschild Metric Tensor, Riemannian curvature tensor R^1_{313} is given by

$$R^1_{331} = -R^1_{313} = -\frac{Msin^2\theta}{r} \quad (38)$$

Clearly, as $r \to 0$, $R^1_{313} = -\infty$ and as $r \to \infty$, R^1_{313} tends to zero.

$$\frac{G}{c^2} = 1 \quad (39)$$

This is the condition for the Riemannian Curvature Tensor R^1_{313} to be the same in the Howusu Metric Tensor and the Schwarzschild Metric Tensor.

Consider the Riemannian curvature tensor R^2_{020} of the Howusu Metric Tensor:

$$R^2_{020} = \frac{GM}{c^4r^3} - \frac{G^2M^2}{c^4r^4} \quad (40)$$

It is clear from (40) that, as $r \to 0$, $R^2_{020} = 0$ and as $r \to \infty$, R^2_{020} tends to zero. For the Schwarzschild Metric Tensor Riemannian curvature tensor R^2_{020}

$$R^2_{020} = \frac{M(-2M+r)}{r^4} \quad (41)$$

Clearly, as $r \to 0$, $R^0_{303} = \infty$ and as $r \to \infty$, R^2_{020} tends to zero.At

$$r = \frac{2M-G^2M}{c^4-G} \quad (42)$$

Equation (42) is the only condition for the Riemannian curvature tensor R^2_{020} to be equal in the Howusu Metric Tensor and the Schwarzschild Metric Tensor.

The Riemannian curvature tensor R^2_{121} of the Howusu Metric Tensor is

$$R^2_{121} = -\frac{GM}{c^2r^3} \quad (43)$$

It is evident from (43) that, as $r \to 0$, $R^2_{121} = -\infty$ and as $r \to \infty$, R^2_{121} tends to zero. One the other hand, for the Schwarzschild Metric Tensor, the Riemannian curvature tensor R^2_{121} is

$$R^2_{121} = \frac{M}{(2M-r)r^2} \quad (44)$$

Clearly, as $r \to 0$, $R^2_{121} = \infty$ and as $r \to \infty$, R^2_{121} tends to zero.At

$$r = \frac{-2GM}{c^2-G} \quad (45)$$

The Riemannian curvature tensor R^2_{121} is equal in the Howusu Metric Tensor and the Schwarzschild Metric Tensor.

Consider the Riemannian curvature tensor R^2_{323} of the Howusu Metric tensor

$$R^2_{323} = \frac{2GMsin^2\theta}{c^2r} - \frac{G^2M^2sin^2\theta}{c^4r^2} - cos^2\theta \tag{46}$$

It is evident from (46) that, as $r \to 0$, $R^2_{323} = -cos^2\theta$ and as $r \to \infty$, R^2_{323} tends to zero. For the Schwarzschild Metric Tensor Riemannian curvature tensor R^2_{323} is

$$R^2_{332} = -R^2_{323} = \frac{2Msin^2\theta}{r} \tag{47}$$

Clearly, as $r \to 0$, $R^2_{323} = \infty$ and as $r \to \infty$, R^2_{323} tends to zero.

Equating the Riemannian curvature tensor of the Howusu Metric Tensor with the Schwarzschild Metric Tensor,

$$r = \frac{-\left(\frac{2GM}{c^2}-2M\right)^2 \pm \sqrt{\left(\frac{2GM}{c^2}-2M\right)^2 - 4\left(\frac{cos^2\theta}{sin^2\theta}\right)\left(\frac{G^2M^2}{c^4}\right)}}{2\left(\frac{cos^2\theta}{sin^2\theta}\right)} \tag{48}$$

Consider the Riemannian curvature tensor R^3_{030} of the Howusu Metric Tensor

$$R^3_{030} = \frac{GM}{c^4r^3} - \frac{G^2M^2}{c^4r^4} \tag{49}$$

It is obvious from (49) that, as $r \to 0$, $R^3_{030} = 0$ and as $r \to \infty$, R^3_{030} tends to zero. In the case the Schwarzschild Metric Tensor, the Riemannian curvature tensor R^3_{030} is

$$R^3_{030} = \frac{M(-2M+r)}{r^4} \tag{50}$$

Clearly, as $r \to 0$, $R^3_{030} = \infty$ and as $r \to \infty$, R^3_{030} tends to zero. If

$$r = \frac{2M-G^2M}{c^4-G} \tag{51}$$

the Riemannian curvature tensor R^3_{030} in the Howusu Metric Tensor will be equal to the Riemannian curvature tensor in the Schwarzschild Metric Tensor.

The Riemannian curvature tensor R^3_{131} of the Howusu Metric is

$$R^3_{131} = -\frac{GM}{c^2r^3} \tag{52}$$

It is clear from (52) that, as $r \to 0$, $R^3_{131} = -\infty$ and as $r \to \infty$, R^3_{131} tends to zero. In the case of the Schwarzschild Metric Tensor, the Riemannian curvature tensor R^3_{131} is

$$R^3_{131} = \frac{M}{(2M-r)r^2} \tag{53}$$

Clearly, as $r \to 0$, $R^3_{131} = -\infty$ and as $r \to \infty$, R^3_{131} tends to zero.

$$r = \frac{2GM}{G-c^2} \tag{54}$$

This is the condition for the Riemannian curvature tensor R^3_{131} to be the same in the Howusu Metric Tensor and the Schwarzschild Metric Tensor.

Consider the Riemannian curvature tensor R^3_{232} of the Howusu Metric

$$R^3_{232} = -\frac{G^2M^2}{c^4r^2} \tag{55}$$

It is clear from (55) that, as $r \to 0$, $R^3_{232} = -\infty$ and as $r \to \infty$, R^3_{232} tends to zero. In the case of the Schwarzschild Metric Tensor, the Riemannian curvature tensor R^3_{232} is

$$R^3_{232} = \frac{2M}{r} \tag{56}$$

Clearly, as $r \to 0$, $R^3_{232} = \infty$ and as $r \to \infty$, R^3_{232} tends to zero. When

$$r = \frac{G^2M}{2c^4} \tag{57}$$

The Riemannian curvature tensor R^3_{232} is the same both in Howusu Metric Tensor and Schwarzschild Metric Tensor.

SUMMARY AND CONCLUSION

In this paper, we formulated the Riemannian Curvature Tensor based on Howusu Metric Tensor for all gravitational fields in nature (10) to (21). We compared the Howusu's Riemannian Curvature Tensor with the well-known Schwarzschild's Riemannian Curvature Tensor and discovered that even though they acted alike at $r \to \infty$ but there was a slight difference and different at $r \to 0$. Even though Howusu's Riemannian Curvature Tensor is different from Schwarzschild's Riemannian Curvature Tensor but it is more generalized because there is no restriction on it unlike the Schwarzschild making it cover more areas. The doors are henceforth open to more verification.

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