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NON-COMMUTATIVE STOPPING TIMES AND TIME PROJECTIONS IN A FILTERED HILBERT SPACE

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Abstract

We present the theory of non-commutative stopping time were the von Neumann algebra is semi-finite that is the von Neumann algebra has a faithful, normal and semi-finite trace. We define a representation mapping and analyze the representation mapping into the existing properties of stopping times and time projection. We also prove the optional stopping theorem.

***Keywords*:** Non-Commutative Stopping Times, Time Projections, Filtered Hilbert Space

1. INTRODUCTION

The theory of quantum probability plays a vital role in the study of operator algebras such as C*-algebra, W*-algebra (also known as von Neumann algebra). An operator $T \in B(H)$ (the space of bounded linear operator (B(H)) acting on a Hilbert space(H)) is a map $B: H \to H$ which is bounded and linear. A von Neumann algebra is a unital sub-algebra of $B(H)$ which is closed with respect to the weak operator topology.

The theory of von Neumann algebras was initiated and developed by J. von Neumann and F. J. Murray in their series of papers [1-4]. Their theories were extensively studied by many authors such as [5] and [6]. Non-commutative stopping time could trace back to the work of [7] where he formulated the concept of stopping time to a non-commutative setting as a projection valued measure.

The "Stopping non-commutative processes" was investigated by [8] where they indicated that the 'formalism' of stopping times carries over to a non-commutative context exist and prove an Optional Stopping Theorem. Following the ideas of [7], a composition operation in the space of stop time to make it a semigroup was introduced by [9], stop time integrals are also introduced and their properties constitute the basic tools for the subject. They imply the strong Markov property of quantum Brownian motion in the boson as well as fermion sense and the Dynkin-Hunt property that the classical Brownian motion begins afresh at each stop time. The stopped Weyl and fermion processes were defined and their properties were studied. Time Projections in a von Neumann algebra was studied by [10]. In their work they characterized some stopped processes in the theory of stopping times and stopping integration within the context of the Clifford Filtration. The work of [8] was further extended by [11] where they proved an analogue of Doob's optional stopping theorem and in the special case of the quasi-free representation of the Canonical Anti-Commutation Relations (CAR) and also proved the random stopping theorem where the underlying von Neumann algebra possesses only a faithful normal state. Random times and their associated time projections within the context of quantum probability theory were discussed by [12]. A stochastic integral representation for time projections was obtained, and their order structure was investigated. Random times, predictable processes and stochastic integration in finite von Neumann algebra were discussed by [13] where they offered a viewpoint for the theory of non-commutative stochastic processes. In support of this view, they considered random times and random stopping as a departure point and constructed a class of predictable processes using random times. They defined various stochastic integrals of these predictable processes and prove some elementary result on a finite von Neumann algebra. Quantum stopping times, quantum stochastic interval, stopping quantum L^1-process by quantum stopping times and the relationship between stopping and Doob-Meyer decomposition of the squares of quantum martingales were discussed by [14].

2. Preliminaries

In this section, we present preliminary results, definitions and fundamentals of von Neumann algebra.

Definition 2.1 Let H be a separable Hilbert space over the complex field ℂ. The identity operator on H is denoted by I_H, or I simply if no confusion arises. We introduce the following locally convex topologies in B(H):

1. uniform (operator) topology: It is the topology generated by the operator norm $\|.\|$.

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2. weak (operator) topology: If a net $a_i \to 0$, then
$\langle a_i\xi, \eta\rangle \to 0\ \forall\ \xi,\eta \in$ H;
3. strong (operator) topology: If a net $a_i \to 0$, then
$\|a_i\xi\| \to 0\ \forall\ \xi \in$ H;
4. σ-weak (operator) topology: If a net $a_i \to 0$, then

$$\sum_n \langle a_i\xi_n, \eta_n\rangle \to 0,\ \forall\ \sum_n(\|\xi_n\|^2 + \|\eta_n\|^2) < \infty;$$

5. σ-strong (operator) topology: If a net $a_i \to 0$, then
$\sum_n \|a_i\xi_n\|^2 \to 0\ \forall\ \sum_n \|\xi_n\|^2 < \infty$.

Definition 2.2 A von Neumann algebra M is a unital self-adjoint sub-algebra of B(H) which is closed in the weak operator topology. That is, I ∈ M, $a = a*$ and the net $a_i \to a$ in the weak operator topology if $\langle a_i\xi, \eta\rangle \to \langle a\xi, \eta\rangle\ \forall\ \xi,\eta \in$ H. Alternatively, a * sub-algebra M of B(H) is called a von Neumann algebra, if $M = M''$,
where M'' = {b ∈ B(H) : ba = ab, ∀a ∈ M}is the commutant of M, and $M'' = (M')'$ is the double commutant of M. Note that M' is also a von Neumann algebra.

Definition2.3 Let H be a Hilbert space. A bounded operator P: H → H is called a projection if
$P^* = P^2 = P$.

Definition 2.4 Let M be a von Neumann algebra and let w: M → k be a linear functional on M. Then w is said to be
i. positive, if w(a) ≥ 0 for any $a \in M^+$ (where $M^+ = \{a \in M \mid \geq 0\}$ the set of all positive elements of M), and is denoted by w ≥ 0,. Moreover, for two linear functionals w, ψ on M, the relation ψ ≤ w means that (w − ψ) ≥ 0. A positive linear functional w is said to be:
ii. faithful, if w(a) = 0 for some $a \in M^+$, then we have a = 0.
iii. normal, if for any bounded increasing net $\{a_i\} \subset M^+$, we have $\sup_i$ w(a) = w($\sup_i a$).
iv. state, if w(I) = 1.
v. trace, if w($a*a$) = w($aa*$) ∀$a \in M^+$.
vi. weight, if w(λa + b) = λw(a) + w(b), ∀λ ≥ 0, a,b ∈ M^+
Consider the set M^+w = {x ∈ M^+ | w(x) < ∞}, We say that a linear functional ω is semi-finite if M^+w is weakly dense in M^+. In addition, a linear functional ω on M is called a normal state (on M), if it is positive, normal and state.

Definition 2.5 A von Neumann algebra M is said to be:
i. σ-finite, if M has a faithful, normal and state.
ii. semi-finite, if M has a faithful, normal and semi-finite trace.
iii. finite, if M has a faithful, normal and tracial state.

Definition 2.6 Let M be a von Neumann algebra on a Hilbert space H. A vector Ω ∈ H is cyclic for M if the set {aΩ : a ∈ M} is dense in H and Ω ∈ H is separating for M if for all $a \in M$, aΩ = 0 for all Ω ∈ H implies a = 0.

Definition 2.7 Let M be a von Neumann algebra and π be a map from M to B(H), then (π, H) is called a *-representation of M if:
i. π(λa + μb) = λπ(a) + μπ(b), ∀ a, b ∈ M, λ,μ ∈C
ii. π(ab) = π(a)π(b), ∀a, b ∈ M
iii. π($a*$) = π(a)*, ∀a ∈ M
For any positive linear functional w on M, we get a *-representation (π_w, H_w) of M and this representation admits a cyclic vector Ω_w that is $\pi_w(M)\Omega_w = H_w$ such that $\omega(a) = < \pi_w(a)\Omega_w, \Omega_w >$ ∀a ∈ M. This is called the GNS construction.

Definition 2.8 Let N be a von Neumann sub-algebra of M. P is called a projection of norm one from M onto N, if P is linear, PM = N, Pb = b, ∀b ∈ N, and $\|Pa\| \leq \|a\|$, ∀a ∈ M.
That is, by a projection of norm one we mean a projection mapping from a Banach space onto its subspace whose norm is one.

Definition 2.9 Let N be a von Neumann sub-algebra of M. A conditional expectation mapping E from M to N is positive contraction E: M → N such that E(b) = b and such that E(bx) = bE(x) and E(xb) = E(x)b ∀ x ∈ M, b ∈ N (E is N-linear).
Therefore, a conditional expectation is a self-adjoint idempotent map from M to M (a projection onto N) of norm one.
The conditional expectation does not always exists, [15] presents a condition for the existence of a conditional expectations as follows

Theorem 2.10 (Takesaki (1972), Theorem 1) Let M be a von Neumann algebra and ω is faithful, semi-finite, normal weight on M+. Let N be a von Neumann sub-algebra of M on which ω is semifinite. Then the following two statements are equivalent:

i. N is invariant under the modular automorphism group σt associated with ω;
ii. There exists a σ-weakly continuous faithful projection E of norm one from M onto N such that
w(x) = w ∘ E(x) ∀x ∈ M
The projection E of norm one of M onto N is called the conditional expectations of M onto N with respect to ω.
Definition 2.11 For each non-negative t, let $\{\overline{M_t\Omega}^{\|\cdot\|} = H_t\}$ be the family of Hilbert subspaces of $\overline{M\Omega}^{\|\cdot\|} = H$ such that
i. if t, s ∈ $\overline{\mathbb{R}}^+$ with s ≤ t then $H_s \subseteq H_t$
ii. $(\cup_{t\geq 0} H_t)^{II} = H$
iii. iii. $\cap_{t>s} H_t = H_s$
From the definition above, the family $\{H_t\}_{t\in\overline{\mathbb{R}}^+}$ is called a filtered Hilbert space.

3. Main Result

In this section, we present an analysis of a representation mapping and thereby introduce the representation mapping to stopping times and time projections.
Let H be a separable Hilbert space and let B(H) be the space of bounded linear operators on H, and let $M \subseteq$ B(H) a von Neumann algebra with a faithful normal state ω and with cyclic and separating vector Ω in H such that ω(x) = $<$ x$\Omega, \Omega >$, ∀x ∈ M. For each non-negative t, let $\{M_t : t \in \overline{\mathbb{R}}^+\}$ be the family of von Neumann sub-algebra of M such that
i. if t, s ∈$\overline{R^+}$ with s ≤ t then $M_s \subseteq M_t$
ii. the von Neumann algebra M is generated by $\cup_{t\geq 0} M_t$ that is $\overline{\cup_{t\geq 0} M_t}^{\|\cdot\|} = M$
iii. $\cap_{t>s} M_t = M_t$
Finally, suppose there exists the family of faithful normal conditional expectations $\{E_t: t \in \overline{\mathbb{R}}^+\}$ from M onto M_t such that
iv. ω ∘ E_t = ω ∀ t ∈ $\overline{\mathbb{R}}^+$
v. $E_t(ax\text{b}) = aE_t(x)\text{b}$ ∀ a, b ∈ M_t, x ∈M
vi. E_t (I) = I
Now we can define an inner product on M by
$< a$, b $>$ = ω(b* a) ∀ a, b ∈ M with $\sqrt{<x,x>} = \|x\|_2,\ x \in M$
$(M, <.,\ .>)$ is a pre-Hilbert space and we denotes its closure with respect to $\|.\|_2$ by $L^2(M) = \overline{M\Omega}^{\|\cdot\|_2} = H$. In a similar way, we can construct $L^2(M_t)$ and so $L^2(M_t) \subseteq L^2(M)$.
The map π: M → B($L^2(M)$) defined by π(a).b = a b for a ∈ M b ∈ $L^2(M)$ and
π(a)b = $\lim_{n\to\infty} ab_n \in L^2(M)$, for $b_n \to b \in L^2(M)$ is a well defined homomorphism since
π(a + b).c = (a + b)c
= a c + a b
= π(a).c + π(b).c
⇒ π(a + b) = π(a) + π(b)
and
π(ab).c = (ab).c
= a(bc)
= π(a)(bc)
= π(a)π(b).c
⇒ π(ab) = π(a)π(b)
Furthermore, π is normal positive map, suppose $a_i \to a$, then by definition of π, we have
π(a_i).b = a_ib → ab = π(a)b (1)
and so (1) imply that we have π(a) = π($\sup_i a_i$) = $\sup_i$(π(a_i))
making π normal. Finally π is injective since π(a) = 0 ⇒ π(a).I = 0 ⇒ a = 0
From the above properties, we also observe that π(M) is a von Neumann algebra using the fact that the unit ball of π(M) is ultra-weakly compact. So π(M) becomes an isometric copy of M. Furthermore, I is a cyclic and separating vector for π(M) since {π(a).I : $a \in M$ } = M is dense in $L^2(M)$, and π(a)I = 0 ⇒ a = 0 ⇒ π(a) = 0
Remark 3.1 Let $\{E_t, t \in \overline{\mathbb{R}}^+\}$ be a family conditional expectation from M onto M_t. Then a map P_t defined on $L^2(M)$ by $P_t(a\Omega) = E_t(a)\Omega$, For each t ∈ $\overline{\mathbb{R}}^+$is a projection from $L^2(M)$ onto $L^2(M_t)$ = $\overline{M_t\Omega}^{\|\cdot\|}$ and ∀ $a \in M$ and P_t lies in the commutant of π(M).
To see this, we need to show that $P_t^2 = P_t$ and $P_t^* = P_t$.
Now

$P_t^2(a\Omega) = P_t(P_t\ (a\Omega))$
$= P_t(E_t(a)\Omega)$
Since $E_t(a)\Omega = \mathrm{a}a\Omega \in L^2(M_t) = \overline{M_t\Omega}^{\|\cdot\|}$. We have
$P_t(E_t(a)\Omega) = P_t(a\Omega) \Rightarrow P_t^2(a\Omega) = P_t(a\Omega)$
and so P_t is an idempotent.
Also for a,b ∈ M we have
$\langle P_t(\mathrm{a}\Omega), \mathrm{b}\Omega\rangle = \langle P_t(\mathrm{a}\Omega), P_t(\mathrm{b}\Omega)\ +\ (1 - P_t)(\mathrm{b}\Omega)\rangle$
$= \langle P_t(\mathrm{a}\Omega), P_t(\mathrm{b}\Omega)\rangle + \langle P_t(\mathrm{a}\Omega), (1 - P_t)(\mathrm{b}\Omega)\rangle$
Since $P_t\ (\mathrm{a}\Omega) \in L^2(M_t)$ then $(1 - P_t)(\mathrm{b}\Omega) \in L^2(M_t)^{\perp} \Rightarrow \langle P_t(\mathrm{a}\Omega), (1 - P_t)(\mathrm{b}\Omega)\rangle = 0$ and so we have $\langle P_t(\mathrm{a}\Omega), \mathrm{b}\Omega\rangle = \langle P_t(\mathrm{a}\Omega), P_t(\mathrm{b}\Omega)\rangle$ (2)
again
$\langle \mathrm{a}\Omega, P_t(\mathrm{b}\Omega)\rangle = \langle P_t(\mathrm{a}\Omega)\ +\ (1 - P_t)(\mathrm{a}\Omega), P_t(\mathrm{b}\Omega)\rangle$
$= \langle P_t(\mathrm{a}\Omega), P_t(\mathrm{b}\Omega)\rangle + \langle (1 - P_t)(\mathrm{a}\Omega), P_t(\mathrm{b}\Omega)\rangle$
also $\langle (1 - P_t)(\mathrm{a}\Omega), P_t(\mathrm{b}\Omega)\rangle = 0$
$\langle \mathrm{a}\Omega, P_t(\mathrm{b}\Omega)\rangle = \langle P_t(\mathrm{a}\Omega), P_t(\mathrm{b}\Omega)\rangle$ (3)
and so from 2 and 3 we have that $\langle P_t(\mathrm{a}\Omega), \mathrm{b}\Omega\rangle = \langle \mathrm{a}\Omega, P_t(\mathrm{b}\Omega)\rangle$
and since $\langle P_t(\mathrm{a}\Omega), \mathrm{b}\Omega\rangle = \langle \mathrm{a}\Omega, P_t^*(\mathrm{b}\Omega)\rangle$
$\Rightarrow P_t^* = P_t$ and so P_t is self-adjoint and hence, P_t an orthogonal projection onto L2(Mt).
Now we show that P_t lies in the commutant of π(M). To see this, we have
$P_t\ (\mathrm{xy}\Omega) = \mathrm{Et(xy)}\Omega,\quad \forall\ \mathrm{x} \in \pi(\mathrm{M}),\ \mathrm{y} \in \pi(\mathrm{M})$
$= \mathrm{Et(x)y}\Omega$
$= \mathrm{yEt(x)}\Omega$
$= \mathrm{y}P_t(\mathrm{x}\Omega)$
Since MΩ is dense in H, then the result follows.
Here, we use the representation mapping to define stopping times, time projections and analyse some of their properties.

Definition 3.2 A stopping time, τ, adapted to the filtration of von Neumann algebras $(\pi(M_t))_{t\in\mathbb{R}^+}$ is an increasing family $\{\pi(q_t)\}_{t\in\mathbb{R}^+}$ of projections in M such that:

i. $\tau(t) = \pi(q_t) \in \pi(M_t)^{proj}$
ii. $\tau(0) = \pi(q_0) = 0$
iii. $\tau(\infty) = \pi(q_\infty) = I$

Definition 3.3 Let $\tau = \pi(q_t)_{t\in\mathbb{R}^+}$ and $\sigma = \pi(r_t)_{t\in\mathbb{R}^+}$ be stopping times. We can define an order $\tau \le \sigma \Leftrightarrow \pi(r_t) \le \pi(q_t)$. We define $\tau \wedge \sigma = \pi(q_t) \vee \pi(r_t)$ and $\tau \vee \sigma = \pi(q_t) \wedge \pi(r_t)$.
Let Θ denote the set of all finite partitions of [0, ∞]. Then for θ ∈ Θ say $\theta = \{t_0,..., t_n\}$, we define an operator $P_{\tau(\theta)}$ on H as

$$P_{\tau(\theta)} = \sum_{i=1}^{n} \pi\ (q_{t_i} - q_{t_{i-1}})P_{t_i} = \sum_{i=1}^{n} \Delta\pi\ (q_{t_i})P_{t_i} \quad (4)$$

Theorem 3.4 Let $\tau(t) = \pi(q_t)$ be a stopping time. Then

(i) $P_{\tau(\theta)}$ is an orthogonal projection
(ii) if $\theta_1, \theta_2 \in \Theta$ with $\theta_2 \supseteq \theta_1$ then $P_{\tau(\theta_2)} \le P_{\tau(\theta_1)}$
(iii) if $\sigma = \pi(r_t)$ is another stopping time with $\tau \le \sigma$ then $P_{\tau(\theta)} \le P_{\sigma(\theta)}$

Proof.
(i) Let $\tau = \pi(q_t)$, ξ ∈ H and $\theta = \{t_0, t_1, . . . , t_n\} \in \Theta$ then

$$P_{\tau(\theta)} \circ P_{\tau(\theta)}\ (\xi) = \sum_{j=1}^{n} \Delta\pi\ \left(q_{t_j}\right) P_{t_j}\ (\ \sum_{i=1}^{n} \Delta\pi\ (q_{t_i})P_{t_i})\ (\xi) \quad \text{(by 4)}$$
$$= \sum_{j=1}^{n} \sum_{i=1}^{n} \Delta\pi\ \left(q_{t_j}\right) P_{t_j}\ \Delta\pi\ (q_{t_i})P_{t_i}\ (\xi)$$
$$= \sum_{j-1}^{n} \sum_{i=1}^{n} P_j\ \Delta\pi\ (q_{t_j})\ \Delta\pi\ (q_{t_i})P_{t_i}\ \xi$$
$$= \sum_{i=1}^{n} P_{t_i}\ \Delta\pi\ (q_{t_i})\xi$$
$$= P_{\tau(\theta)}\ (\xi)$$

Since $\Delta\pi\ (q_{t_j})\ \Delta\pi\ (q_{t_i}) = 0$ for i ≠ j, then we have
$P_{\tau(\theta)} \circ P_{\tau(\theta)} = P_{\tau(\theta)}$ that is $P^2{}_{\tau(\theta)} = P_{\tau(\theta)}$. Thus $P_{\tau(\theta)}$ is an idempotent.
Also for ξ, η ∈ H, we have
$< P_{\tau(\theta)}\eta, \xi > = < \sum_{i=1}^{n} \Delta\pi\ (q_{t_i})P_{t_i}\eta, \xi >$

$= \sum_{i=1}^{n} < \eta, P_{t_i}\Delta\pi\,(q_{t_i})\xi >$
$= \sum_{i=1}^{n} < \eta, \Delta\pi\,(q_{t_i})P_{t_i}\xi >$
$= < \eta, \sum_{i=1}^{n} P_{t_i}\,\Delta\pi\,(q_{t_i})\xi >$
$= < \eta,\ P_{\tau(\theta)}\xi >$

that is $< P_{\tau(\theta)}\eta, \xi > = < \eta, P_{\tau(\theta)}\xi >$ (5)

but

$< P_{\tau(\theta)}\eta, \xi > = < \eta, P^{*}{}_{\tau(\theta)}\xi >$ (6)

From (5) and (6) we see that $P^{*}{}_{\tau(\theta)} = P_{\tau(\theta)}$. Hence, $P_{\tau(\theta)}$ is a self-adjoint establishing (i).

(ii) Suppose $\theta_2 = \theta_1 \cup \{s\}$ where $\theta_1 = \{t_0 < \ldots t_n\}$ with $t_i < t_{i+1}$ and $s \in \{t_r, t_{r+1}\}$, $r + 1 < n$. Then for $\xi \in H$,

$$P_{\tau(\theta_1)} \circ P_{\tau(\theta_2)}(\xi) = \sum_{j=1}^{n} \Delta\pi\left(q_{t_j}\right)P_{t_j}\left(\sum_{i=1}^{r}\Delta\pi\,(q_{t_i})P_{t_i}(\xi)\right)$$
$$+ \sum_{j=1}^{n}\Delta\pi\left(q_{t_j}\right)P_{t_j}\ \pi\,(q_{t_s} - q_{t_r})P_{t_s}(\xi)$$
$$+ \sum_{j=1}^{n}\Delta\pi\left(q_{t_j}\right)P_{t_j}\ \pi\,(q_{t_{r+1}} - q_{t_s})P_{t_{r+1}}(\xi)$$
$$+ \sum_{j=1}^{n}\Delta\pi\left(q_{t_j}\right)P_{t_j}\left(\sum_{j=1}^{n}\Delta\pi\left(q_{t_j}\right)P_{t_j}(\xi)\right)$$

Using the fact that P_t lies in the commutant of $\pi(M)$ and the orthogonality of $\Delta\pi(q_{t_j})$ and $\Delta\pi(q_{t_i})$ and for $\theta_1 \subseteq \theta_2$ we have

$$P_{\tau(\theta_1)} \circ P_{\tau(\theta_2)}(\xi) = \sum_{j=1}^{n}\Delta\pi\left(q_{t_j}\right)P_{t_j}(\xi) + \pi\left(q_{t_s} - q_{t_r}\right)P_{t_s}(\xi)$$
$$+ \pi\left(q_{t_{r+1}} - q_{t_s}\right)P_{t_{r+1}}(\xi) + \sum_{j=1}^{n}\Delta\pi\left(q_{t_j}\right)P_{t_j}(\xi)$$
$$= P_{\tau(\theta_2)}(\xi) \Rightarrow P_{\tau(\theta_1)} \circ P_{\tau(\theta_2)} = P_{\tau(\theta_2)}$$

and so $P_{\tau(\theta_2)} \leq P_{\tau(\theta_1)}$

$\Rightarrow (P_{\tau(\theta)})\ \theta \in \Theta$ is a decreasing net of orthogonal projections and so, the infimum exists and let $P_\tau = \inf_{\theta\in\Theta} P_{\tau(\theta)}$.

Definition 3.5 For a stopping time $\tau = (\pi(q_t))_{t\in\overline{\mathbb{R}}^{+}}$ we can define the time projection at τ, P_τ, as

$P_\tau = \inf_{\theta\in\Theta} \sum_{i=1}^{n}\Delta\pi\,(q_{t_i})P_{t_i}(\xi) = \inf_{\theta\in\Theta}P_{\tau(\theta)}$

(iii) Given $\sigma \geq \tau$, let $\sigma = \pi(r_t)$. So that $\pi(r_t) \leq \pi(q_t)$ for each $t \in \overline{\mathbb{R}}^{+}$. Let $\theta \in \Theta$ be as in (ii) above say, then

$\sum_{i=1}^{n}\Delta\pi\,(q_{t_i}) = \sum_{i=1}^{n}\pi\,(q_{t_i} - q_{t_{i-1}})$
$= \pi\left(q_{t_1}\right) - \pi(q_{t_0}) + \pi\left(q_{t_2}\right) - \pi(q_{t_1}) + \pi(q_{t_3}) - \pi(q_{t_2}) + \pi(q_{t_4}) - \pi(q_{t_3})$
$+ \ldots + \pi\left(q_{t_{n-3}} - \pi(q_{t_{n-2}}\right) + \pi\left(q_{t_{n-1}}\right) - \pi(q_{t_{n-2}}) + \pi\,(q_{t_n}) - \pi(q_{t_{n-1}})$
$= \pi\,(q_{t_n}) - \pi(q_{t_0})$
$= I - 0$ where $\pi(q_{t_0}) = 0$ and $\pi(q_{t_n}) = I$
$= I$

and so we have

$I = \sum_{i=1}^{n}\Delta\pi\,(q_{t_i}) = \sum_{i=1}^{n}\Delta\pi\,(r_{t_i})$

Now $P_{\tau(\theta)} \circ P_{\sigma(\theta)} = \sum_{i=1}^{n}\Delta\pi\,(q_{t_i})\,P_{t_i}(\sum_{j=1}^{n}\Delta\pi\,(r_{t_j})P_{t_j})$

But $\Delta\pi\left(q_{t_i}\right)P_{t_i}\sum_{j=1}^{n}\Delta\pi\,(r_{t_j})P_{t_j} = \sum_{j=1}^{n}\Delta\pi\left(q_{t_i}\right)P_{t_i}\Delta\pi\left(r_{t_j}\right)$

from the fact that P_t lies in the commutant of $\pi(M)$ and observing that $\Delta\pi(r_{t_j}) \leq \Delta\pi(q_{t_i})$ for $j \leq i - 1$, we see that

$$P_{\tau(\theta)} \circ P_{\sigma(\theta)} = \sum_{i=1}^{n}\sum_{j=1}^{n}\Delta\pi\left(q_{t_i}\right)P_{t_i}\,\Delta\pi\,(r_{t_j})P_{t_j}$$
$$= \sum_{i=1}^{n}(\sum_{j=1}^{n}\Delta\pi\left(q_{t_i}\right)P_{t_i}\,\Delta\pi\left(r_{t_j}\right) + \Delta\pi\left(q_{t_i}\right)P_{t_i}(I - \sum_{k=1}^{n}\Delta\pi\,(r_{t_k}))$$
$$= \sum_{i=1}^{n}\Delta\pi\,(q_{t_i})\,P_{t_i}$$
$$= P_{\tau(\theta)}$$

Hence $P_{\tau(\theta)} \leq P_{\sigma(\theta)}$ establishing (iii)

Theorem 3.6 Let $\tau = \pi(q_t)$ and $\sigma = \pi(r_t)$ be stopping times, and let $\theta \in \Theta$. Then

$P_{(\sigma\vee\tau)(\theta)} = P_{\sigma(\theta)} \vee P_{\tau(\theta)}$ and $P_{(\sigma\wedge\tau)(\theta)} = P_{\sigma(\theta)} \wedge P_{\tau(\theta)}$

Proof. Suppose that $\theta = \{t_0, t_1, \ldots, t_n\} \in \Theta$, then for any $\xi \in H$, we have

$P^{\perp}{}_{\sigma(\theta)}\xi = \sum_{i=1}^{n}\pi\,(r_{t_i})\,\Delta P_{t_i}\xi$
$= \sum_{i=1}^{n}\pi\left(r_{t_i}\right)(P_{t_i} - P_{t_{i-1}})\xi$

and hence

$P^{\perp}{}_{\tau(\theta)} \circ P^{\perp}{}_{\sigma(\theta)}\xi = \sum_{j=1}^{n}\pi\left(q_{t_j}\right)\Delta P_{t_j}(\sum_{i=1}^{n}\pi\,(r_{t_i})\,\Delta P_{t_i}\xi)$
$= \sum_{j=1}^{n}\sum_{i=1}^{n}\pi\left(q_{t_j}\right)\Delta P_{t_j}\,\pi\,(r_{t_i})\,\Delta P_{t_i}\xi$

$= \sum_{j=1}^{n} \sum_{i=1}^{n} \pi(q_{t_j}) \Delta E_{t_j} (\pi (r_{t_i})) \Delta P_{t_j} \Delta P_{t_i} \xi$

Where $\Delta E_{t_j} = E_{t_j} - E_{t_{j-1}}$ and the conditional expectation is $E_t = \sum_{j=1}^{n} \pi(r_{t_j}) \pi(q_{t_j}) \Delta P_{t_j} \xi$

since $\Delta P_{t_i} \Delta P_{t_j} = 0$ for $i \neq j$ It follows that

$(P^{\perp}{}_{\tau(\theta)} \circ P^{\perp}{}_{\sigma(\theta)})^k \xi = \sum_{j=1}^{n} \left(\pi(r_{t_j}) \pi(q_{t_j})\right)^k \Delta P_{t_j} \xi$

for any k=1, 2, ...

Letting $k \to \infty$, we obtain

$P^{\perp}{}_{\tau(\theta)} \wedge P^{\perp}{}_{\sigma(\theta)} \xi = \sum_{j=1}^{n} (\pi(r_{t_j}) \wedge \pi(q_{t_j})) \Delta P_{t_j} \xi$

$= P_{(\tau \vee \sigma)(\theta)} \xi$

Taking the orthogonal complements, we see that

$P_{\tau(\theta)} \vee P_{\sigma(\theta)} = P_{(\tau \vee \sigma)(\theta)}$.

For the infimum, we begin with

$P_{\sigma(\theta)} \xi = P_0 \xi + \sum_{i=1}^{n} \pi(r_{t_i}{}^{\perp}) \Delta P_{t_i} \xi$

As above we see that

$(P_{\tau(\theta)} \circ P_{\sigma(\theta)})^k \xi = \sum_{i=1}^{n} \left(\pi(q_{t_i}{}^{\perp}) \pi(r_{t_i}{}^{\perp})\right)^k \Delta P_{t_i} \xi$

Letting $k \to \infty$, we get

$P_{\tau(\theta)} \wedge P_{\sigma(\theta)} = P_{(\tau \wedge \sigma)(\theta)}$

Theorem 3.7 (optional stopping) For stopping times τ, σ with $\tau \leq \sigma$, we have $P_\tau \leq P_\sigma$.

Proof.

From theorem 3.4 (iii) $P_{\tau(\theta)} \leq P_{\sigma(\theta)}$ ∀θ. The result now follows by taking the limit as θ refines.

Corollary 3.8 For any stopping times τ, σ we have $P_\tau \wedge P_\sigma = P_{(\tau \wedge \sigma)}$.

Proof. we have $\tau \wedge \sigma \leq \tau$ and so $P_{(\tau \wedge \sigma)} \leq P_\tau$ (by optional stopping theorem).

Similarly, $P_{(\tau \wedge \sigma)} \leq P_\sigma$. Hence $P_{(\tau \wedge \sigma)} \leq P_\tau \wedge P_\sigma$. On the other hand, for any $\theta \in \Theta$,

$P_{\tau(\theta)} \wedge P_{\sigma(\theta)} = P_{(\tau \wedge \sigma)(\theta)}$ (By theorem 3.5)

Hence

$P_\tau \wedge P_\sigma \leq P_{\tau(\theta)} \wedge P_{\sigma(\theta)} = P_{(\tau \wedge \sigma)(\theta)} \quad \forall \theta \in \Theta$

giving

$P_\tau \wedge P_\sigma \leq P_{(\tau \wedge \sigma)}$

Conclusion

In conclusion, we present the theory of non-commutative stopping time and time projection on a filtered Hilbert space where we define our representation mapping on the filtered Hilbert space using the general definition of filtration and thereby introduce the representation mapping on stopping times and time projection. We also prove the optional stopping theorem.

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