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NEW RESULTS ON A CERTAIN CLASS OF ANALYTIC FUNCTIONS

Folashade Mistura Jimoh[1] and Mohammed Umar Faruq[2]

[1]Department of Physical Sciences, Al-Hikmah University, Ilorin, Kwara State, Nigeria
[2]Department of Computer Science, Kwara State College of Arabic and Islamic Studies, Ilorin, Kwara State, Nigeria

Abstract

The class $J_n^\alpha(\beta)$ is a generalized class of analytic functions that was first studied in [1]. In that paper, it was proved that analytic functions belonging to $J_n^\alpha(\beta)$ are univalent for $n \geq 1$. Results on sufficient inclusion criteria for functions to be in $J_n^\alpha(\beta)$ were also establised. In this article, we provide more characterizations for the class $J_n^\alpha(\beta)$ of analytic functions. Specifically, using the well-known Jack's Lemma and some properties of functions with positive real part, we obtain another sufficient inclusion result for functions to be in $J_n^\alpha(\beta)$, angular estimates and sufficient conditions for analytic functions $f(z)$ to be strongly starlike of order β and strongly convex functions of order β in the open unit disk are also derived.

***Keywords*:** Analytic functions, univalent functions, angular estimates, strongly starlike and strongly convex functions.

1. Introduction

Let E be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and A the class of analytic functions in E, which have the form $f(z) = z + a_2 z^2 + \ldots$

A function $f \in A$ is called starlike of order β if and only if it satisfies the inequality

$$Re\,\frac{zf'(z)}{f(z)} > \beta,\ 0 \leq \beta < 1,\ z \in E.$$

This class is often denoted by $S^*(\beta)$.

If $f \in A$ satisfies

$$\left| arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\beta,\quad z \in E,$$

then f is said to be stongly starlike. The class of strongly starlike functions f is denoted by $\bar{S}^*(\beta)$.

Furthermore, a function $f \in A$ is convex of order β denoted by $C(\beta)$ if and only if it satisfies the condition

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta,\ 0 \leq \beta < 1,\ z \in E.$$

The Class $\bar{C}(\beta)$ is the class of strongly convex function and consists of analytic functions satisfying the inequality

$$\left| arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\beta,\ z \in E.$$

In [1], the class $J_n^\alpha(\beta)$ of analytic functions satisfying the condition

$$Re\left(\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1} f(z)}{D^n f(z)}\right) > \beta \qquad (1)$$

Correspondence Author: Folashade M.J., Email: Folashittu@yahoo.com, Tel: +2348033607932

where D^n is the Salagean derivative and β is a real number such that $0 \le \beta < 1$ was investigated and shown to consist of univalent functions for $n \ge 1$.
In this work, we obtain an inclusion result for functions to be in the class $J_n^\alpha(\beta)$. This was achieved by considering a subclass of $J_n^\alpha(\beta)$ of analytic function $f(z)$ satisfying

$$\left|\frac{D^n f(z)^n}{\alpha^n z^\alpha}\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right| < 1-\beta. \quad (2)$$

Clearly, if $f(z)$ satisfies (2) then

$$\beta < \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1} f(z)}{D^n f(z)} < 2-\beta$$

which in turn implies that $f(z)$ satisfies (1). An angular estimate for functions in the class $J_n^\alpha(\beta)$ was also obtained.

2. Preliminary Lemmas

Lemma 1. *[2] [Jack's Lemma] Let the function* $w(z)$ *be analytic in the open unit disk* E *with* $w(0) = 0$ *and* $|w(z) < 1|\,(z \in E)$. *Then if* $|w(z)|$ *attains its maximum on the circle* $|z| = r < 1$ *at a point* $z_0 \in E$, *we have*

$$z_0 w'(z_0) = k w(z_0)$$

where $k \ge 1$ is a real number.

Lemma 2. *[3] Let* $f(z) \in A$, *and* $\alpha > 0$ *be real. If* $\frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha}$ *takes a value which is independent of* n, *then*

$$\frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha} = \alpha\frac{D^{n+1} f(z)}{D^n f(z)}.$$

Lemma 3. *[4] Let the function* $p(z)$ *be analytic in* E *with* $p(0) = 1$ *and* $p(z) \ne 0$ $(z \in E)$. *If there exists a point* $z_0 \in E$ *such that*

$$|arg\ p(z)| < \frac{\pi}{2}\beta \text{ for } |z| < |z_0|$$

and

$$|arg\ p(z_0)| = \frac{\pi}{2}\beta$$

with $0 < \beta$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta$$

where

$$p(z_0)^{\frac{1}{\beta}} = \pm ai.$$

and

$$k\begin{cases} \ge \frac{1}{2}\left(a + \frac{1}{a}\right) \ge 1, & \text{when } arg\ p(z_0) = \frac{\pi}{2}\beta, \\ \le -\frac{1}{2}\left(a + \frac{1}{a}\right) \le -1, & \text{when } arg\ p(z_0) = -\frac{\pi}{2}\beta. \end{cases}$$

3. Main Results

Theorem 1. *If* $f(z) \in A$ *satisfies*

$$\left|\frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha} + \frac{D^{n+2} f(z)}{D^{n+1} f(z)} - \frac{D^{n+1} f(z)}{D^n f(z)}\right| < \frac{(\alpha+1)(1-\beta)}{2-\beta}, \quad (3)$$

then $f(z) \in J_n^\alpha(\beta)$.

Proof. Let

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1} f(z)}{D^n f(z)} = 1 + (1-\beta)w(z), \quad (w(z) \neq 1) \qquad (4)$$

Then $w(z)$ is analytic in the unit disk E and $w(0) = 0$. Differentiating (4) logarithmically we obtain

$$\frac{D^{n+1} f(z)^\alpha}{zD^n f(z)^\alpha} + \frac{D^{n+2} f(z)}{zD^{n+1} f(z)} - \frac{D^{n+1} f(z)}{zD^n f(z)} - \frac{\alpha}{z} = \frac{(1-\beta)w'(z)}{1+(1-\beta)w(z)}$$

so that

$$\frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha} + \frac{D^{n+2} f(z)}{D^{n+1} f(z)} - \frac{D^{n+1} f(z)}{D^n f(z)} = \frac{\alpha + \alpha(1-\beta)w(z) + (1-\beta)zw'(z)}{1+(1-\beta)w(z)}.$$

Suppose there exists z_0 in E such that

$$\max_{|z|<|z_0|} |w(z)| = |w(z_0)| = 1,$$

then by Lemma 1, we have $z_0 w'(z_0) = kw(z_0)$, so that

$$\left|\frac{D^{n+1} f(z_0)^\alpha}{D^n f(z_0)^\alpha} + \frac{D^{n+2} f(z_0)}{D^{n+1} f(z_0)} - \frac{D^{n+1} f(z_0)}{D^n f(z_0)}\right| = \left|\frac{\alpha + \alpha(1-\beta)w(z_0) + (1-\beta)kw(z_0)}{1+(1-\beta)w(z_0)}\right|$$

$$= \left|\frac{\alpha + (\alpha+k)(1-\beta)w(z_0)}{1+(1-\beta)w(z_0)}\right|$$

$$\geq \left|\frac{(\alpha+1)(1-\beta)w(z_0)}{1+(1-\beta)w(z_0)}\right|$$

$$\geq \left|\frac{(\alpha+1)(1-\beta)}{2-\beta}\right|.$$

This contradicts the assumption of the theorem as given in (3) . Therefore, there is no $z_0 \in E$ such that $|w(z)| = 1$ and so we have that $|w(z)| < 1$. That is

$$\left|\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right| < 1-\beta,$$

so that $Re\left(\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1} f(z)}{D^n f(z)}\right) > \beta$ and $f(z) \in J_n^\alpha(\beta)$ as required.

Theorem 2. *If* $f(z) \in A$ *satisfies*

$$\left|arg\left(\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1} f(z)}{D^n f(z)}\right)\right| < \frac{\pi}{2}\delta,$$

then

$$\left|arg\ \frac{D^n f(z)}{\alpha^n z^\alpha}\right| < \frac{\pi}{2}\beta$$

for $0 < \beta < 1$ and $\alpha > 0$, where $\delta = \beta + \frac{2}{\pi} tan^{-1}\frac{\beta}{\alpha}$.

Proof. Define

$$p(z) = \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}. \qquad (5)$$

Then $p(z)$ is analytic in E with $p(0) = 1$ and $p(z) \neq 0$. Differentiating both sides of (5) logarithmically gives

$$\frac{zp'(z)}{p(z)} + \alpha = \frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha}. \qquad (6)$$

Since the left hand side of (6) is independent of n, we use Lemma 2 to obtain

$$\frac{zp'(z)}{\alpha p(z)} + 1 = \frac{D^{n+1} f(z)}{D^n f(z)}. \qquad (7)$$

Subsequently, it follows from (5) and (7) that

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1} f(z)}{D^n f(z)} = p(z)\left(1+\frac{zp'(z)}{\alpha p(z)}\right).$$

Suppose there exists a point $z_0 \in E$ such that $|arg\ \ p(z)| \le \frac{\pi}{2}\beta$ for $|z| < |z_0|$ and $|arg\ \ p(z_0)| = \frac{\pi}{2}\beta$, then by Lemma 3 we have $\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta$.

If $arg\ \ p(z_0) = \frac{\pi}{2}\beta$, then

$$\begin{aligned} arg\left(\frac{D^n f(z_0)^\alpha}{\alpha^n z_0^\alpha}\frac{D^{n+1} f(z_0)}{D^n f(z_0)}\right) &= arg\ \ p(z_0)\left(1+\frac{z_0 p'(z_0)}{\alpha p(z_0)}\right) \\ &= arg\ \ p(z_0) + arg\left(1+\frac{z_0 p'(z_0)}{\alpha p(z_0)}\right) \\ &= \frac{\pi}{2}\beta + arg\left(1+\frac{ik\beta}{\alpha}\right) \\ &\ge \frac{\pi}{2}\beta + tan^{-1}\left(\frac{\beta}{\alpha}\right). \\ &= \frac{\pi}{2}\left(\beta + \frac{2}{\pi}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right). \end{aligned}$$

That is

$$\arg\left(\frac{D^n f(z_0)^\alpha}{\alpha^n {z_0}^\alpha}\frac{D^{n+1} f(z_0)}{D^n f(z_0)}\right) \ge \frac{\pi}{2}\left(\beta + \frac{2}{\pi}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right). \qquad (8)$$

If $arg\ \ p(z_0) = -\frac{\pi}{2}\beta$, then

$$\arg\left(\frac{D^n f(z_0)^\alpha}{\alpha^n {z_0}^\alpha}\frac{D^{n+1} f(z_0)}{D^n f(z_0)}\right) \le -\frac{\pi}{2}\left(\beta + \frac{2}{\pi}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right). \qquad (9)$$

Inequalities (8) and (9) contradict the assumption of the theorem, therefore

$$\left|arg\ \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\right| < \frac{\pi}{2}\beta.$$

Theorem 3. *Let* $p(z)$ *be analytic in* E *such that* $p(0) = 1$ *and* $p'(0) \ne 0$ *in* E, *also suppose that*

$$\left|arg\ \left(p(z) + z\rho(z)p'(z)\right)\right| < \frac{\pi}{2}\beta + tan^{-1}\left(\frac{\beta Re\ \rho(z)}{1+\beta\ |im\ \rho(z)|}\right),$$

then

$$|arg\ \ p(z)| \le \frac{\pi}{2}\beta,$$

where $\beta(0 < \beta < 1)$ is real and

$$\rho(z) = \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1} f(z)}{D^n f(z)}.$$

Proof. Suppose there is a point $z_0 \in E$ such that $|arg\ \ p(z)| \le \frac{\pi}{2}\beta$ for $|z| < |z_0|$. Consider

$$arg\left(p(z_0) + \frac{z_0 D^n f(z_0)^\alpha}{\alpha^n z_0^\alpha}\frac{D^{n+1} f(z_0)}{D^n f(z_0)}p'(z_0)\right)$$

$$= arg\ \ p(z_0) + arg\left(1 + ik\beta\frac{D^n f(z_0)^\alpha}{\alpha^n z_0^\alpha}\frac{D^{n+1} f(z_0)}{D^n f(z_0)}\right). \qquad (10)$$

Now,

$$im\left(1+ik\beta\frac{D^n f(z_0)^\alpha}{\alpha^n z_0^\alpha}\frac{D^{n+1}f(z_0)}{D^n f(z_0)}\right) = k\beta Re\frac{D^n f(z_0)^\alpha}{\alpha^n z_0^\alpha}\frac{D^{n+1}f(z_0)}{D^n f(z_0)} = k\beta Re\ \rho(z_0).$$

and

$$Re\left(1+ik\beta\frac{D^n f(z_0)^\alpha}{\alpha^n z_0^\alpha}\frac{D^{n+1}f(z_0)}{D^n f(z_0)}\right) = 1-k\beta\, im\ \rho(z_0).$$

Using the last two equations with Lemma 3 when $arg\ \ p(z_0)=\frac{\pi}{2}\beta$, $k\geq 1$, in equation (10) we obtain

$$argp(z_0)+arg\left(1+ik\beta\frac{D^n f(z_0)^\alpha}{\alpha^n {z_0}^\alpha}\frac{D^{n+1}f(z_0)}{D^n f(z_0)}\right) \geq \frac{\pi}{2}\beta+\tan^{-1}\left(\frac{\beta Re\ \rho(z_0)}{1+\beta|im\ \rho(z_0)}\right). \qquad (11)$$

Also, when $arg\ \ p(z_0)=-\frac{\pi}{2}\beta$, $k\leq -1$, then

$$arg\ \ p(z_0)+arg\left(1+ik\beta\frac{D^n f(z_0)^\alpha}{\alpha^n z_0^\alpha}\frac{D^{n+1}f(z_0)}{D^n f(z_0)}\right) \leq -\frac{\pi}{2}\beta-tan^{-1}\left(\frac{\beta Re\ \rho(z_0)}{1+\beta\ |\ im\ \rho(z_0)\,|}\right) = -\left(\frac{\pi}{2}\beta+tan^{-1}\left(\frac{\beta Re\ \rho(z_0)}{1+\beta\ |\ im\ \rho(z_0)\,|}\right)\right).$$

That is

$$argp(z_0)+arg\left(1+ik\beta\frac{D^n f(z_0)^\alpha}{\alpha^n {z_0}^\alpha}\frac{D^{n+1}f(z_0)}{D^n f(z_0)}\right) \leq -\left(\frac{\pi}{2}\beta+\tan^{-1}\left(\frac{\beta Re\ \rho(z_0)}{1+\beta|im\ \rho(z_0)}\right)\right) \qquad (12)$$

The inequalities (11) and (12) contradict the assumption of the theorem therefore $|\,arg\ \ p(z)\,|\leq\frac{\pi}{2}\beta$.

Corollary 1. *Let* $f\in J_n^\alpha(\beta)$ *satisfies*

$$\left|arg\left(\frac{zf'(z)}{f(z)}+\frac{zD^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1}f(z)}{D^n f(z)}\left(\frac{(zf'(z))'}{f(z)}-z\left(\frac{f''(z)}{f(z)}\right)^2\right)\right)\right| < \frac{\pi}{2}\beta+tan^{-1}\left(\frac{\beta Re\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1}f(z)}{D^n f(z)}}{1+\beta\ |\ im\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1}f(z)}{D^n f(z)}\ |}\right).$$

Then

$$\left|arg\ \frac{zf'(z)}{f(z)}\right|\leq\frac{\pi}{2}\beta.$$

Corollary 2. *If* $f\in A$ *satisfies*

$$\left|arg\left(\left(1+\frac{zf''(z)}{f'(z)}\right)+\frac{zD^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1}f(z)}{D^n f(z)}\left(\frac{(zf''(z))'}{f'(z)}-z\left(\frac{f''(z)}{f'(z)}\right)^2\right)\right)\right| < \frac{\pi}{2}\beta+tan^{-1}\left(\frac{\beta Re\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1}f(z)}{D^n f(z)}}{1+\beta\ |\ im\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1}f(z)}{D^n f(z)}\ |}\right).$$

Then

$$\left|arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right|\leq\frac{\pi}{2}\beta.$$

That is $f(z)$ is strongly convex of order β .

Corollary 3. *If* $f \in A$ *satisfies*

$$\left|arg \frac{zf'(z)}{f(z)}\left(1+(zf'(z))'-\frac{zf'(z)^2}{f(z)}\right)\right| < \frac{\pi}{2}\beta + tan^{-1}\left[\frac{\beta Re\ f'(z)}{1+\beta\ |\ im\ f'(z)\ |}\right].$$

Then

$$\left|arg \frac{zf'(z)}{f(z)}\right| \leq \frac{\pi}{2}\beta.$$

That is $f(z)$ is strongly starlike of order β .

Corollary 4. *If* $f \in A$ *satisfies*

$$\left|arg\left(\left(1+\frac{zf''(z)}{f'(z)}\right)+z\left((zf''(z))'-\frac{zf''(z)^2}{f'(z)}\right)\right)\right|$$

$$< \frac{\pi}{2}\beta + tan^{-1}\left[\frac{\beta Re\ f'(z)}{1+\beta\ |\ im\ f'(z)\ |}\right].$$

Then

$$\left|arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| \leq \frac{\pi}{2}\beta.$$

That is $f(z)$ is strongly convex of order β .

4. Summary and Conclusion

In this paper, more properties of the class $J_n^{\alpha}(\beta)$ of analytic functions were investigated. Particularly, we establish a condition for analytic functions to be in the class $J_n^{\alpha}(\beta)$. Angular estimates for functions in $J_n^{\alpha}(\beta)$ were also obtained and interesting result on the condition for an analytic function to be strongly starlike and strongly convex of order β in the open unit disk were obtained.

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