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CONVOLUTION PROPERTIES AND COEFFICIENT INEQUALITIES OF λ -PSEUDO ANALYTIC UNIVALENT FUNCTIONS

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Abstract

In this paper, we isolate some new and interesting classes of λ -pseudo starlike $G_{\lambda}^{*}(\omega,\alpha)$ and λ -pseudo analytic $G_{\lambda}^{**}(\omega,\alpha)$ univalent functions in the unit disk $U=\{z:|z|<1\}$. Properties such as coefficient inequalities, extremal functions and their convolution to each of the new subclass were derived using techniques based on Holder's inequalities.

Keywords: *Analytic functions, univalent functions, convolution, starlike functions, convex functions.*

1. Introduction

Let A denote the class of the functions

$$f(z)=z+a_2z^2+a_3z^3+\ldots \qquad (1)$$

which are analytic in the unit disk $U=\{z:|z|<1\}$ and by S the subclass of A which consist of univalent functions only. Furthermore, let $R(\beta)$ and $S^*(\beta)$ be the well known subclasses of S consisting of functions which are respectively of bounded turning and starlike of order $\beta, 0\le\beta<1$ in U. That is, functions satisfying respectively $\operatorname{Re} f^1(z)>\beta$ and $\operatorname{Re} zf^1(z)/f(z)>\beta$ in U. Singh in [1] studied a subclass of S denoted by $B_1(\alpha)$ consisting of functions which are a special case of Bazilevic functions which consists only univalent functions. The functions in $B_1(\alpha)$ satisfy the geometric condition

$$\operatorname{Re}\left(\frac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}}\right)>0, \qquad z\in U \qquad (2)$$

for $\alpha>0$ is real and this class of functions are called or known as Bazilevic functions of type α. This class of functions include the starlike and bounded turning functions as the case $\alpha=0$ and $\alpha=1$ shows.

In 1999, Kanas and Ronning [2] introduced a new concept of analytic functions which they define as $A(\omega)\subset A$ denote the class of function of the form

$$f(z)=(z-\omega)+\sum_{k=2}^{\infty}a_k(z-\omega)^k \qquad (3)$$

which are analytic in the unit disk $U=\{z:|z|<1\}$ and normalized with $f(\omega)=f'(\omega)-1=0$ where ω is an arbitrary fixed point in U and also $S(\omega)\subset S$. By using (3), they studied the classes of

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$$S(\omega) = \{ f \in A(\omega) : f \text{ is univalent in } U \} \qquad (4)$$

and

$$S^*(\omega) = \left\{ f \in S(\omega) : \operatorname{Re} \frac{(z-\omega) f'(z)}{f(z)} > 0 \right\} \qquad (5)$$

which are respectively the classes of univalent and ω-starlike functions and ω is an arbitrary fixed point in *U*.

Acu and Owa in [3] also considered the class $\operatorname{Re} f'(z) > 0$. Also, several authors [4-8] have dealt so much with these classes of functions and they obtained valuable results. Therefore, for the purpose of this work, we say that $f \in A(\omega)$ is a Bazilevic function of type α and order β if and only if

$$\operatorname{Re} \frac{f(z)^{\alpha-1} f'(z)}{f(z-\omega)^{\alpha-1}} > \beta, z \in U \qquad (6)$$

For f is of form (3), and we denote the class of such functions by $B_1(\omega, \alpha, \beta)$

Definition 1.1 [9]: Let $f \in A(\omega)$, $0 \le \alpha < 1$ and $\lambda \ge 1$ is real. Then $f(z)$ belongs to the class $G_\lambda^*(\omega, \alpha)$ of $\omega - \lambda -$ pseudo starlike functions of order α in the unit disk *U* if and only if

$$\operatorname{Re} \left\{ \frac{(z-\omega)(f'(z))^\lambda}{f(z)} \right\} > \alpha, \quad z \in U \qquad (7)$$

and all powers mean principal determinations only

Definition 1.2: Let $f \in A(\omega)$, $0 \le \alpha < 1$ and $0 \le \lambda \le 1$. Then $f(z)$ belongs to the class $G_\lambda^{**}(\omega, \alpha)$ of $\lambda -$ pseudo analytic functions of order α in the unit disk U if and only if

$$\operatorname{Re} \left\{ \frac{(z-\omega)(f'(z))^\lambda}{f(z)} \left(1 + \frac{\lambda (z-\omega) f''(z)}{f'(z)} - \frac{(z-\omega) f'(z)}{f(z)} \right) \right\} > \alpha, \ z \in U \qquad (8)$$

And all powers mean principal determinations only

Also, for functions $f_i(z) \in A(\omega) (i = 1, 2, ..., m)$ given by

$$f_i(z) = (z-\omega) + \sum_{k=p}^{\infty} \alpha_{k,i} (z-\omega)^k, \ (i = 1, 2, ..., m) \qquad (9)$$

Where ω is a fixed point in *U* and the Hadamard product (or convolution) is defined by

$$(f_1 * ... * f_m)(z) = (z-\omega) + \sum_{k=p}^{\infty} \left(\prod_{i=1}^{m} \alpha_{k,i} \right) (z-\omega)^k \qquad (10)$$

Finally, we define the function $(f'(z))^\lambda$ as

$$z\,(f'(z))^\lambda = z \left(1 + \sum_{j=1}^{\infty} \lambda_j (2a_2 z + 3a_3 z^2 + ...)^j \right) \qquad (11)$$

Where $\lambda_j = \binom{\lambda}{j}, \ j = 1, 2, \cdots$

Our intention in this present work is to extend studies on the classes of functions introduced in [3,6] by deriving some coefficient inequalities, extremal functions and convolution properties for the classes of functions $G_\lambda^*(\omega, \alpha)$ and $G_\lambda^{**}(\omega, \alpha)$.

2. Coefficient Inequalities

Theorem 2.1: A function $f(z) \in A(\omega)$ is in the class $G_\lambda^*(\omega, \alpha)$ if and only if

$$\sum_{k=2}^{\infty} (\lambda_j k + \alpha - 2)(r+d)^{k-1} a_k \le 1 - \alpha \quad \text{where} \quad |z| = r < 1 \quad \text{and} \quad |\omega| = d$$

Proof: Assuming the inequality holds true, then

$$\left|\frac{(z-\omega)(f'(z))^{\lambda}}{f(z)}-1\right|=\left|\frac{\sum_{k=2}^{\infty}(\lambda_j k-1)a_k(z-\omega)^{k-1}}{1+\sum_{k=2}^{\infty}a_k(z-\omega)^{k-1}}\right|\leq\frac{\sum_{k=2}^{\infty}(\lambda_j k-1)a_k(r+d)^{k-1}}{1+\sum_{k=2}^{\infty}a_k(r+d)^{k-1}}\leq 1-\alpha \quad (12)$$

Clearly, we can see that $\frac{(z-\omega)(f'(z))^{\lambda}}{f(z)}$ lies in the circle centre ω where ω is a fixed point in U whose radius is $1-\alpha$. Therefore, $f(z)$ is in the class $G_{\lambda}^{*}(\omega,\alpha)$.

To prove the converse, assume that $f(z)$ is in the $G_{\lambda}^{*}(\omega,\alpha)$, then

$$\mathrm{Re}\left(\frac{(z-\omega)(f'(z))^{\lambda}}{f(z)}\right)=\mathrm{Re}\left(\frac{1+\sum_{k=2}^{\infty}\lambda_j k a_k(z-\omega)^{k-1}}{1+\sum_{k=2}^{\infty}a_k(z-\omega)^{k-1}}\right)>\alpha \quad (13)$$

For ω is a fixed point in U. Choosing values of z on the real axis so that $\frac{(z-\omega)(f'(z))^{\lambda}}{f(z)}$ is real. Clearing the denominator from equation (13) and let $z\to 1$, we have

$$\alpha\left(1+\sum_{k=2}^{\infty}(r+d)^{k-1}ak\right)\leq 1+\sum_{k=2}^{\infty}\lambda_j k a_k(r+d)^{k-1} \quad (14)$$

Finally, we note that the theorem is sharp with the extremal function

$$f(z)=(z-\omega)+\frac{1-\alpha}{(\lambda_j k+\alpha-2)(r+d)^{k-1}},\ (z-\omega)^k,\quad k\geq 2 \quad (15)$$

Corollary 2.1: Let $f(z)\in A(\omega)$ be in the class $G_{\lambda}^{*}(\omega,\alpha)$, then we have

$$a_k\leq\frac{1-\alpha}{(\lambda_j k+\alpha-2)(r+d)^{k-1}},\ k\geq 2 \quad (16)$$

where $d=|\omega|$

Theorem 2.2: A function $f(z)\in A(\omega)$ is in the class $G_{\lambda}^{**}(\omega,\alpha)$ if and only if

$$\sum_{j=1}^{\infty}k\ (\lambda_j k+\alpha-2)(r+d)^{k-1}a_k\leq 1-\alpha$$

Proof: The proof follows the same technique as in theorem (2.1) but the extremal function in this case is

$$f(z)=(z-\omega)+\frac{1-\alpha}{k(\lambda_j k+\alpha-2)(r+d)^{k-1}}(z-\omega)^k,\quad k\geq 2 \quad (17)$$

Corollary 2.2: Let $f\in A(\omega)$ be in the class $G_{\lambda}^{**}(\omega,\alpha)$, then

$$a_k\leq\frac{1-\alpha}{k(\lambda_j k+\alpha-2)(r+d)^{k-1}} \quad (18)$$

where $d=|\omega|$

3. Convolution Properties for Functions in the Class $G_{\lambda}^{*}(\omega,\alpha)$

We first prove the Hadamard product (or convolution) defined by (10)

Theorem 3.1: If $f_i(z)\in G_{\lambda}^{*}(\omega,\alpha_i)(i=1,2,....,m)$ then $(f_i*...f_m)(z)\in G_{\lambda}^{*}(\omega,\beta)$ where

$$\beta=1-\frac{(\lambda_j p-1)\prod_{i=1}^{m}(1-\alpha_i)}{(1-d)^{k-1}\prod_{i=1}^{m}(\lambda_j p+\alpha_i-2)+\prod_{i=1}^{m}(1-\alpha_i)}$$

The result is very sharp for the functions $f_i(z)$ $(i=1,2,....,m)$ given by

$$f_i(z)=(z-\omega)+\frac{1-\alpha_i}{(\lambda_j p+\alpha_i-2)(r+d)^{k-1}}(z-\omega)^p \quad (19)$$

Proof: Principle of mathematical induction will be used to prove theorem (3.1)

Let $f_1(z)\in G_{\lambda}^{*}(\omega,\alpha_1)$ and $f_2(z)\in G_{\lambda}^{*}(\omega,\alpha_2)$, then the inequality

$$\sum_{k=p}^{\infty}(\lambda_j k+\alpha_i-2)(r+d)^{k-1}a_{k,i}\le 1-\alpha_i$$

implies that

$$\sum_{k=p}^{\infty}\sqrt{\frac{(\lambda_j k+\alpha_i-2)(r+d)^{k-1}}{1-\alpha_i}}a_{k,i}\le 1 \quad (20)$$

Thus, by applying the Cauchy-Schwarz inequality, we have

$$\left|\sum_{k=p}^{\infty}\sqrt{\frac{(\lambda_j k+\alpha_1-2)(\lambda_j k+\alpha_2-2)(r+d)^{k-1}}{(1-\alpha_1)(1-\alpha_2)}}(a_{k,1})(a_{k,2})\right|^2 \quad (21)$$

$$\le (r+d)^{k-1}\left(\sum_{k=p}^{\infty}\frac{(\lambda_j k+\alpha_1-2)}{1-\alpha_1}a_{k,1}\right)\left(\sum_{k=p}^{\infty}\frac{(\lambda_j k+\alpha_2-2)}{1-\alpha_2}a_{k,2}\right)\le 1 \quad (22)$$

hence, i

$$\sum_{k=p}^{\infty}\frac{(\lambda_j k+\delta-2)}{1-\delta}(a_{k,1})(a_{k,2})\le\sum_{k=p}^{\infty}\sqrt{\frac{(\lambda_j k+\alpha_1-2)(\lambda_j k+\alpha_2-2)(r+d)^{k-1}}{(1-\alpha_1)(1-\alpha_2)}}(a_{k,1})(a_{k,2}) \quad (23)$$

that is

$$\sqrt{(a_{k,1})(a_{k,2})}\le\frac{1-\delta}{\lambda_j k+\delta-2}\sqrt{\frac{(\lambda_j k+\alpha_1-2)(\lambda_j k+\alpha_2-2)(r+d)^{k-1}}{(1-\alpha_1)(1-\alpha_2)}} \quad (24)$$

then $(f_1 * f_2)(z)\in G_\lambda^*(\omega,\delta_i)$

we note that the inequality (20) gives

$$\sqrt{a_{k,i}}\le\sqrt{\frac{1-\alpha_i}{(\lambda_j k+\alpha_i-2)(r+d)^{k-1}}},\ (i=1,\ 2;\ k=p,\ p+1,\ p+2,\ldots) \quad (25)$$

Consequently, if

$$\sqrt{\frac{(1-\alpha_1)(1-\alpha_2)}{(\lambda_j k+\alpha_1-2)(\lambda_j k+\alpha_2-2)(r+d)^{k-1}}}\le\frac{1-\delta}{\lambda_j k+\delta-2}\sqrt{\frac{(\lambda_j k+\alpha_1-2)(\lambda_j k+\alpha_2-2)(r+d)^{k-1}}{(1-\alpha_1)(1-\alpha_2)}} \quad (26)$$

that is

$$\frac{\lambda_j k+\delta-2}{1-\delta}\le\frac{(\lambda_j k+\alpha_1-2)(\lambda_j k+\alpha_2-2)(r+d)^{k-1}}{(1-\alpha_1)(1-\alpha_2)}(k=p,\ p+1,\ p+2,\ldots) \quad (27)$$

then we have $(f_1 * f_2)(z)\in S^*(\omega,\ \delta)$. From (27), we have

$$\delta\le 1-\frac{(\lambda_j k-1)(1-\alpha_1)(1-\alpha_2)}{(r+d)^{k-1}(\lambda_j k+\alpha_1-2)(\lambda_j k+\alpha_2-2)+(1-\alpha_1)(1-\alpha_2)}=\Gamma(k),\ (k=p,\ p+1,\ p+2,\ldots) \quad (28)$$

since $\Gamma(k)$ is increasing for $k\ge p$ we have

$$\delta\le 1-\frac{(\lambda_j k-1)(1-\alpha_1)(1-\alpha_2)}{(r+d)^{k-1}(\lambda_j k+\alpha_1-2)(\lambda_j k+\alpha_2-2)+(1-\alpha_1)(1-\alpha_2)} \quad (29)$$

which shows that $(f_1 * f_2)(z)\in G_\lambda^*(\omega,\delta)$, where

$$\delta=1-\frac{(\lambda_j p-1)(1-\alpha_1)(1-\alpha_2)}{(r+d)^{k-1}(\lambda_j p+\alpha_1-2)(\lambda_j p+\alpha_2-2)+(1-\alpha_1)(1-\alpha_2)} \quad (30)$$

Next, if $(f_1 *\ldots* f_m)(z)\in G_\lambda^*(\omega,\beta)$, where

$$\beta=1-\frac{(\lambda_j p-1)\prod_{i-1}^{m}(1-\alpha_i)}{(1-d)^{k-1}\prod_{i-1}^{m}(\lambda_j p+\alpha_i-2)+\prod_{i-1}^{m}(1-\alpha_i)} \quad (31)$$

then, by the same process above, we can show that $(f_1 *\ldots* f_{m+1})(z)\in G_\lambda^*(\omega,\alpha)$, where

$$\alpha = 1 - \frac{(\lambda_j p - 1)(1-\beta)(1-\beta_{m+1})}{(r+d)^{k-1}(\lambda_j p + \beta - 2)(\lambda_j p + \alpha_{m+1} - 2) + (1-\beta)(1-\alpha_{m+1})} \tag{32}$$

Since

$$(1-\beta)(1-\alpha_{m+1}) = 1 - \frac{(\lambda_j p - 1)\prod_{i-1}^{m}(1-\alpha_i)}{(1-d)^{k-1}\prod_{i-1}^{m}(\lambda_j p + \alpha_i - 2) + \prod_{i-1}^{m}(1-\alpha_i)} \tag{33}$$

and

$$(\lambda_j p + \beta - 2)(\lambda_j p + \alpha_{m+1} - 2) = 1 - \frac{(\lambda_j p - 1)\prod_{i-1}^{m+1}(\lambda_j p + \alpha_i - 2)}{(1-d)^{k-1}\prod_{i-1}^{m}(\lambda_j p + \alpha_i - 2) + \prod_{i-1}^{m}(1-\alpha_i)} \tag{34}$$

Equation (32) shows that

$$\alpha = 1 - \frac{(\lambda_j p - 1)\prod_{i-1}^{m+1}(1-\alpha_i)}{(1+d)^{k-1}\prod_{i-1}^{m+1}(\lambda_j p + \alpha_i - 2) + \prod_{i-1}^{m+1}(1-\alpha_i)} \tag{35}$$

Finally, for functions $f_i(z)(i = 1, 2, ..., m)$ given by (19), we have

$$(f_1 * ... * f_m)(z) = (z-\omega) + \left(\prod_{i-1}^{m}\left(\frac{1-\alpha_i}{(\lambda_j p + \alpha_i - 2)(r+d)^{k-1}}\right)\right)(z-\omega)^p = (z-\omega) + A_p(z-\omega)^p \tag{36}$$

where

$$A_p = \prod_{i-1}^{m}\left(\frac{1-\alpha_i}{(\lambda_j p + \alpha_i - 2)(r+d)^{k-1}}\right) \tag{37}$$

It follows that

$$\sum_{k=p}^{\infty}\frac{(\lambda_j k + \alpha - 2)(r+d)^{k-1}}{1-\alpha_i}A_k = 1 \tag{38}$$

and this completes the proof.

Corollary 3.1: If $f_i(z) \in G_\lambda^*(\omega, \beta)(i = 1, 2, ..., m)$ then $(f_1 * ... * f_{m+1})(z) \in G_\lambda^*(\omega, \alpha)$, where

$$\alpha = 1 - \frac{(\lambda_j p - 1)(1-\beta)^m}{(r+d)^{m(k-1)}(\lambda_j p + \beta - 2)^m + (1-\beta)^m} \tag{39}$$

The result is sharp for the functions $f_i(z)(i = 1, 2, ..., m)$ given by

$$f_1(z) = (z-\omega) + \left(\frac{1-\beta}{(r+d)^{k-1}(\lambda_j p + \beta - 2)}\right)(z-\omega)^p, \quad (i = 1, 2, ..., m) \tag{40}$$

and ω is a fixed point in *U*.

4. Convolution Properties for Functions in the Class $G_\lambda^{**}(\omega, \alpha)$

Theorem 4.1: If $f_i(z) \in G_\lambda^{**}(\omega, \alpha_i)(i = 1, 2, ..., m)$ then $(f_1 * ... * f_m)(z) \in G_\lambda^{**}(\omega, \beta)$, where

$$\beta = 1 - \frac{(\lambda_j p - 1)\prod_{i-1}^{m}(1-\alpha_i)}{p^{m-1}(1+d)^{k-1}\prod_{i-1}^{m}(\lambda_j p + \alpha_i - 2) + \prod_{i-1}^{m}(1-\alpha_i)}$$

The result is sharp for the functions $f_i(z)(i = 1, 2, ..., m)$ given by

$$f_1(z) = (z-\omega) + \left(\frac{1-\alpha}{(r+d)^{k-1}p(\lambda_j p + \alpha_i - 2)}\right)(z-\omega)^p, \ (i = 1, 2, ..., m) \tag{41}$$

Proof: Let $f_1(z) \in G_\lambda^{**}(\omega, \alpha_1)$ and $f_2(z) \in G_\lambda^{**}(\omega, \alpha_2)$. By similar process in theorem 3.1, the following inequality holds

$$\sum_{k=p}^{\infty}\frac{k(\lambda_j k+\delta-2)(r+d)^{k-1}}{1-\sigma}(a_{k,1})(a_{k,2})\le 1 \qquad (42)$$

which shows that $(f_1 * f_2)(z)\in G_{\lambda}^{**}(\omega,\delta)$.

Following the same process in theorem 3.1, we obtain that

$$\delta\le 1-\frac{(\lambda_j k-1)(1-\alpha_1)(1-\alpha_2)}{(r+d)^{k-1}k(\lambda_j k+\alpha_1-2)(\lambda_j k+\alpha_2-2)+(1-\alpha_1)(1-\alpha_2)}\ (k=p,\ p+1,\ p+2,\ldots) \qquad (43)$$

The right hand side of (43) takes its minimum at k=p because it is an increasing function of $k\ge p$. This shows that $(f_1 * f_2)(z)\in G_{\lambda}^{**}(\omega,\delta)$, where

$$\delta=1-\frac{(\lambda_j p-1)(1-\alpha_1)(1-\alpha_2)}{(r+d)^{k-1}p(\lambda_j p+\alpha_1-2)(\lambda_j p+\alpha_2-2)+(1-\alpha_1)(1-\alpha_2)} \qquad (44)$$

Now, assuming that $(f_1 * \ldots * f_m)(z)\in G_{\lambda}^{**}(\omega,\beta)$ where

$$\beta=1-\frac{(\lambda_j p-1)\prod_{i-1}^{m}(1-\alpha_i)}{p^{m-1}(r+d)^{k-1}\prod_{i-1}^{m}(\lambda_j p+\alpha_i-2)+\prod_{i-1}^{m}(1-\alpha_i)} \qquad (45)$$

hence, we have $(f_1 * \ldots * f_m)(z)\in G_{\lambda}^{**}(\omega,\alpha)$ where

$$\alpha=1-\frac{(\lambda_j p-1)(1-\beta)(1-\alpha_{m+1})}{(r+d)^{k-1}p(\lambda_j p+\beta-2)(\lambda_j p+\alpha_{m+1}-2)+(1-\beta)(1-\alpha_{m+1})} \qquad (46)$$

$$=1-\frac{(\lambda_j p-1)\prod_{i-1}^{m+1}(1-\alpha_i)}{p^{m}(r+d)^{k-1}\prod_{i-1}^{m+1}(\lambda_j p+\alpha_i-2)+\prod_{i-1}^{m+1}(1-\alpha_i)} \qquad (47)$$

By taking the function $f_1(z)$ given by (41), we can easily verify that the result is sharp.

By letting $\alpha_i=\alpha\ (i=1,2,\ldots,m)$ in theorem 4.1, we obtain

Corollary 4.1: If $f_i(z)\in G_{\lambda}^{**}(\omega,\alpha)\ (i=1,2,\ldots,m),$ then $(f_1 * \ldots * f_m)(z)\in G_{\lambda}^{**}(\omega,\beta)$, where

$$\beta=1-\frac{(\lambda_j p-1)(1-\alpha)^m}{(r+d)^{m(k-1)}p^{m-1}(\lambda_j p+\alpha-2)^m+(1-\alpha)^m} \qquad (48)$$

The result is sharp for the functions $f_i(z)(i=1,2,\ldots,m)$ given by

$$f_1(z)=(z-\omega)+\left(\frac{1-\alpha}{(r+d)^{k-1}p(\lambda_j p+\alpha-2)}\right)(z-\omega)^p,\ (i=1,2,\ldots,m) \qquad (49)$$

and ω is a fixed point in *U*.

Lemma 4.1: If $f(z)\in G_{\lambda}^{*}(\omega,\alpha)$ and $g(z)\in G_{\lambda}^{**}(\omega,\beta)$, then $(f * g)\in G_{\lambda}^{*}(\omega,\gamma)$, where

$$\gamma=1-\frac{(\lambda_j p-1)(1-\alpha)(1-\beta)}{(r+d)^{k-1}p(\lambda_j p+\alpha-2)(\lambda_j p+\beta-2)+(1-\alpha)(1-\beta)} \qquad (50)$$

The result is sharp for the functions $f(z)$ and $g(z)$ given by

$$f(z)=(z-\omega)+\left(\frac{1-\alpha}{(r+d)^{k-1}(\lambda_j p+\alpha-2)}\right)(z-\omega)^p \qquad (51)$$

and

$$g(z)=(z-\omega)+\left(\frac{1-\beta}{(r+d)^{k-1}p(\lambda_j p+\beta-2)}\right)(z-\omega)^p \qquad (52)$$

where ω is a fixed point in *U*.

Proof: Let

$$f(z)=(z-\omega)+\sum_{k=p}^{\infty}a_k\ (z-\omega)^k \qquad (53)$$

and

$$g(z)=(z-\omega)+\sum_{k=p}^{\infty} b_k \ (z-\omega)^k \qquad (54)$$

then, by theorem 2.1, it is sufficient to show that

$$\sum_{k=p}^{\infty}\frac{(r+d)^{k-1}(\lambda_j k+\gamma-2)}{1-\gamma}(a_k)(b_k)\le 1 \qquad (55)$$

for $(f*g)(z)\in G^{*}_{\lambda,p}(\omega,\gamma)$, since

$$\sum_{k=p}^{\infty}\frac{(r+d)^{k-1}(\lambda_j k+\alpha-2)}{1-\alpha}(a_k)\le 1 \qquad (56)$$

and

$$\sum_{k=p}^{\infty}\frac{(r+d)^{k-1}k(\lambda_j k+\beta-2)}{1-\beta}(b_k)\le 1 \qquad (57)$$

If we assume that

$$\sum_{k=p}^{\infty}\frac{(r+d)^{k-1}(\lambda_j k+\gamma-2)}{1-\gamma}(a_k)(b_k)\le \sum_{k=p}^{\infty}\sqrt{\frac{k(\lambda_j k+\alpha-2)(\lambda_j k+\beta-2)(r+d)^{k-1}}{(1-\alpha)(1-\beta)}}(a_k)(b_k) \qquad (58)$$

so that

$$\sqrt{(a_k)(b_k)}\le \frac{1-\gamma}{(r+d)^{k-1}(\lambda_j k+\gamma-2)}\sqrt{\frac{k(\lambda_j k+\alpha-2)(\lambda_j k+\beta-2)(r+d)^{k-1}}{(1-\alpha)(1-\beta)}}(a_k)(b_k) \qquad (59)$$

then we show that $(f*g)(z)\in G^{*}_{\lambda}(\omega,\gamma)$, if γ satisfies the inequality

$$\gamma\le 1-\frac{(\lambda_j k-1)(1-\alpha)(1-\beta)}{k(\lambda_j k+\alpha-2)(\lambda_j k+\beta-2)(r+d)^{k-1}+(1-\alpha)(1-\beta)} \qquad (60)$$

then $(f*g)(z)\in G^{*}_{\lambda,p}(\omega,\gamma)$. By theorem 2.1, theorem 4.1 and lemma 4.1, we arrive at

Theorem 4.2: If $f_i(z)\in G^{*}_{\lambda}(\omega,\alpha_i)\ (i=1,2,\dots x)$ and $g_i(z)\in G^{**}_{\lambda}(\omega,\beta_i)\ (i=1,2,\dots y)$, then $(f_1*\dots f_x*g_1*\dots*g_y)(z)\in G^{*}_{\lambda}(\omega,\gamma)$, where

$$\gamma=1-\frac{(\lambda_j p-1)(1-\alpha)(1-\beta)}{p(\lambda_j p+\alpha-2)(\lambda_j p+\beta-2)(r+d)^{k-1}+(1-\alpha)(1-\beta)} \qquad (61)$$

$$\alpha=1-\frac{(\lambda_j p-1)\prod_{i=1}^{x}(1-\alpha_i)}{(r+d)^{k-1}\prod_{i=1}^{x}(\lambda_j p+\alpha_i-2)+\prod_{i=1}^{x}(1-\alpha_i)} \qquad (62)$$

and

$$\beta=1-\frac{(\lambda_j p-1)\prod_{i=1}^{y}(1-\beta_i)}{p^{y-1}(r+d)^{k-1}\prod_{i=1}^{y}(\lambda_j p+\beta_i-2)+\prod_{i=1}^{y}(1-\beta_i)} \qquad (63)$$

The result is sharp for the functions $f_i(z)\ (i=1,2,\dots x)$ and $g_i(z)\ (i=1,2,\dots y)$ given by

$$f_i(z)=(z-\omega)+\left(\frac{1-\alpha_i}{(r+d)^{k-1}(\lambda_j p+\alpha_i-2)}\right)(z-\omega)^p,\ (i=1,2,\dots,x) \qquad (64)$$

and

$$g_i(z)=(z-\omega)+\left(\frac{1-\beta_i}{(r+d)^{k-1}p(\lambda_j p+\beta_i-2)}\right)(z-\omega)^p,\ (i=1,2,\dots,y) \qquad (65)$$

for $\alpha_i=\alpha\ (i=1,2,\dots x)$ and $\beta_i=\beta\ (i=1,2,\dots y)$. Theorem 4.2 yields the next corollary

Corollary 4.2: If $f_i(z)\in G^{*}_{\lambda}(\omega,\alpha)\ (i=1,2,\dots x)$ and $g_i(z)\in G^{**}_{\lambda}(\omega,\beta)\ (i=1,2,\dots y)$, then $(f_1*\dots*f_x*g_1*\dots*g_y)(z)\in G^{*}_{\lambda,p}(\omega,\gamma)$, where

$$\gamma = 1 - \frac{(\lambda_j p)(1-\alpha)^x (1-\beta)^y}{p^y (r+d)^{k-1}(\lambda_j p + \alpha - 2)^x (\lambda_j p + \beta - 2)^y + (1-\alpha)^x (1-\beta)^y} \qquad (66)$$

The result is sharp for the functions $f_i(z)$ $(i = 1, 2, \ldots x)$ and $g_i(z)$ $(i = 1, 2, \ldots y)$ given by

$$f_i(z) = (z-\omega) + \left(\frac{1-\alpha}{(r+d)^{k-1}(\lambda_j p + \alpha - 2)}\right)(z-\omega)^p, \; (i = 1, 2, \ldots, x) \qquad (67)$$

and

$$g_i(z) = (z-\omega) + \left(\frac{1-\beta}{(r+d)^{k-1} p(\lambda_j p + \beta - 2)}\right)(z-\omega)^p, \; (i = 1, 2, \ldots, y) \qquad (68)$$

In conclusion, this work has established the coefficient inequalities, extremal functions and their convolution to each of the new subclass derived.

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