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CONDITIONAL EXPECTATIONS ON PARTIAL GENERALIZED VON NEUMANN ALGEBRA

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Abstract

We have constructed Partial Generalized von Neumann Algebra and Conditional expectations have been defined on it; where we considered weak and unbounded Conditional Expectations and we found out that the unbounded conditional expectation is a proper subspace of the weak Conditional Expectation.

***Keywords*:** Partial Generalized von Neumann Algebra, Conditional Expectations, von Neumann Algebra, vectors, commutants, multipliers, Partial *-algebras, partial O*-algebras.

1.1 Introduction

In probability theory, Conditional Expectation plays a very vital role. Conditional expectation is a map which takes values from an algebra onto its sub- algebra. This implies that the domain of conditional expectation is not equal to the algebra in which it is acting upon. The study of conditional expectations for O*- algebras was first carried out by [1]. Let M be an O*-algebra on a dense subspace D of a complex separable Hilbert space H with cyclic and separating vector Ω_0 and N an O*-subalgebra of M. In their study, [1] defined conditional expectation as a map $A \to P_N A\Omega_0$ of M into the closed subspace H_N of H, where P_N is the orthogonal projection of H onto H_N . We call this the vector conditional expectation given by (N, Ω_0). It is important to check the existence of a Conditional expectation. In fact, [2] has shown that conditional expectation does not necessarily exist for a general von Neumann algebra. But for semi finite von Neumann algebras, here conditional expectation exists if and only if $\Delta_{\Omega 0}^{it} N \Delta_{\Omega 0}^{-it} = N$ where Δ is the modular automorphism group.

[3, 4] have studied Unbounded Conditional Expectations for operator algebras and O*-algebras. [5] studied Unbounded Conditional Expectations for Partial O*-algebras where he defined it as a positive linear map E of a Partial O*-algebra M onto its Partial O*-subalgebra N; thereby generalizing conditional expectations in Operator algebras and O*-algebras for Partial O*-algebras. In this paper we shall define Conditional Expectations on Partial Generalized von Neumann Algebra M onto its Partial Generalized von Neumann subalgebra N and compare two classes of Conditional Expectations; Weak Conditional Expectations and Unbounded Conditional Expectations. Hence the work of [5] has been extended.

2.1 Preliminaries

In order to make the paper self contained, we reproduce the definitions of partial *- algebras, partial O*-algebras and Partial Generalized von Neumann Algebra. For more details on the subject we refer the reader to [6].

-algebra:** A **-algebra is an algebra $\mathfrak{A}$, together with an involution which enjoys the following properties;
(i) $(x+y)^* = y^* + x^*$, (ii) $(x.y)^* = y^*.x^*$, (iii) $x^{**} = x$, (iv) $(\alpha x)^* = \bar{\alpha} x^*$, for all $x, y \in \mathfrak{A}$, $\alpha \in C$.

Partial *-algebra: A partial *-algebra is a complex vector space $\mathfrak{A}$ with an involution $x \to x^*$ (that is a bijection $x^{**} = x$) and a subset $\Gamma \subset \mathfrak{A} \times \mathfrak{A}$ (a binary relation) such that

(i) $(x, y) \in \Gamma$ implies $(y^*, x^*) \in \Gamma$
(ii) (x, y_1), $(x, y_2) \in \Gamma$ implies $(x, \alpha y_1 + \beta y_2) \in \Gamma$, for all $\alpha, \beta \in \boldsymbol{C}$;
(iii) whenever $(x, y) \in \Gamma$, there exists a product $x.y \in \mathfrak{A}$ with the usual properties of the multiplication: $x.(y + \alpha z) = x.y + \alpha(x.z)$ and $(x.y)^* = y^*.x^*$ for $(x, y), (x, z) \in \Gamma$ and $\alpha \in$

The element e of $\mathfrak{A}$ is called a unit if $e^* = e, (e, x) \in \Gamma$ for all $x \in \mathfrak{A}$ and $e.x = x.e = x$, for all $x \in \mathfrak{A}$. Notice thathe partial multiplication is not required to be associative. Whenever $(x, y) \in \Gamma$, x is called the left multiplier of y and y is called the right multiplier of x and we write $x \in L(y)$ and $y \in R(x)$. For a subset $\mathfrak{N} \subset \mathfrak{A}$, we write
$L(\mathfrak{N}) = \bigcap_{x \in \mathfrak{N}} L(x)$, $R(\mathfrak{N}) = \bigcap_{x \in \mathfrak{N}} R(x)$.

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Note that if $\mathfrak{A}$ has no unit, it may always be embedded into a larger partial *-algebra with unit in the standard fashion.

Partial O*-Algebra: A partial O^*-algebra is a *-subalgebra M of $L_w^\dagger(D,H)$, with identity satisfying the following properties:

(i) $X_1 + X_2,\ X_1, X_2 \in M$, (ii) $\alpha X,\ \alpha \in \boldsymbol{C}, X \in M$. (iii) $X \to X^\dagger = X^* \upharpoonright D$, (iv) $X_1 \square X_2 = X_1^{\dagger *} X_2$, defined whenever $X_1 \in L^w(X_2)$ or $X_2 \in R^w(X_1)$, that is if and only if $X_2 D \subset D(X_1^{\dagger *})$ and $X_1^* D \subset D(X_2^*)$, for all $X^\dagger \in M, X_1, X_2 \in M$

***-Representation:** A *- representation of a partial *- algebra $\mathfrak{A}$ is a *- homomorphism of $\mathfrak{A}$ into $L^\dagger(D,H)$, satisfying $\pi(e) = I$, whenever $e \in \mathfrak{A}$, that is,

(i) π is linear;

(ii) $x \in L^w(y)$ in $\mathfrak{A}$ implies $\pi(x) \in L^w(\pi(y))$ and $\pi(x)\square\pi(y) = \pi(xy)$;

(iii) $\pi(x^*) = \pi(x)^\dagger$ for every $x \in \mathfrak{A}$

2.2 Properties of Conditional Expectations on von Neumann Algebra

Here we state the properties of Conditional Expectations on von Neumann Algebra. For the properties of the classical Conditional Expectations, [8] has done it extensively.

Let M be a von Neumann Algebra on a separable Hilbert space H with a faithful normal state ω and a cyclic vector Ω_0 in H; let N be a von Neumann subalgebra of $\mathfrak{M}$. Then a map E of M onto N is said to be a Conditional Expectation of M onto N if it satisfies the following properties:

1. E is linear,
2. $E(A)^* = E(A^*)$ for all $A \in M$
3. $E(X) = X$, for all $X \in N$,
4. $E(A^* A) \geq 0$, for all $A \in M$
5. $E(A^* A) \leq E(A^*)\, E(A)$, for all $A \in M$.
6. $E(XAY) = XE(A)Y$, for all $A \in M,\ X, Y \in N$
7. $E(E(A)X) = E((A)E(X)) = E(A)E(X)$, for all $A \in M, X \in N$ or $X \in M, A \in N$.
8. $\omega_{\Omega_0}(E(A)) = \omega_{\Omega_0}(A)$, for all $A \in M$.

Remark: [7] has proved that every projection of norm one of a C^*-algebra onto its C^*-subalgebra enjoys properties 4-6.

2.3 Existence of Conditional Expectation in von=--‘]
Neumann Algebra

Let M be a von Neumann Algebra on a separable Hilbert space H with a faithful normal weight ω on M_+let N be a von Neumann subalgebra of M in which ω is semifinite. Then the following two statements are equivalent.

i. N is invariant under the modular automorphism group σ_t associated with ω,

ii. There exists a σ -weakly continuous faithful projection E of norm one from M onto N such that $\omega(X) = \omega \circ E(X)$, for every M_ω.

2.4 Construction of Partial Generalized von Neumann Algebra

In order to construct a Partial Generalized von Neumann Algebra, we equip the †-invariat vector space $L^\dagger(D,H)$ with the partial multiplication denoted by □ as follows; X is a left multiplier of Y or $X \in L^w(Y)$ if and only if $YD \subset D(X^{\dagger\star})$ and $X^\dagger D \subset D(Y^\star)$ and then $X\square Y = X^{\dagger\star}Y$. Also $X \in L^w(Y)$ if and only if $Y^\dagger \in L^w(X^\dagger)$ and then $(X\square \mathrm{Y})^\dagger = Y^\dagger \square \mathrm{X}^\dagger$. If $X \in L^w(Y) \cap L^w(Z)$, then $X \in L^w(\alpha Y + \beta Z)$ for all $\alpha, \beta \in C$, and $X\square(\alpha Y + \beta Z) = \alpha(X\square \mathrm{Y}) + \beta(X\square \mathrm{Z})$. A †-invariat subspace of $L^\dagger(D,H)$ is called a partial $O^\star$-algebra on D if $X\square \mathrm{Y} \in M$, whenever $X, Y \in M$ with $X \in L^w(Y)$.

Definition 2.4.1 Let M_0 be a von Neuman Algebra on H such that $M_0' D \subset D$. A partial $O^\star$-algebra M on D is called a Partial Generalized von Neuman Algebra on D over M_0' if $D = \cap_{X\in M} D(\overline{X})$, and $M = [M_0 \upharpoonright D]^{s*}$. Supposed that M is a Partial Generalized von Neuman Algebra on D over M_0'. Then it follows that;

$$M_{w\sigma}'' = \{X \in L^\dagger(D,H) : < CX\xi, \eta > = < C\xi, \mathrm{X}^\dagger \eta >, \text{ for each } C \in M_c',\ \xi, \eta \in M\} \equiv \{X \in L^\dagger(D,H) : \bar{X}\eta M_0\}.$$

2.5 Conditional Expectations in Partial Generalized von Neumann Algebra

In this section, let M be a Partial Generalized von Neumann Algebra on D over M_0(where M_0 is a von Neumann Algebra on H) with a strongly cyclic and separating vector $\Omega_0 \in D$ and let N be a Partial Generalized von Neumann subalgebra of M over N_0 (where N_0 is a von Neumann subalgebra over M_0).

2.5.1 Weak Conditional Expectation

Let $N \subseteq M$ be a Partial Generalized von Neumann Algebra on D with a strongly cyclic and separating vector $\Omega_0 \in D$ such that $(N \cap R^w(M))\, \Omega_0$ is dense in $H_N \equiv \overline{N\Omega_0}$. Then the following lemma is immediate:

Lemma 2.5.1 Put
$\pi_N: (N \cap R^w(M))\,\Omega_0 \longrightarrow L^{\dagger}(D,H)$
$XY\Omega_0 \longmapsto (X\square Y)\,\Omega_0$
$D(\pi_N) = (N \cap R^w(M))\,\Omega_0$,
$\pi_N(X)Y\Omega_0 = (X\square Y)\,\Omega_0, \forall X \in N, \forall Y \in N \cap R^w(M)$.
Then π_N is a *-representation of N in the Hilbert space $H_N \equiv \overline{D(\pi_N)}$. We denote by P_N the projection of $L^2(M) \equiv H_M$ onto $L^2(N) \equiv H_N$.
This projection plays a vital roll in this Research.
Lemma 2.5.2 If P_N and π_N are defined as
$P_N: L^2(M) \longrightarrow L^2(N) \xrightarrow{\pi_N} L^2(N)$
$X\Omega_0 \longmapsto P_N X\Omega_0 = XP_N\Omega_0$.
Then it holds that
$P_N D \subset D^*(\pi_N)$
And
$\pi_N^{\star}(X)\, P_N\Omega_0 = P_N X\Omega_0, \forall\, X \in N, \Omega_0 \in D$.

Proof.
$\langle (X\square Y)\,\Omega_0 | P_N\Omega\rangle = \langle\, X^{\dagger}Y\Omega_0 \big| \Omega\rangle = \langle\, Y\Omega_0 | X\Omega\rangle = \langle\, Y\Omega_0 | P_N X\Omega\rangle$
And so
$P_N D \subset D^*(\pi_N)$
And
$\pi_N^*(X)\, P_N\Omega_0 = P_N X\Omega_0, \forall\, X \in N, \Omega_0 \in D$.

Definition 2.5.1 A map E_N of M into $L^{\dagger}(D(\pi_N), H_N)$ is said to be a weak Conditional Expectations of (M, Ω_0) with respect to N if it satisfies
$\langle E_N(AX\Omega_0) | Y\Omega_0\rangle = \langle P_N(AX)\Omega_0) | Y\Omega_0\rangle, \forall A \in M, \forall X, Y \in N \cap R^w(M)$.
For weak Conditional Expectations, we have the following theorem;
Theorem 2.5.2 There exists a unique weak Conditional Expectation E_N of (M, Ω_0) with respect to N and $E_N(A) = P_N A \restriction D(\pi_N), \forall A \in M$.
The weak Conditional Expectation E_N of (M, Ω_0) with respect to N has the following properties

1. E_N is linear,
2. E_N is a projection, that is $E_N(A)^{\dagger} = E_N(A^{\dagger}), \forall A \in M$,
3. $E_N(X) = X, \forall\, X \in N$,
4. $E_N(A^{\dagger}\square A) \geq 0, \forall A \in M$ such that $A^{\dagger}\square A$ is well-defined,
5. $E_N(A^{\dagger}\square A) = E_N(A^{\dagger})\square E_N(A), \forall A \in M$ such that $A^{\dagger}\square A$ and $E_N(A^{\dagger})\square E_N(A)$, are well-defined,
6. $E_N(A\square X) = E_N(A)\square X$, for any $A \in M, X \in N \cap R^w(M)$ and $E_N(A)\square X$ is well-defined,
7. $E_N(X\square A) = X\square E_N(A)$, for any $A \in M \cap R^w(N), X \in N$,
8. $\omega_{\Omega_0}(E_N(A)) = \omega_{\Omega_0}(A)$, for all $A \in M$.

Proof. We know that $E_N(A)$ is a linear map of $D(\pi_N)$ into $D^{\star}(\pi_N)$ for any $A \in M$, and furthermore, we have $E_N(A)^{\dagger} = E_N(A^{\dagger})$, for all $A \in M$. So E_N is a map of M into $L^{\dagger}(D(\pi_N), H_N)$.
Since
$\langle E_N(AX\Omega_0) | Y\Omega_0\rangle = \langle P_N(AX)\Omega_0) | Y\Omega_0\rangle, \forall A \in M, \forall X, Y \in N \cap R^w(M)$.
E_N is a weak Conditional Expectation of (M, Ω_0) with respect to N, $E(A) = E_N(A)$, for each $A \in M$. Thus we have shown the existence and uniqueness of weak Conditional Expectation. The conditions 3-5 follow, since $E_N(A) = P_N A \restriction D(\pi_N), \forall A \in M$. This completes the proof.

2.6 Unbounded Conditional Expectation in Partial Generalized von Neumann algebra
Let $N \subseteq M$ be a Partial Generalized von Neumann algebra on D in H with a strongly cyclic and separating vector $\Omega_0 \in D$ for M such that $(N \cap R^w(M))\,\Omega_0$ is dense in H_N.

Definition 2.6.1 A map $E: D(E) \subseteq M$ onto N is said to be an Unbounded Conditional Expectation of (M, Ω_0) with respect to N if

i. The domain $D(E)$ of E is a †-invariant subspace of M containg N,

ii. E is a projection, that is hermitian $E(A)^{\dagger} = E(A^{\dagger})$, for $A \in D(E)$ and $E(X) = X, \forall X \in N$,

iii. $E(A\square X) = E(A)\square X$, for any $A \in D(E), X \in N \cap R^{w}(M)$

iv. $E(X\square A) = X\square E(A)$, for any $A \in D(E) \cap R^{w}(N), X \in N$

v. $\omega_{\Omega_0} E(A)) = \omega_{\Omega_0}(A)$, for all $A \in D(E)$,

Remark 2.6.1 If $D(E) = M$ then E is said to be a weak Conditional Expectation of (M, Ω_0) with respect to N.

Note that the Unbounded Conditional Expectation E is a subspace of the weak Conditional Expectation E_N of (M, Ω_0) with respect to N. That is if $E_N: M \longrightarrow N$, then $E: D(E) \subset M \longrightarrow N$, also $E_N \upharpoonright D(E) = E$.

For Unbounded Conditional Expectation, we have the following lemma

Lemma 2.6.1 Let E be an Unbounded Conditional Expectation of (M, Ω_0) with respect to N. Then $E(AX)\Omega_0 = P_N AX\Omega_0, \forall A \in D(E), X \in N \cap R^{w}(M)$.

Proof.

$$\langle E(AX)\Omega_0 | Y\Omega_0 \rangle = \langle E(A\square X)\Omega_0 | Y\Omega_0 \rangle = \langle E(Y^{\dagger}\square A\square X)\Omega_0 \big| \Omega_0 \rangle = \langle (Y^{\dagger}\square A\square X)\Omega_0 \big| \Omega_0 \rangle = \langle (A\square X)\Omega_0 | Y\Omega_0 \rangle$$
$$= \langle (AX)\Omega_0 | Y\Omega_0 \rangle = \langle (AX)\Omega_0 | P_N Y\Omega_0 \rangle = \langle P_N AX\Omega_0 | Y\Omega_0 \rangle$$

Hence,

$E(AX)\Omega_0 = P_N AX\Omega_0, \forall A \in D(E), X \in N \cap R^{w}(M)$. □

Let $\mathfrak{J}$ be the set of all Unbounded Conditional Expectation of (M, Ω_0) with respect to N. Then $\mathfrak{J}$ is an ordered set with the following order $\subset$:

$E_1 \subset E_2$ if and only if $D(E_1) \subset D(E_2)$, $E_1(A) = E_2(A), \forall A \in D(E_1)$.

Theorem 2.6.1 There exists a maximal Conditional Expectation of (M, Ω_0) with respect to N, and it is denoted by $\mathcal{E}_n$.

Proof.

We put

$D(\mathcal{E}_0) \equiv \{A \in M: A \upharpoonright_{(N \cap R^{w}(M))\Omega_0} \in N \upharpoonright_{(N \cap R^{w}(M))\Omega_0}\}$.

Then for any $A \in D(\mathcal{E}_0)$, there exists a unique map $\mathcal{E}_0$ such that

$\mathcal{E}_0 (AX)\Omega_0 = P_N AX\Omega_0 = E(AX)\Omega_0, \forall X \in N \cap R^{w}(M)$.

It is easily shown that $\mathcal{E}_0$ is an Unbounded Conditional Expectation of (M, Ω_0) with respect to N. Moreover, $\mathcal{E}_0$ is maximal in $\mathfrak{J}$. Indeed, let $E \in \mathfrak{J}$. Take an arbitrary $A \in D(E)$. Then by lemma **2.6.1** we see that

$E(AX)\Omega_0 = P_N AX\Omega_0 = E_N(AX)\Omega_0, \forall X \in N \cap R^{w}(M)$.

Which implies that $E(AX) \upharpoonright_{(N \cap R^{w}(M))\Omega_0}$. Hence $E \subset \mathcal{E}_0$ and $\mathcal{E}_0$ is maximal in $\mathfrak{J}$. □

Thus we remark for the weak and for the Unbounded Conditional Expectations E_N and E that $E_N = N$, $E(D(E)) \neq N$ and $E(D(E)) \subset N$.

2.7 Existence of Conditional Expectations on Partial Generalized von Neumann Algebra

For the existence of Conditional Expectations in von Neumann Algebra, Takesaki has obtained the following:

Theorem 2.7.1 Let M be a von Neumann Algebra on a Hilbert space H with a separating and cyclic vector Ω_0 and N a von Neumann subalgebra of M. Then E_N is a Conditional Expectation of M onto N with respect to Ω_0 if and only if $\Delta_{\Omega_0}^{it}(N)\, \Delta_{\Omega_0}^{-it} = N, \forall t \in R$, where Δ_{Ω_0} is the modula operator for the left Hilbert Algebra $M\Omega_0$.

Then our extension is as follows:

Theorem 2.7.2 Let M be a Partial Generalized von Neumann Algebra on D in H with a strongly cyclic and separating vector $\Omega_0 \in D$, and let N be a Partial Generalized von Neumann subalgebra of M satisfying $N_w' \widehat{D}(N) \subset \widehat{D}(N), (N \cap R^{w}(M))\, \Omega_0$ is essentially self-adjoint for N and $E_N(A) = P_N A \upharpoonright P_N\, D, \forall A \in M_{wc}''$. Then

1. E_N is linear,
2. E_N is hermitian, that is $E_N(A)^{\dagger} = E_N(A^{\dagger}), \forall A \in M$,
3. $E_N(X) = X, \forall X \in N$,
4. $E_N(A^{\dagger}\square A) \geq 0, \forall A \in M$ such that $A^{\dagger}\square A$ is well-defined,
5. $E_N(A^{\dagger}\square A) = E_N(A^{\dagger})\square E_N(A), \forall A \in M$ such that $A^{\dagger}\square A$ and $E_N(A^{\dagger})\square E_N(A)$, are well-defined,

6. $E_N(A\Box X) = E_N(A)\Box X$, for any $A \in M, X \in N \cap R^w(M)$ and $E_N(A)\Box X$ is well-defined,
7. $E_N(X\Box A) = X\Box E_N(A)$, for any $A \in M \cap R^w(N), X \in N$,
8. $\omega_{\Omega_0}(E_N(A)) = \omega_{\Omega_0}(A)$, for all $A \in M$.
9. $\Delta_{\Omega_0}^{\prime\prime it}(N_w')' \, \Delta_{\Omega_0}^{\prime\prime -it} = (N_w')', \forall t \in R$ where Δ_{Ω_0}'' is the modular operator for the full Hilbert Algebra $(M_w')'$.

Proof.
Let
$D(E_N) = \{A \in M : P_N A\Omega_0 \in N\Omega_0\}$
Then we see that
$P_N A\Omega_0 = E_N(A)\,\Omega_0 \in N\Omega_0$, for each $A \in M$. Hence $D(E_N) \subset M$.
Since Ω_0 is strongly cyclic and separating vector for M. It follows that for any $A \in D(E_N)$. There exists a unique element $E_N(A)$ of N such that $P_N A\Omega_0 = E_N(A)\,\Omega_0$.
Take arbitrary $X \in N$, then $\bar{X}$ is affiliated with the von Neumann Algebra $(N_w')'$. And so
$N_w' = N_{qw}'$.
By the self-adjointness of M and $(N \cap R^w(M))\,\Omega_0$ being dense in H_N, it follows that
$N(N \cap R^w(M))\,\Omega_0 \subset \overline{(N \cap R^w(M))\,\Omega_0} = \overline{N\Omega_0}$,
where $(N \cap R^w(M))\,\Omega_0$ is a reducing subspace for N. Since $(N \cap R^w(M))\,\Omega_0$ is essentially self-adjoint for N, $P_N \in N_w', P_N\widehat{D}(N) \subset \widehat{D}(N)$.
Now since $\overline{X\eta}(N_w')'$, for each $X \in N$, we have $N\Omega_0 = \overline{(N_w')'\Omega_0}$ that is $P_N = P((N_w')')$.
Let $S_{\Omega_0}$and S_{Ω_0}'' be the closures of the maps
$S_{\Omega_0}A\Omega_0 = A^\dagger\Omega, A \in M$
$S_{\Omega_0}''B\Omega_0 = B^*\Omega, B \in (M_w')'$
And so, $S_{\Omega_0} \subset S_{\Omega_0}''$ and $S_{\Omega_0} \neq S_{\Omega_0}''$ in general.
But then
$\Delta_{\Omega_0}^{\prime\prime it}(N_w')' \, \Delta_{\Omega_0}^{\prime\prime -it} = (N_w')', \forall t \in R$ where Δ_{Ω_0}'' is the modular operator for the full Hilbert Algebra $(M_w')'$.
This implies
$P((N_w')')S_{\Omega_0}'' \subset S_{\Omega_0}''P((N_w')')$
And there exists a Conditional Expectation E_N'' of the Partial Generalized von Neumann Algebra $((M_w')', \Omega_0)$ with respect to $(N_w')'$.
And so
$E_N'(A^\dagger)\Omega_0 = P_N A^\dagger\Omega_0 = P_N S_{\Omega_0}A\Omega_0 = P_N S_{\Omega_0}''A\Omega_0 = S_{\Omega_0}''P_N A\Omega_0 = S_{\Omega_0}''E_N(A)\Omega_0 = S_{\Omega_0}E_N(A)\Omega_0 = E_N(A)^\dagger\Omega_0$, for each $A \in M$ which implies by the separateness of Ω that E_N is hermitian.
It is clear that $E(X) = X, \forall X \in N$.
Now take arbitrary $A \in M$ and $X \in N \cap R^w(M)$.
Since E_N is hermitian, it follows that $A\Box X \in M$ and $X \in N \cap R^w(M)$.
Obviously,
$\omega_{\Omega_0}(E_N(A)) = \omega_{\Omega_0}(A)$, for all $A \in M$.
Therefore $E_N$is a Conditional Expectation of (M, Ω_0) with respect to N. □

Theorem 2.7.3 Let M be a Partial Generalized von Neumann Algebra on D in H with a strongly cyclic and separating vector $\Omega_0 \in D$, and let N be a Partial Generalized von Neumann subalgebra of M satisfying the following conditions

i. $\overline{N\Omega_0} = H_N$
ii. $N_w'\widehat{D}(N) \subset \widehat{D}(N)$
iii. $\overline{N\Omega_0}$ is essentially self-adjoint for N.
iv. $\Delta_{\Omega_0}^{\prime\prime it}(N_w')' \, \Delta_{\Omega_0}^{\prime\prime -it} = (N_w')', \forall t \in R$ where Δ_{Ω_0}'' is the modular operator for the full Hilbert Algebra $(M_w')'$.

Then there exists a Conditional Expectation of (M, Ω_0) with respect to N if and only if $P_N M\Omega_0 = N\Omega_0$.

Proof.
Since $N\Omega_0 = \overline{N\Omega_0}^{t_N} = P_N D$. It follows that $\overline{E_N}(A) = \widehat{E_N}(A)$, for each $A \in M$, and
$\overline{E_N}(A) \subset (N_{P_N})_{wc}''$. □

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