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SCATTERING, BOUND, AND QUASI-BOUND STATES SOLUTION OF DIRAC PARTICLES WITH DEFORMED HYLLERAAS PLUS WOODS-SAXON POTENTIAL.

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Abstract

The Dirac particle with the deformed Hylleraas plus Woods-Saxon potentials has been solved analytically in terms of hypergeometric functions. The scattering and,bound, and quasi-bound state solutions have also been obtained using the properties of the equation of continuity of the wave functions. We calculated in detail the analytical solutions of the scattering phase shift.

Keywords: Dirac particles; Quasi-bound states; Bound states; Scattering states; Hylleraaspotential.

1. Introduction

The tremendous breakthrough made in several areas of human endeavour may be traced to quantum mechanical principles. This is because several phenomena that could not be accounted for using the Newtonian law of classical electromagnetism became possible with quantum mechanical principles. Harnessing such principles has assisted in explaining the behaviour of particles at the atomic domain, macroscopic particles, bandgap energy in metals and semiconductors, blackbody radiation, wave-particle dualities among others.Relativistic wave equations have attracted tremendous interest from researchers in recent years [1-9] and other references therein. The interest in this regard is not only to enrich our knowledge of the basic principles of quantum mechanics but the numerous applications to diverse areas of human endeavours ranging from molecular physics [10-13] to condensed matter physics [14-16]. The scattering and bound state solutions of relativistic and non-relativistic particles have also attracted several research attentions [4, 17-22] due to the importance of the study of x-ray diffraction and the collision of a particle within the atomic domain[23, 24].

For instance, Mijatovic*et al*. [18] studied the Scattering and bound states of a nonrelativistic neutral spin-l/2 particle in a magnetic field using a single rectangular potential barrier. The tunneling with spin, band structure and change of polarization during transmission and reflection were studied in detail. The Scattering, bound, and quasi-bound states of the generalized symmetric Woods-Saxon potential has also be studied by Lütfüoğlu*et al.* [25]. The Scattering and bound states of fermions in the modified Hulthen potential has also been studied in ref. [4]. Also in ref. [26], the Scattering and Bound States of the Dirac Particle for q-Parameter Hyperbolic
Poschl-Teller Potential has also been studied.

The quasi-bound state solutions have gain attention in recent times due to its applications in resonant tunneling [27] and for this reason, call for essential interest.

The deformed Hylleraas plus Woods-Saxon (dHWS) potential is given by

$$V(r)=\frac{v_0}{b(1+e^{-\alpha r})}+\frac{v_0^2(A+Te^{-\alpha r})^2}{b^2(1+e^{-\alpha r})^2}, \qquad (1)$$

where v_0 is the potential depth parameter, A, T are Hylleraas parameter, and b is adjustable constant.

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The organization of this paper consists of four sections: In Sec. 2, a review of the basic Dirac equations will be done while In Sec. 3 the Dirac equation with the deformed Hylleraas plus Woods-Saxon potential will be carried. The bound, quasi-bound, scattering and scattering phase shift solutions will also be obtained analytically. Finally, the concluding remark in Sec. 4.

2 Dirac equation

Let us recall that the Dirac equation with the scalar potential $S(x)$ and vector potential $V(r)$, $(\hbar = c = 1)$. is given by [47-52]

$$\left[\vec{\sigma}.\vec{\Re} + \chi\{M + S(r)\}\right]\varphi(r) = (E - V(r))\varphi(r), \quad (2)$$

where E is the relativistic energy term, $\vec{\Re}$ is the momentum operator, and σ, χ are the Dirac Matrices given by

$$\vec{\sigma} = \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix},\ \chi = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (7)$$

so that τ and I represents the $2X2$ Pauli and unitary matrix.

The Dirac spinor $\varphi(r)$ in eq.(2) has a solution for the upper combination $F(r)$ and the lower term $G(x)$ and can be written as

$$\varphi_{n,k}(r) = \begin{pmatrix} \dfrac{F_{n,k}(r)}{r} Y^{l}_{j,m}(\theta,\phi) \\ \dfrac{iG_{n.k}(r)}{r} Y^{\tilde{l}}_{j,m}(\theta,\phi) \end{pmatrix}, \quad (8)$$

where $Y^{l}_{j,m}(\theta,\phi)$ and $Y^{\tilde{l}}_{j,m}(\theta,\phi)$ are the spin and pseudospin spherical harmonics, n is the radial quantum number, m is the angular momentum quantum number, $\tilde{l} = l + 1$, and $\kappa = \pm\left(j + 1/2\right)$, with $j = l + 1/2$.

On substituting eq. (8) into eq. (6) we have two coupled differential equation given by

$$G^{'}(r) = (E - M - \Sigma(r)F(r),$$
$$F^{'}(r) = (-E - M + \Delta(r)G(r), \quad (9)$$

where the constant $\Delta(r)$ and $\Sigma(r)$ is given by

$$\Delta(r) = V(r) - S(r),$$
$$\Sigma(r) = V(r) + S(r). \quad (10)$$

Eliminating one component in favour of the other yield the decoupled equations

$$\left\{\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} - \frac{\frac{d\Delta(r)}{dr}}{(-M - E + \Delta(r))}\frac{d}{dr} + (-M - E + \Delta(r))(M - E + \Sigma(r))\right\}F_{n,\kappa}(r) = 0, \quad (11)$$

for $\kappa(\kappa+1) = l(+1)$,

$$\left\{\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - \frac{\frac{d\Sigma(r)}{dr}}{(-E + M + \Sigma(r))}\frac{d}{dr} + (M + E - \Delta(r))(E - M - \Sigma(r))\right\}G_{n,\kappa}(r) = 0, \quad (12)$$

for $\kappa(\kappa-1) = \tilde{l}(\tilde{l}+1)$.

The upper and the lower component were considered here for two different wave functions $\varphi(r)$, and satisfy the boundary conditions $G_{n,\kappa}(0) = F_{n,\kappa}(0) = 0$, and $G_{n,\kappa}(r) = F_{n,\kappa}(r) \to 0$ at infinity.

3. TheDirac equation with dHWS potential

We consider a case for which the Dirac scalar potential $S(r)$ is equal to the vector potential $V(r)$ and eq. (11) reduces to

$$\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - 2(E_{n,k} + M)V(r) - M^2 + E^2_{n,k}\right]F_{n,k}(r) = 0. \quad (13)$$

The upper part of the Dirac wave function in eq. (9) becomes

$$G_{n,k}(r) = \frac{1}{M + E_{n,k}}\left[\frac{d}{dr} + \frac{k}{r}\right]F_{n,k}(r). \tag{14}$$

To deal with the centrifugal term in eq. (13) we use the following approximation

$$\frac{1}{r^2} \approx \frac{2\alpha e^{-\alpha r}}{(1+e^{-\alpha r})^2}. \tag{15}$$

Substituting eqs. (1) and (15) into eq. (13) we obtain

$$\left[\frac{d^2}{dr^2} - 2\alpha^2 k(k+1)\frac{e^{-\alpha r}}{(1+e^{-\alpha r})^2} - 2(E_{n,k}+M)\left[\frac{v_0^2(A+Te^{-\alpha r})^2}{b^2(1+e^{-\alpha r})^2} - \frac{v_0}{b(1+e^{-\alpha r})}\right] - M^2 + E_{n,k}^2\right]F_{n,k}(r) = 0. \tag{16}$$

3.1 Bound and Quasi-bound states solution

The bound states solution is obtained by mapping the potential parameter $v_0 = -v_0$ in eq. (1). By making use of a new variable $y = -e^{-\alpha r}$ turns eq. (16) to a form of hypergeometric equation given by

$$y(1-y)\frac{d^2F_{n,k}(y)}{dy^2} + (1-y)\frac{dF_{n,k}(y)}{dy} + \frac{1}{y(1-y)}\left\{\omega_1 y^2 + \omega_2 y + \omega_3\right\}F_{n,k}(y) = 0, \tag{17}$$

where

$$\left.\begin{aligned}
\omega_1 &= \frac{(M+E_{n,k})}{\alpha^2}\left[(E_{n,k}-M) - \frac{2v_0^2}{b^2}\right],\\
\omega_2 &= \frac{1}{\alpha^2}\left[2\alpha^2 k(k+1) + 2(E_{n,k}+M)\left\{\frac{2ATv_0^2}{b^2} - (E_{n,k}-M) - \frac{v_0}{b}\right\}\right],\\
\omega_3 &= \frac{(M+E_{n,k})}{\alpha^2}\left[\frac{2v_0}{b}\left(1 - \frac{2A^2v_0}{b}\right) - (M-E_{n,k})\right]
\end{aligned}\right\}. \tag{18}$$

Equation (17) has singularities at $z = 0,\ z = 1$,and $z = \infty$, so that the trial wave function may be defined as

$$F_{n,k}(y) = y^{\mu}(1-y)^{\nu}\varphi_{n,k}(z). \tag{19}$$

which turns into a hypergeometric differential equation of the form [4, 26, 30]

$$y(1-y)\frac{d^2\varphi_{n,k}(y)}{dy^2} + \left[1 + 2\mu - (2\mu + 2\nu + 2)y\right]\frac{d\varphi_{n,k}(y)}{dy} - \left(\mu + \nu + i\sqrt{\omega_1}\right)\left(\mu + \nu - i\sqrt{\omega_1}\right)\varphi_{n,k}(y) = 0, \tag{20}$$

where,

$$\left.\begin{aligned}
\mu &= \frac{i}{\alpha}\sqrt{(M+E_{n,k})\left[\frac{2v_0}{b}\left(1 - \frac{2A^2v_0}{b}\right) - (M-E_{n,k})\right]},\\
\nu &= \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{2v_0^2(E_{n,k}+M)\left(2AT - T^2 - A^2\right)}{\alpha^2 b^2} + \frac{1}{2}k(k+1)}\right)
\end{aligned}\right\}. \tag{21}$$

The solution to equation (20) is the second type of hypergeometric function given by [4, 26, 30]

$$\varphi_{n,k}(y) = {}_2F_1\left(a_{n,k}, b_{n,k}, c_{n,k}; y\right), \tag{22}$$

with the parameters $a_{n,k}, b_{n,k}$, and $c_{n,k}$ given as

$$\begin{aligned}
a_{n,k} &= \mu + \nu + i\sqrt{\omega_1},\\
b_{n,k} &= \mu + \nu - i\sqrt{\omega_1},\\
c_{n,k} &= 1 + 2\mu.
\end{aligned} \tag{23}$$

The lower-spinor component of the Dirac wavefunction is then obtained from Eq. (19) as

$$F_{n,k}(y) = -P_{n,k}e^{-\alpha\mu r}\left(1 + e^{-\alpha r}\right)^{\nu}{}_2F_1\left(a_{n,k}, b_{n,k}, c_{n,k}; -e^{-\alpha r}\right), \tag{24}$$

where $P_{n,k}$ is the normalization constant. The lower component is obtained from Eq. (14) as

$$G_{n,k}(r)=-\frac{1}{M+E_{n,k}}\left[\frac{d}{dr}+\frac{k}{r}\right]P_{n,k}e^{-\alpha\mu r}\left(1+e^{-\alpha r}\right)^{\nu}{}_2F_1\left(a_{n.k},b_{n,k},c_{n,k};-e^{-\alpha r}\right). \quad (25)$$

So that,

$$\begin{aligned}G_{n,k}(r)=\frac{P_{n,k}}{M+E_{n,k}}&\left[\frac{ke^{-\alpha r}}{r}\left(1+e^{-\alpha r}\right)^{\nu}{}_2F_1\left(a_{n.k},b_{n,k},c_{n,k};-e^{-\alpha r}\right)\right.\\&+\alpha\mu e^{-\alpha\mu r}\left(1+e^{-\alpha r}\right)^{\nu}{}_2F_1\left(a_{n.k},b_{n,k},c_{n,k};-e^{-\alpha r}\right)\\&+\alpha\nu e^{-\alpha(\mu+1)r}\left(1+e^{-\alpha r}\right)^{\nu-1}{}_2F_1\left(a_{n.k},b_{n,k},c_{n,k};-e^{-\alpha r}\right)\\&\left.-\frac{a_{n.k}b_{n,k}}{c_{n,k}}e^{-\alpha\mu r}\left(1+e^{-\alpha r}\right)^{\nu}{}_2F_1\left(a_{n.k}+1,b_{n,k}+1,c_{n,k}+1;-e^{-\alpha r}\right)\right].\end{aligned} \quad (26)$$

The energy of the bound states for the upper component of the Dirac particle with the dHWSpotential is obtained as

$$b_{n,k}=\mu+\nu-i\sqrt{\frac{(M+E_{n,k})}{\alpha^2}\left[(E_{n,k}-M)-\frac{2v_0^2}{b^2}\right]}=-n, \qquad n=0,1,2,3,....... \quad (27)$$

The quasi-bound state solution is obtained when the energy term is positive ($v_0=+v_0$) and has a complex energy eigenvalue so that

$E_{n,k}^{quasi}=E_r-iE_i$, (28)

where $E_r \gg E_i$, and $E_{n,k}^{quasi}>0$.

Using the same transformation turns the hypergeometric of Eq. (17) into the following

$$y(1-y)\frac{d^2F_{n,k}(y)}{dy^2}+(1-y)\frac{dF_{n,k}(y)}{dy}+\frac{1}{y(1-y)}\left\{\chi_1y^2+\chi_2y+\chi_3\right\}F_{n,k}(y)=0, \quad (29)$$

where

$$\begin{aligned}\chi_1&=\frac{(E_{n,k}-M)}{\alpha^2}\left[(E_{n,k}-M)+\frac{2v_0^2}{b^2}\right],\\\chi_2&=\frac{1}{\alpha^2}\left[2\alpha^2k(k+1)+2(M-E_{n,k})\left\{\frac{2ATv_0^2}{b^2}-(M-E_{n,k})+\frac{v_0}{b}\right\}\right],\\\chi_3&=\frac{(E_{n,k}-M)}{\alpha^2}\left[\frac{-2v_0}{b}\left(1-\frac{2A^2v_0}{b}\right)-(M+E_{n,k})\right].\end{aligned} \quad (30)$$

The trial wavefunction for Eq.(29) is given as $F_{n,k}(y)=y^{\varsigma}(1-y)^{\tau}\Phi_{n,k}(z)$, which turns into a hypergeometric differential equation of the form [27]

$$y(1-y)\frac{d^2\Phi_{n,k}(y)}{dy^2}+\left[1+2\varsigma-(2\varsigma+2\tau+2)y\right]\frac{d\Phi_{n,k}(y)}{dy}-\left(\varsigma+\tau+i\sqrt{\chi_1}\right)\left(\varsigma+\tau-i\sqrt{\chi_1}\right)\Phi_{n,k}(y)=0, \quad (31)$$

where,

$$\begin{aligned}\varsigma&=\frac{1}{\alpha}\sqrt{-(M+E_{n,k})\left[\frac{2v_0}{b}\left(1+\frac{2A^2v_0}{b}\right)-(M+E_{n,k})\right]},\\\tau&=\frac{1}{2}\left(1\pm\sqrt{1+\frac{2v_0^2(M-E_{n,k})\left(2AT-T^2-A^2\right)}{\alpha^2b^2}+\frac{1}{2}k(k+1)}\right).\end{aligned} \quad (32)$$

The solution to equation (31) is the second type of hypergeometric function given by

$\Phi_{n,k}(y)={}_2F_1\left(a_{n.k},b_{n,k},c_{n,k};y\right)$, (33)

with the parameters $a_{n,k},b_{n,k}$, and $c_{n,k}$ given as

$$\begin{aligned}a_{n,k}&=\varsigma+\tau+i\sqrt{\chi_1},\\b_{n,k}&=\varsigma+\tau-i\sqrt{\chi_1},\\c_{n,k}&=1+2\varsigma.\end{aligned} \quad (34)$$

The quasi-bound stateenergy for the upper component of the Dirac particle with the dHWS potential is obtained as

$$b_{n,k} = \varsigma + \tau - \sqrt{\frac{-(E_{n,k} - M)}{\alpha^2}\left[(M - E_{n,k}) + \frac{2v_0^2}{b^2}\right]} = -n, \qquad n = 0,1,2,3,....... \quad (35)$$

3.2 The scattering states and phase-shift solution

To obtain the scattering states, we make use of a new variable

$$z = 1 - e^{-\alpha r}, \qquad z = 1 - y. \quad (36)$$

and obtain

$$z(1-z)\frac{d^2F_{n,k}(z)}{dz^2} + (1-z)\frac{dF_{n,k}(z)}{dz} + \frac{1}{z(1-z)}\left\{\omega_1 z^2 + \omega_2 z + \omega_3\right\}F_{n,k}(z) = 0. \quad (37)$$

By defining the wavefunction in Eq. (36) as $F_{n,k}(z) = z^{\eta}(1-z)^{\gamma}\varphi_{n,k}(z)$ which turns into a hypergeometric differential equation of the form [4, 26, 30]

$$z(1-z)\frac{d^2\varphi_{n,k}(z)}{dz^2} + \left[1 + 2\eta - (2\eta + 2\gamma + 2)z\right]\frac{d\varphi_{n,k}(z)}{dz} - \left(\eta + \gamma + i\sqrt{\omega_1}\right)\left(\eta + \gamma - i\sqrt{\omega_1}\right)\varphi_{n,k}(z) = 0, \quad (38)$$

where,

$$\eta = \frac{i}{\alpha}\sqrt{(M + E_{n,k})\left[\frac{2v_0}{b}\left(1 - \frac{2A^2v_0}{b}\right) - (M - E_{n,k})\right]},$$

$$\gamma = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{2v_0^2(E_{n,k} + M)\left(2AT - T^2 - A^2\right)}{\alpha^2 b^2} + \frac{1}{2}k(k+1)}\right), \qquad (\gamma > 0). \quad (39)$$

The wavefunction in Eq. (37) is then written as

$$F_{n,k}(z) = N_{n,k} z^{\eta}(1-z)^{\gamma}\,{}_2F_1\left(a_{n.k}, b_{n,k}, c_{n,k}; -e^{-\alpha r}\right), \quad (40)$$

where

$N_{n,k}$ is the normalization constant. The lower component of the wavefunction is given by

$$G_{n,k}(r) = -\frac{1}{M + E_{n,k}}\left[\frac{d}{dr} + \frac{k}{r}\right]N_{n,k}e^{-\alpha\eta r}\left(1 + e^{-\alpha r}\right)^{\gamma}{}_2F_1\left(a_{n.k}, b_{n,k}, c_{n,k}; -e^{-\alpha r}\right). \quad (41)$$

So that the total wavefunction of the scattering state is

$$\left.\begin{aligned} \varphi_{n,k}(r) &= \begin{pmatrix} \dfrac{F_{n,k}(r)}{r}Y^{l}_{j,m}(\theta,\phi) \\ \dfrac{iG_{n.k}(r)}{r}Y^{\tilde{l}}_{j,m}(\theta,\phi) \end{pmatrix} \\ &= \frac{N_{n,k}}{r}\begin{pmatrix} \dfrac{-i}{E_{n,k} + M} & \dfrac{1}{(\sigma.\hat{r})\Upsilon(r)} \end{pmatrix} X\ e^{-\alpha\eta r}\left(1 - e^{-\alpha r}\right)^{\gamma} Y^{l}_{jm}(\theta,\varphi)\ X\ {}_2F_1\left(a_{n.k}, b_{n,k}, c_{n,k}; -e^{-\alpha r}\right) \end{aligned}\right\}, \quad (42)$$

where,

$$\Upsilon(r) = \frac{k}{r} + \alpha\gamma + \alpha\eta e^{-\alpha r}\left(1 + e^{-\alpha r}\right)^{-1} - \frac{a_{n.k}b_{n,k}}{c_{n,k}}\ X\ \frac{{}_2F_1\left(a_{n.k}+1, b_{n,k}+1, c_{n,k}+1; -e^{-\alpha r}\right)}{{}_2F_1\left(a_{n.k}, b_{n,k}, c_{n,k}; -e^{-\alpha r}\right)}. \quad (43)$$

The energy eigenvalue for the scattering state is written according to Eq. (27) as follow:

$$\eta + \gamma - i\sqrt{\frac{(M + E_{n,k})}{\alpha^2}\left[(E_{n,k} - M) - \frac{2v_0^2}{b^2}\right]} = -n. \quad (44)$$

At this point, we obtained the scattering phase shift by finding the asymptotic form of the wave function for $r \to 0 \ (z \to 1)$. And by using the properties of the hypergeometric function [28, 29] gives the following

$$
\begin{aligned}
{}_2F_1(a,b,c;x) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} X \ {}_2F_1(a,b,a+b-c+1;1-x) \\
&+(1-x)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} X \ {}_2F_1(a,b,a+b-c+1;1-x),
\end{aligned} \tag{45}
$$

From Eq. (37) and using the fact that ${}_2F_1(a,b,c,0)=1$ as $r \to \infty$, we have that

$$
\lim_{r\to\infty} {}_2F_1(a,b,c;1-e^{-\alpha r}) \square \left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} + e^{-2ik_2\alpha r}\left(\frac{\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\right)\right] \tag{46}
$$

So that the asymptotic wave number becomes

$$
k_2 = \sqrt{\frac{(M+E_{n,k})}{\alpha^2}\left[\frac{2v_0}{b}\left(1-\frac{2A^2v_0}{b}\right)-(M-E_{n,k})\right]}, \ \eta = ik_2. \tag{47}
$$

By using the upper component of the wavefunction of Eq. (40) and substituting Eq. (46) we obtain

$$
\underset{r\to\infty}{F_{n,k}(r)} \to \underset{r\to\infty}{N_{n,k}} z^{\eta}(1-z)^{\gamma} X \left[\frac{\Gamma(c_{n,k})\Gamma(c_{n,k}-a_{n,k}-b_{n,k})}{\Gamma(c_{n,k}-a_{n,k})\Gamma(c_{n,k}-b_{n,k})} + e^{-2ik_1\alpha r}\left(\frac{\Gamma(c_{n,k})\Gamma(a_{n,k}+b_{n,k}-c_{n,k})}{\Gamma(a_{n,k})\Gamma(b_{n,k})}\right)\right]. \tag{48}
$$

So that

$$
\begin{aligned}
\underset{r\to\infty}{F_{n,k}(r)} \to\ & N_{n,k}\Gamma(c_{n,k}) X \left|\frac{\Gamma(c_{n,k}-a_{n,k}-b_{n,k})}{\Gamma(c_{n,k}-a_{n,k})\Gamma(c_{n,k}-b_{n,k})}\right| X \left(e^{-i(k_2\alpha r-\delta} + e^{i(k_2\alpha r-\delta}\right) \\
&= 2N_{n,k}\Gamma(c_{n,k}) X \left|\frac{\Gamma(c_{n,k}-a_{n,k}-b_{n,k})}{\Gamma(c_{n,k}-a_{n,k})\Gamma(c_{n,k}-b_{n,k})}\right| X \cos(k_2\alpha r-\delta) \\
& 2N_{n,k}\Gamma(c_{n,k}) X \left|\frac{\Gamma(c_{n,k}-a_{n,k}-b_{n,k})}{\Gamma(c_{n,k}-a_{n,k})\Gamma(c_{n,k}-b_{n,k})}\right| \\
& X \ sin\left(\delta - k_2\alpha r + \frac{\pi}{2}\right).
\end{aligned} \tag{49}
$$

The boundary condition imposed for the asymptotic behaviour of the scattering wave is given by [28, 29]

$$
f(r) = 2 sin\left(k_2 r - \frac{\pi}{2} l + \delta_l\right). \tag{50}
$$

Then

$$
\begin{aligned}
\underset{r\to\infty}{F_{n,k}(r)} \to\ & 2N_{n,k}\Gamma(c_{n,k}) X \left|\frac{\Gamma(c_{n,k}-a_{n,k}-b_{n,k})}{\Gamma(c_{n,k}-a_{n,k})\Gamma(c_{n,k}-b_{n,k})}\right| \\
& X \ sin\left(k_2\alpha r + \frac{\pi}{2} + \arg\left(\frac{\Gamma(c_{n,k}-a_{n,k}-b_{n,k})}{\Gamma(c_{n,k}-a_{n,k})\Gamma(c_{n,k}-b_{n,k})}\right)\right).
\end{aligned} \tag{51}
$$

So that

$$
\begin{aligned}
N_{n,k} &= \frac{1}{(1+2\mu)^2} X \\
& \left|\frac{\Gamma(1-2\nu)}{\Gamma\left(1+\mu-\nu-\frac{1}{\alpha}\sqrt{\frac{(M+E_{n,k})}{\alpha^2}\left[(M-E_{n,k})-\frac{2v_0^2}{b^2}\right]}\right)\Gamma\left(1+\mu-\nu+\frac{1}{\alpha}\sqrt{\frac{(M+E_{n,k})}{\alpha^2}\left[(M-E_{n,k})-\frac{2v_0^2}{b^2}\right]}\right)}\right|
\end{aligned} \tag{52}
$$

And the upper component of the scattering phase shift of the wave function becomes

$$\delta_{l,n,k} = \frac{\pi}{2}(l+1) + \arg\Gamma(1-2\nu)$$
$$-\arg\Gamma\left(1+\mu-\nu-\frac{1}{\alpha}\sqrt{\frac{(M+E_{n,k})}{\alpha^2}\left[(M-E_{n,k})-\frac{2v_0^2}{b^2}\right]}\right) \quad (53)$$
$$-\arg\Gamma\left(1+\mu-\nu+\frac{1}{\alpha}\sqrt{\frac{(M+E_{n,k})}{\alpha^2}\left[(M-E_{n,k})-\frac{2v_0^2}{b^2}\right]}\right).$$

So that the lower component using Eq. (41) becomes

$$\underset{r\to\infty}{G_{n,k}(r)} \to \underset{r\to\infty}{\frac{N_{n,k}}{E_{n,l}+M}}\Upsilon(r)(1-z)^{\gamma}\ X\ e^{-ik_1\alpha r}\left[\frac{\Gamma(c_{n,k})\Gamma(c_{n,k}-a_{n,k}-b_{n,k})}{\Gamma(c_{n,k}-a_{n,k})\Gamma(c_{n,k}-b_{n,k})}\right. \quad (54)$$
$$\left.+e^{2ik_1\alpha r}\left(\frac{\Gamma(c_{n,k})\Gamma(a_{n,k}+b_{n,k}-c_{n,k})}{\Gamma(a_{n,k})\Gamma(b_{n,k})}\right)\right].$$

$$\lim_{r\to\infty}\Upsilon(r) = \lim_{r\to\infty}\left[\frac{k}{r}+\alpha\gamma+\alpha\eta e^{-\alpha r}\left(1+e^{-\alpha r}\right)^{-1}-\frac{a_{n.k}b_{n,k}}{c_{n,k}}\right.$$
$$\left. X\ \frac{{}_2F_1\left(a_{n.k}+1, b_{n,k}+1, c_{n,k}+1; -e^{-\alpha r}\right)}{{}_2F_1\left(a_{n.k}, b_{n,k}, c_{n,k}; -e^{-\alpha r}\right)}\right] \quad (55)$$
$$=\alpha\gamma-\frac{a_{n.k}b_{n,k}}{c_{n,k}}.$$

So that,

$$\underset{r\to\infty}{G_{n,k}(r)} \to \frac{N_{n,k}}{E_{n,l}+M}\left(\alpha\gamma-\frac{a_{n.k}b_{n,k}}{c_{n,k}}\right)\Gamma(c_{n,k})\left|\frac{\Gamma(c_{n,k}-a_{n,k}-b_{n,k})}{\Gamma(c_{n,k}-a_{n,k})\Gamma(c_{n,k}-b_{n,k})}\right| \quad (56)$$
$$X\left(e^{-i(k_2\alpha r-\delta)}+e^{i(k_2\alpha r-\delta)}\right).$$

With Eqs. (49), (50), (51), and (55) we obtain the following

$$\underset{r\to\infty}{G_{n,k}(r)} \to \frac{2N_{n,k}}{E_{n,l}+M}\left(\alpha\gamma-\frac{a_{n.k}b_{n,k}}{c_{n,k}}\right)\Gamma(c_{n,k})X\left|\frac{\Gamma(c_{n,k}-a_{n,k}-b_{n,k})}{\Gamma(c_{n,k}-a_{n,k})\Gamma(c_{n,k}-b_{n,k})}\right| \quad (57)$$
$$X\ Sin\left(k_2\alpha r+\frac{\pi}{2}+\arg\left(\frac{\Gamma(c_{n,k}-a_{n,k}-b_{n,k})}{\Gamma(c_{n,k}-a_{n,k})\Gamma(c_{n,k}-b_{n,k})}\right)\right).$$

So that

$$\delta_{l,n,k} = \frac{\pi}{2}(l+1) + \arg\Gamma(c_{n,k}-a_{n,k}-b_{n,k}) \quad (58)$$
$$-\arg\Gamma\left(c_{n,k}-a_{n,k}\right)-\arg\Gamma\left(c_{n,k}-b_{n,k}\right)$$

4. Concluding Remarks

We have studied analytically the Dirac particles with deformed Hylleraas plus Woods-Saxon potential and obtained the bound, quasi-bound, and scattering phase shift of this problem by using the properties of the hypergeometric function. This study can find its applications in resonant tunneling reactions in atomic and nuclear physics.

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