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# ANALYTICAL SOLUTIONS OF THE KLEIN-GORDON EQUATION IN THE PRESENCE OF COULOMB-ECKART POTENTIAL

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***Abstract***

***The Nikiforov-Uvarov (NU) method is employed to calculate the approximate analytical solutions of the Klein-Gordon equation with a newly proposed potential known as the Coulomb-Eckart potential. The new potential is a combination of the Coulomb potential and the Eckart potential. The energy equation, eigenvalues and energy spectra of HCl, LiH and CH diatomic molecules were obtained. Also, the ro-vibrational energy for different energy states were computed. By varying some potential parameters, energy equations of some other potentials like the Hellmann potential and the Yukawa potential were obtained.***

## 1. Introduction

The continuous interest in the study of the relativistic and non-relativistic wave equations in quantum mechanics had increased over the years because of its importance in physics today. The non-relativistic regime is described using the Schrödinger equation while the Klein-Gordon and the Dirac equation are used to describe the relativistic effect for spin 0 and spin-1/2 particles respectively when considering the relativistic effect. The solution of these waves can either be exact or approximate depending of the potential under consideration. Over the years, several traditional techniques have been developed to obtain these solutions. Examples include the Nikiforov-Uvarov (NU) method [1-3], the supersymmetric quantum mechanics and shape invariance method [4-9], asymptotic iteration method (AIM) [10-15], the variational method [16] and others. Most of these techniques can be used effectively for approximate solution i.e. when $l \neq 0$, for different physical potential models. However, for $l = 0$, the solution of these equations can be obtain with only few potentials such as Manning-Rosen potential [4,17,18], Hulthén potential [19, 20], Kratzer potential [21], Wood-Saxon potential [22, 23] and the Pöschl-Teller potential [24]. Thus, a lot of works have focus on the approximate solutions with some of these potential models. Some of the reported works are Yahya and Oyewumi [25], who solved the Klein-Gordon equation with the Pöschl-Teller-type potential. In their study, they obtained the thermodynamic properties for the Pöschl-Teller-type potential in the classical limit. Jia and Cao [26] obtained the molecular energy of spinless Morse potential under the Klein-Gordon equation. They also studied the vibrational transition frequencies using the methodology of supersymmetric and shape invariance approach. Recently, Ebomwonyi et al. [27], studied arbitrary $l$ solutions of the Schrödinger equation in the presence of a new potential formed from a combination of Hellmann potential and Deng-Fan potential model. In the present work, we desire to study the approximate solution of the Klein-Gordon for a case when $V(r) = S(r)$ and the Schrödinger equations with a combined potential called the Coulomb-Eckart potential model. These potentials were individually studied under different in the past [28, 29]. However, a combination of these potential will give more applications and interesting results; and to the best of our knowledge, this combination is novel in the literature. The Coulomb-Eckart potential is of the form

$$V(r) = -\frac{A}{r} + \frac{Be^{-\delta r}}{1-e^{-\delta r}} - \frac{4Ce^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2} \tag{1}$$

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where $r$ is internuclear separation, α is the screening parameter related to the range of the potential, $A$, $B$ and $C$ are the potential strengths.

## 2. Parametric Nikiforov-Uvarov method

The Nikiforov-Uvarov (NU) method is based upon reducing the second-order linear differential equation to a hypergeometric type equation [30]. By introducing an appropriate transformation $s = s(x)$, we can then write an equation of the form

$$\psi_n''(s) + \frac{\bar{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\bar{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0, \tag{2}$$

where $\sigma(s)$ and $\bar{\sigma}(s)$ are polynomials of at most second degree two and $\bar{\tau}(s)$ is a polynomial of degree one at most. It is discovered that the conventional NU method has tedious and logical calculations. In order to simplify these calculations, Tezcan and Sever [31] in their study, derived the following equation from the conventional NU method [30-32]

$$\frac{d^2\psi(s)}{ds^2} + \left(\frac{\lambda_1 - \lambda_2 s}{s(1-\lambda_3 s)}\right)\frac{d\psi(s)}{ds} + \left(\frac{-\varsigma_1 s^2 + \varsigma_2 s - \varsigma}{s(1-\lambda_3 s)}\right)\psi(s) = 0. \tag{3}$$

which they called parametric Nikiforov-Uvarov method. From the parametric NU method, the bound state energy condition is obtained as [31]

$$n\lambda_2 - (2n+1)\lambda_5 + (2n+1)\left(\sqrt{\lambda_9} + \lambda_3\sqrt{\lambda_8}\right) + n(n-1)\lambda_3 + \lambda_7 + 2\lambda_3\lambda_8 + 2\sqrt{\lambda_8\lambda_9} = 0, \tag{4}$$

with the wave function as

$$\psi_{n,l}(s) = N_{n,l}\, s^{\lambda_{12}}(1-\lambda_3 s)^{-\lambda_{12}-\frac{\lambda_{13}}{\lambda_3}} P_n^{\left(\lambda_{10}-1, \frac{\lambda_{11}}{\lambda_3}-\lambda_{10}-1\right)}(1-2\lambda_3 s), \tag{5}$$

where the parameters in Eq. (4) and Eq. (5) are given as follows

$$\left.\begin{aligned} &\lambda_4 = \frac{1-\lambda_1}{2}, \lambda_5 = \frac{\lambda_2 - 2\lambda_3}{2}, \lambda_6 = \lambda_5^2 + \zeta_1, \lambda_7 = 2\lambda_4\lambda_5 - \zeta_2, \lambda_8 = \lambda_4^2 + \zeta_3, \\ &\lambda_9 = \lambda_3(\lambda_7 + \lambda_3\lambda_8) + \lambda_6, \lambda_{10} = \lambda_1 + 2\lambda_4 - 2\sqrt{\lambda_8}, \lambda_{11} = \lambda_2 - 2\lambda_5 + 2\left(\sqrt{\lambda_9} - \lambda_3\sqrt{\lambda_8}\right), \\ &\lambda_{12} = \lambda_4 - \sqrt{\lambda_8}, \lambda_{13} = \lambda_5 - \left(\sqrt{\lambda_9} - \lambda_3\sqrt{\lambda_8}\right) \end{aligned}\right\}. \tag{6}$$

## 3. Bound state solution

### 3.1 Solution to the Klein-Gordon equation

The Klein-Gordon equation with scalar potential $S(r)$ and vector potential $V(r)$ of a particle of mass $M$ with relativistic energy $E$ is given as

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + (S(r)+M)^2 - (E - V(r))^2\right]\varphi(r) \tag{7}$$

Eq. (7) has a potential $2V$ in the nonrelativistic limit. Substituting Eq. (1) into Eq. (7), we have

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + \left(-\frac{A}{r} + \frac{Be^{-\delta r}}{1-e^{-\delta r}} - \frac{4Ce^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2} + M\right)^2 + \left(E - \left(V(r) = -\frac{A}{r} + \frac{Be^{-\delta r}}{1-e^{-\delta r}} - \frac{4Ce^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}\right)\right)^2\right]\varphi(r) \tag{8}$$

Due to the centrifugal term $\frac{l(l+1)}{r^2}$ in Eq. (8), we need to apply a suitable approximation scheme to deal with the centrifugal term. For a short potential range, we apply the following Greene-Aldrich [33] approximate type

$$\frac{1}{r^2} \approx \frac{\delta^2}{\left(1-e^{-\delta r}\right)^2} \tag{9}$$

Where we have taken $\delta = 2\alpha$

Substituting Eq. (9) into Eq. (8), we have

$$\left[\frac{d^2}{dr^2} - \frac{4l(l+1)\alpha^2}{\left(1-e^{-2\alpha r}\right)^2} - \frac{2A(M+E)2\alpha}{1-e^{-2\alpha r}} + \frac{2B(M+E)e^{-2\alpha r}}{1-e^{-2\alpha r}} - \frac{8C(M+E)e^{-2\alpha r}}{1-e^{-2\alpha r}} + M^2 - E^2\right]\varphi(r) = 0 \tag{10}$$

In order to use the parametric Nikiforov-Uvarov method, it is important we define a variable of the form $s = e^{-2\alpha r}$ and substitute it into Eq. (10) to have

$$\left[\frac{d^2}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{d}{ds}+\frac{1}{s^2(1-s^2)}\left\{-\kappa_1 s^2+\kappa_2 s-\kappa_3\right\}\right]R(s)=0, \tag{11}$$

where

$$\kappa_1=\frac{B(E+M)}{2\alpha^2}+\frac{E^2-M^2}{4\alpha^2},\kappa_2=\frac{B(E+M)}{2\alpha^2}-\frac{2C(E+M)}{\alpha^2}+\frac{A(E+M)}{\alpha}+\frac{E^2-M^2}{2\alpha^2},\kappa_3=\frac{A(E+M)}{\alpha}+\frac{E^2-M^2}{4\alpha^2}+l(l+1). \tag{12}$$

Comparing Eq. (11) with Eq. (3), we obtain the following parametric constants

$$\begin{aligned}&\lambda_1=\lambda_2=\lambda_3=1,\lambda_4=0,\lambda_5=-\frac{1}{2},\lambda_6=\frac{1}{4}+\kappa_1,\lambda_7=-\kappa_2,\lambda_8=\kappa_3,\\&\lambda_9=-\kappa_2+\kappa_3+\kappa_1+\frac{1}{4},\lambda_{10}=1+2\sqrt{\kappa_3},\lambda_{11}=2\left(1-\sqrt{\lambda_8}+\sqrt{\lambda_9}\right),\\&\lambda_{12}=-\sqrt{\kappa_3},\lambda_{13}=-\frac{1}{2}-\sqrt{\lambda_9}+\sqrt{\lambda_8}.\end{aligned} \tag{13}$$

Using these parametric constants in Eq. (13) in Eq. (4), the energy equation is obtained as

$$\frac{1}{4\alpha^2}\left(\left(E^2-M^2\right)+2A\alpha(E+M)-4\alpha^2L\right)=\left[\frac{\left(n+\frac{1}{2}\right)\sqrt{\frac{8C}{\alpha^2}(E+M)-(2l+1)^2}+n(n-1)+\left(\frac{A}{2\alpha}+\frac{2C}{\alpha^2}-\frac{B}{2\alpha^2}\right)(E+M)-2L-\frac{1}{2}}{2n+1+\sqrt{\frac{8C}{\alpha^2}(E+M)-(2l+1)^2}}\right]^2 \tag{14}$$

### 3.2 Solution to the Schrödinger equation

In order to obtain the solution of the Schrödinger-like equation in the nonrelativistic limit with the Coulomb-Eckart potential in Eq. (1), we write the radial Schrödinger equation of the form [29]

$$\left[\frac{d^2}{dr^2}+\frac{2\mu}{\hbar^2}\left\{E-V(r)-\frac{L\hbar^2}{2\mu r}\right\}\right]R(r)=0, \tag{15}$$

where $L=l(l+1)$, $\mu$ is the reduced particle mass, $E$ is the non-relativistic energy and $V(r)$ is the interacting potential. Now substituting the Eq. (1) and Eq. (9) into Eq. (15), we have

$$\left[\frac{d^2}{dr^2}+\frac{2\mu}{\hbar^2}\left\{E+\frac{2\alpha A}{\left(1-e^{-2\alpha r}\right)}-\frac{Be^{-2\alpha r}}{1-e^{-2\alpha r}}+\frac{4Ce^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}-\frac{L\hbar^2 4\alpha^2}{2\mu\left(1-e^{-2\alpha r}\right)^2}\right\}\right]R(r)=0. \tag{16}$$

Using the same transformation as in the Klein-Gordon equation and substitute it into Eq. (16) we, have

$$\left[\frac{d^2}{ds^2}+\frac{(1-s)d}{s(1-s)ds}+\frac{1}{s^2(1-s)^2}\left\{-A_1s^2+A_2s-A_3\right\}\right]R(s)=0, \tag{17}$$

where

$$A_1=\frac{-\mu E-\mu B}{2\alpha^2\hbar^2}, \tag{18}$$

$$A_2=\frac{-\mu(2E+B+2A\alpha-4C)}{2\alpha^2\hbar^2}, \tag{19}$$

$$A_3=L-\frac{\mu(B+E)}{2\alpha^2\hbar^2}.\cdot \tag{20}$$

On comparing Eq. (17) with Eq. (3), we obtain

$$\left.\begin{aligned}&\theta_1=\theta_2=\theta_3=1,\theta_4=0,\theta_5=-\frac{1}{2},\theta_6=\frac{1}{4}+A_1,\theta_7=-A_2,\theta_8=A_3,\\&\theta_9=\frac{1}{4}\left(\left(1+2l\right)^2-\frac{8\mu C}{\alpha^2\hbar^2}\right),\theta_{10}=1+2\sqrt{A_3},\theta_{11}=2\left(1+\sqrt{\theta_9}+\sqrt{A_3}\right),\\&\theta_{12}=\sqrt{A_3},\theta_{13}=-\frac{1}{2}-\sqrt{\theta_9}-\sqrt{A_3}\end{aligned}\right\} \tag{21}$$

Now substituting the appropriate parametric constants in Eq. (21) into Eq. (4), the nonrelativistic energy equation for the Coulomb-Eckart potential is obtained as

$$E = \frac{2\alpha^2\hbar^2}{\mu}\left[\ell(\ell+1) - \left\{\frac{\frac{\mu(4C+2\alpha A-B)}{2\alpha^2\hbar^2} - 2\ell(\ell+1) - n(n+1) - \frac{1}{2} - \left(n+\frac{1}{2}\right)\sqrt{(1+2\ell)^2 - \frac{8\mu C}{\alpha^2\hbar^2}}}{1+2n+\sqrt{(1+2\ell)^2 - \frac{8\mu C}{\alpha^2\hbar^2}}}\right\}^2\right] - 2\alpha A \quad (22)$$

Table 1: Eigenvalues for $S = V$ for different values of $A, B, C$

| state | $\alpha$ | A=3,B=2,C=1 | B=3,C=2,A=1 | C=3,A=2,B=1 |
|---|---|---|---|---|
| ***2p*** | 0.20 | -0.121573037 | 1.596974043 | 4.984537368 |
| | 0.25 | -0.246720366 | 1.624127558 | 4.978625478 |
| | 0.30 | -0.363398574 | 1.657963870 | 4.975411249 |
| ***3p*** | 0.20 | 0.213878422 | 2.082085681 | 5.636945503 |
| | 0.25 | 0.169641448 | 2.224864947 | 5.781081216 |
| | 0.30 | 0.128749939 | 2.370101913 | 5.922107193 |
| ***3d*** | 0.20 | 0.025865441 | 1.741322672 | 5.072496125 |
| | 0.25 | -0.021735002 | 1.850123256 | 5.115069275 |
| | 0.30 | -0.043422302 | 1.985866087 | 5.171178676 |
| ***4p*** | 0.20 | 0.593179477 | 2.612922085 | 6.329231686 |
| | 0.25 | 0.651121652 | 2.894666417 | 6.645895549 |
| | 0.30 | 0.711457496 | 3.179277529 | 6.958763198 |
| ***4d*** | 0.20 | 0.343592799 | 2.213527357 | 5.720139919 |
| | 0.25 | 0.362751334 | 2.426451131 | 5.908640340 |
| | 0.30 | 0.396922423 | 2.656923872 | 6.103087016 |
| ***4f*** | 0.20 | 0.270800284 | 1.968924914 | 5.208860058 |
| | 0.25 | 0.376177899 | 2.216949678 | 5.330623926 |
| | 0.30 | 0.571857003 | 2.538014138 | 5.487800922 |

Table 2: Energy spectra for HCl, LiH and CH for $2p$ to $4f$ states; $\hbar c = 1973.29 eV\dot{A}$, $\mu_{HCl} = 0.9801045 amu,$, $\mu_{LiH} = 0.8801221 amu,$, $\mu_{CH} = 0.929931 amu,$, $2B = A = 2 = -2C$.

| state | $\alpha$ | $HCl$ | $LiH$ | $CH$ |
|---|---|---|---|---|
| | | | | |
| ***2p*** | 0.10 | 3.306050903 | 3.507385413 | 3.402421902 |
| | 0.15 | 5.387403793 | 5.880970851 | 5.624331604 |
| | 0.20 | 8.578879671 | 9.478067148 | 9.010895426 |
| ***3p*** | 0.10 | 1.483263394 | 1.632546315 | 1.554374747 |
| | 0.15 | 3.090964529 | 3.480121112 | 3.277708186 |
| | 0.20 | 5.618459629 | 6.328062076 | 5.959465221 |
| ***3d*** | 0.10 | 8.456705783 | 9.221052317 | 8.823892712 |
| | 0.15 | 16.75546869 | 18.50518821 | 17.59646044 |
| | 0.20 | 28.61378177 | 31.74110263 | 30.11719323 |
| ***4p*** | 0.10 | 1.243370524 | 1.450429206 | 1.342467664 |
| | 0.15 | 3.456891290 | 3.948961455 | 3.693484310 |
| | 0.20 | 6.685976721 | 7.556229436 | 7.104608190 |
| ***4d*** | 0.10 | 5.388504535 | 5.939705114 | 5.653093141 |
| | 0.15 | 11.39664223 | 12.67160740 | 12.00941455 |
| | 0.20 | 20.02753155 | 22.30707360 | 21.12342071 |
| ***4f*** | 0.10 | 16.56907552 | 18.23954454 | 17.37194455 |
| | 0.15 | 34.94067213 | 38.72825884 | 36.76153418 |
| | 0.20 | 60.90013035 | 67.64852340 | 64.14467373 |

Table 3: Comparison of ro-vibrational Hellmann energy $\left(-E_{n\ell}\right)$ for 2s, 2p, 3s, 3p and 3d with $\hbar = 2\mu = 1$, $B = \delta$ and $A = 2$.

| state | $\delta$ | Present | Hamzavi et al.[34] | Onate et al.[35] |
|---|---|---|---|---|
| ***2s*** | 0.001 | 0.064001 | 0.063243 | 0.064250 |
| | 0.005 | 0.070025 | 0.067106 | 0.071256 |
| | 0.010 | 0.077600 | 0.071689 | 0.080025 |
| ***2p*** | 0.001 | 0.063750 | 0.063495 | 0.063999 |
| | 0.005 | 0.068756 | 0.067377 | 0.069975 |
| | 0.010 | 0.075025 | 0.072020 | 0.077400 |
| ***3s*** | 0.001 | 0.029280 | 0.028283 | 0.029611 |
| | 0.005 | 0.035334 | 0.031993 | 0.036951 |
| | 0.010 | 0.043003 | 0.036142 | 0.046136 |
| ***3p*** | 0.001 | 0.029168 | 0.028765 | 0.029499 |
| | 0.005 | 0.034756 | 0.032480 | 0.036356 |
| | 0.010 | 0.041803 | 0.036645 | 0.044869 |
| ***3d*** | 0.001 | 0.028945 | 0.028767 | 0.029274 |
| | 0.005 | 0.033617 | 0.032526 | 0.035184 |
| | 0.010 | 0.039469 | 0.036814 | 0.042403 |

**Discussion**

Table 1 shows the energy eigenvalues for different values of the potential strengths, *A*, *B* and *C*. It can be seen that as the screening parameter increases across the states, the energy eigenvalue decreases for *2p*, *3p* and *3d*. However, the reverse is the case in *4p*, *4d* and *4f*. It is also observed in Table 1 that energy increases linearly with the potential strength C. In Table 2, we present the energy spectrum for three diatomic molecules. It is also observed that as the screeening parameter increases, the energy also increases.

In Table 3, the ro-vibrational energies for *2p*, *3s*, *3p* and *3d* states were computed and it can be observed that our results compare favourably with the works of Hamzavi et al, 2013 and Onate et al., 2017.

One very important and interesting feature of our proposed potential is that, the results of other potentials can be deduced from it. For instance, when we put $A = 0$, the potential reduced to Eckart potential

$$V(r) = -\frac{A}{r} + \frac{Be^{-\delta r}}{1-e^{-\delta r}} - \frac{4Ce^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}, \tag{23}$$

and the energy equation becomes

$$E = \frac{2L\alpha^2\hbar^2}{\mu} - \frac{2\alpha^2\hbar^2}{\mu}\left\{\frac{\frac{-4\mu C+\mu B}{2\alpha^2\hbar^2} + 2L + n(n+1) + \left(n+\frac{1}{2}\right)\sqrt{\left(1+2l\right)^2 - \frac{8\mu C}{\alpha^2\hbar^2}} - \frac{1}{2}}{2n+1+\sqrt{\left(1+2l\right)^2 - \frac{8\mu C}{\alpha^2\hbar^2}}}\right\}^2. \tag{24}$$

Similarly, when we put $C = 0$, the potential turns to Hellmann potential

$$V(r) = -\frac{A}{r} + \frac{Be^{-\delta r}}{1-e^{-\delta r}}, \tag{25}$$

with energy equation as

$$E = \frac{2L\alpha^2\hbar^2}{\mu} - 2A\alpha - \frac{2\alpha^2\hbar^2}{\mu}\left\{\frac{\frac{-2\mu\alpha A+\mu B}{2\alpha^2\hbar^2} + 2L + n(n+1) + \left(n+\frac{1}{2}\right)(1+2l) - \frac{1}{2}}{2(1+n+l)}\right\}^2. \tag{26}$$

Putting $B = C = 0$, the potential becomes Coulomb like potential

$$V(r) = -\frac{A}{r} \tag{27}$$

with energy equation as

$$E = \frac{2L\alpha^2\hbar^2}{\mu} - 2A\alpha - \frac{2\alpha^2\hbar^2}{\mu}\left(\frac{\frac{\mu\alpha A}{\alpha^2\hbar^2} + 2L + n(n+1) + \left(n+\frac{1}{2}\right)(1+2l) - \frac{1}{2}}{2(1+n+l)+}\right)^2 . \quad (28)$$

Putting $A = C = 0$, the potential turns to a Yukawa like potential

$$V(r) = \frac{Be^{-\delta r}}{1-e^{-\delta r}} \quad (29)$$

and the energy equation becomes

$$E = \frac{2L\alpha^2\hbar^2}{\mu} - \frac{2\alpha^2\hbar^2}{\mu}\left(\frac{\frac{\mu B}{2\alpha^2\hbar^2} + 2L + n(n+1) + \left(n+\frac{1}{2}\right)(1+2l) - \frac{1}{2}}{2(1+n+l)}\right)^2 . \quad (30)$$

**Conclusion**

The solutions of the Coulomb-Eckart potential (a combination of the Coulomb potential and the Eckart potential) for the Klein-Gordon equation have been obtained via the Nikiforov-Uvarov method. The energy equation and the eigenvalues for different values of the potential strengths were obtained. The energy spectra of HCl, LiH and CH diatomic molecules for *2p* to *4f* states were also obtained. Also, the ro-vibrational energies for different states that we computed compare favourably and are in agreement with other works in literature. The Hellmann potential and the Yukawa potential were recovered from the energy equation by adjusting the potential parameters.

**References**

[1] Ikot, A.N. and Akpabio, L.E. (2010). Approximate analytical solution of the Schrödinger equation with Hulthén potential for arbitrary *l*-state. International Review of Physics 4, pp 224-228.

[2] Antia, A.D., Ikot, A.N. and Akpabio, L.E. (2010). Exact solution of the Schrödinger equation with Manning-Rosen potential plus a ring-shaped like potential by Nikiforov-Uvarov method. European Journal of Scientific Research 46, pp 107-118.

[3] Ikhdair, S.M. and Sever, R. (2006). A pertubative treatment for the energy levels of neutral atoms. International Journal of Modern Physics A 21, pp 6465-6476.

[4] Wei, G.F. and Dong, S.H. (2010). Pseudospin symmetry in the relativistic Manning-Rosen potential including a Pekeris-type approximation to the pseudo-centrifugal term. Physics Letters B 686, pp 288-292.

[5] Jia, C.S., Chen, T. and He, S. (2013). Bound state solutions of the Klein-Gordon equation with the improved expression of the Manning-Rosen potential energy model. Physics Letters A 377, pp 682-686.

[6] Onate, C. (2015). Bound state solution of Duffin-Kemmer-Petaiu equation with Yukawa potential. African Review of Physics 9, pp 259-264.

[7] Hassanabadi, H., Zarrinkamar, S. and Rahimov, H. (2011). Approximate solution of the D-dimensional Klein-Gordon equation with Hulthén-type potential via SUSYQM. Communications in Theoretical Physics 56, pp 423-428.

[8] Hassanabadi, H., Lu, L.L., Zarrinkamar, S., Wu, G. and Rahimov, H. (2012). Approximate solutions of Schrödinger equation under Manning-Rosen potential in arbitrary dimension via SUSYQM. Acta Physica Polonica A 122, pp 650-654.

[9] Hassanabadi, H., Maghsoodi, E., Zarrinkamar, S. and Rahimov, H. (2012). Actual and general Manning-Rosen potentials under spin and pseudospin symmetries of the Dirac equation. Canadian Journal of Physics 90, pp 633-646.

[10] Bayrak, O. and Boztosun, I. (2007). The pseudospin symmetric solution of the Morse potential for any *k*-state. Journal of Physics A: Mathematical and Theoretical 40, pp 11119-11127.

[11] Soylu, A., Bayrak, O. and Boztosun, I. (2008). *k*-state solutions of the Dirac equation for the Eckart potential with pseudospin and spin symmetry. Journal of Physics A: Mathematical and Theoretical 41, pp 065308 (1-8).

[12] Soylu, A., Bayrak, O. and Boztosun, I. (2007). An approximate solution of Dirac-Hulthén problem with pseudospin and spin symmetry for any *k*-state. Journal of Physics A: Mathematical and Theoretical 48, pp 082302-082309.

[13] Oyewumi, K.J., Falaye, B.J., Onate, C.A., Oluwadare, O.J. and Yahya, W.A. (2014). Thermodynamic properties and the approximate solution of the Schrödinger equation with the shifted Deng-Fan potential model. Journal of Molecular Physics 112, pp 127-141.

[14] Oyewumi, K.J., Falaye, B.J., Onate, C.A., Oluwadare, O.J. and Yahya, W.A. (2014). *k*-state solutions for the fermionic massive spin-$^1/_2$ particles interacting with double ring-shaped Kratzer and oscillator potentials. International Journal of Modern Physics E 23, pp 145000 (1-6).

[15] Behere, S.H. (1982). Anharmonicity constants of diatomic molecules for external Rydberg and Dimitreva-Zenerich potential functions. Indian Journal of Physics 55 B, pp 272-277.

[16] Batha, B. (2009). A variational iteration method for solving the nonlinear Klein-Gordon equation. Australian Journal of Basic and Applied Science 3, pp 3876-3890.

[17] Qiang, W.C. and Dong, S.H. (2009). The Manning-Rosen potential studied by a new approximate scheme to the centrifugal term. Physica Scripta 79, pp 045004 (1-5).

[18] Manning, M.F. (1933). Minutes of the Middletown meeting. Physical Review 44 (11), pp 951-954.

[19] Ikot, A.N., Akpabio, L.E. and Uwah, E.J. (2011). Bound state solutions of the Klein-Gordon equation with Hulthén potential. Electronic Journal of Theoretical Physics 8 (25), pp 225-232.

[20] Egrifes, H. and Sever, R. (2005). Bound states of the Dirac equation for the PT-symmetric generalized Hulthén potential by the Nikiforov-Uvarov method. Physics Letters A 344 (2), pp 117-126.

[21] Qiang, W.C. (2004). Bound state of the Klein-Gordon equation of a ring-shaped Kratzer-type potential. Chinese Journal of Physics 13 (5), pp 575-578.

[22] Guo, J.Y. and Shen, Z.Q. (2005). Solution of the Dirac equation for the Wood-Saxon potential with spin and pseudospin symmetry. Physics Letters A 338 (2), pp 90-96.

[23] Berkdemir, C., Berkdemir, A. and Sever, R. (2006). Symmetrical approach to the exact solution of the Dirac equation for a deformed form of the Wood-Saxon potential. Journal of Physics A: Mathematical and General 39 (43), pp 13455-13463.

[24] Jia, C.S., Chen, T. and Cui, L.G. (2009). Approximate analytical solutions of the Dirac equation with the generalized Pöschl-Teller potential including the pseudo-centrifugal term. Physics Letters A 373 (18), pp 1621-1626.

[25] Yahya, W.A. and Oyewumi, K.J. (2016). Thermodynamic properties and approximate solutions of the *l*-state Pöschl-Teller-type potential. Journal of the Association of Arab Universities for Basic and Applied Sciences 21, pp 53-58.

[26] Jia, C.H. and Cao, S.Y. (2013). Molecular spinless energies of the Morse potential energy model. Bulletin of the Korean Chemical Society 34, pp 3425-3428.

[27] Ebomwonyi, O., Onate, C.A., Onyeaju, M.C. and Ikot, A.N. (2017). Any *l*-state solutions of the Schrödinger equation interacting with Hellmann-generalized Morse potential model. Karbal International Journal of Modern Science 3, pp 59-68.

[28] Roy, B. (1986). Shifted 1/N expansion for energy eigenvalues of the exponential cosine screened Coulomb potential. Physical Review A 34, pp 5108-5111.

[29] Dong, S.H., Qiang, W.C., Sun, G.H. and Bezerra, V.R. (2007). Analytical approximations to the *l*-wave solutions of the Schrodinger equation with the Eckart potential. *Journal of Physics A: Mathematical and Theoretical* 40, pp 10535-10540.

[30] Nikiforov, A.F. and Uvarov, V.B. (1988). Special functions of mathematical physics. Birkhauser, Verlag, Basel.

[31] Tezcan, C. and Sever, R. (2009). A general approach for the exact solution of the Schrödinger equation. International Journal of Theoretical Physics 48, pp 337-350.

[32] Cooper, F., Khare, A. and Sukhatme, U. (1995). Supersymmetry and quantum mechanics. Physics Reports 25, pp 267-385.

[33] Greene, R.L. and Aldrich, C. (1976). Variational wave functions for a screened Coulomb potential. Physical Review A 14, pp 2363-2366.
[34] M. Hamzavi, K.E. Thylwe and A.A. Rajabi. Commun. Theor. Phys. 60 (2013) 1
[35] C.A. Onate, M.C. Onyeaju, A.N. Ikot and O. Ebomwonyi. Eur. Phys. J. Plus 132 (2017) 462