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AN APPLICATION OF AIRY FUNCTIONS TO THE SOLUTION OF ONE-DIMENSIONAL SCHRÖDINGER EQUATION

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Abstract

In this work, reported in this paper, we evaluate the Airy functions required in the approximation of second order linear differential equations which arises in the treatment of multiple turning point and energy curve-crossing problems in quantum mechanics. In the framework of the deformed quantum mechanics with a minimal length, we consider the motion of a non-relativistic particle in a homogeneous external field. In this study, we derive the Airy differential equation from the stationary state Schrödinger's equation using a simple linear potential force field. Pairs of numerically linearly independent solutions $Ai(y)$ and $Bi(y)$ are thereafter constructed from the fundamental leading asymptotic series expansion of the Airy differential equation. We then find the series representation for the physically acceptable wave function in the position representation. The asymptotic expansions of the wave functions at large positive argument are also shown. In another development, we employ the leading asymptotic series expansion to derive the Airy functions, which proceeds from classically forbidden region to classically allowed one through a turning point. We also show that if the slope of the potential at a turning point is too steep, the Airy functions are no longer valid around the turning point.

***Keywords*:** Schrödinger's equation, Airy equations, Airy functions, series solution.

1.0 INTRODUCTION

In the physical sciences, the Airy function is a special function named after the British astronomer George Biddell Airy [1]. Airy functions play an important role in the theory of asymptotic representations of various special functions, they have diverse applications in mathematical physics [2].

In quantum mechanics, three approximate methods are commonly applied: perturbation theory, (which produces a series expansion for quantities of interest); the variational method, (which allows a best estimate from a trial solution); and the semiclassical approximation which supposes that $\hbar$ is small compared to the action function in the corresponding classical problem [3].

The Airy function is a solution to Schrodinger's equation for a particle confined within a triangular potential well and for a particle in a one-dimensional constant force field. The triangular potential well solution is directly relevant for the understanding of many semiconductor devices [5]. For the same reason, it also serves to provide uniform semi-classical approximations near a turning point in the WKB approximation, when the potential may be locally approximated by a linear function of position.

The Airy function also underlies the form of the intensity near an optical directional caustic, such as that of the rainbow. Historically, this was the mathematical problem that led Airy to develop this special function. Airy differential equation is important in microscopy and astronomy, it describes the pattern, due to diffraction and interference, produced by a point source of light (one which is much smaller than the resolution limit of a microscope or telescope) [6].

Airy built two partial solutions and for the first equation in the form of a power series. The solutions were named the Airy functions. Much later, H. Jeffreys (1928 – 1942) investigated these functions more deeply. The current notation $Ai(y)$ and $Bi(y)$ were proposed by Miller [7].

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The functions $Ai(y)$ and the related $Bi(y)$ are linearly independent solutions to the Airy differential equation: $\varphi'' \pm xy = 0$, known as Airy equation or the Stokes equation [1].
Although, the homogeneous field potential $V(x) = -Fx$ is not studied so intensively as the quantum well, it has an important application in theoretical physics. In the ordinary quantum mechanics, the solutions to the Schrodinger equation with the linear potential are Airy functions, which are essential to derive the WKB connection formulas through a turning point [4]. This motivates us to study the linear potential in the deformed quantum mechanics.
Physically, Airy functions can be thought of as modeling a spring (elasticity), vibrating with no damping but with increasing spring constant $x \to 0$ (e,g. spring vibrating in a rapidly, chilling room) [7]. Generally, the Airy functions can be applied in solving equations with asymptotic properties, that is, for solving asymptotic approximate problems in the neighborhood of critical points. Applications of Airy functions include quantum mechanics of linear potential, electrodynamics, electromagnetism, optical theory of the rainbow, solid state physics, radiative transfer, and semiconductors in electric fields [8].
This paper is outlined as follows. Section 1, illustrates the basic concept of the work under study. The mathematical theory is presented in section 2. While in section 3, we present the computed and the discussion of the results obtained is given in section 4. The conclusion of this work is shown in section 5 and this is immediately followed by list of references.

2.0 Mathematical Theory

In this section we are going to derive the Airy equations from the Schrödinger's equation. let us consider one-dimensional motion of a particle in a homogenous constant force field, specifically in a field with the potential, $V(x)$. Here we take the direction of the force along the axis of $-x$ and let E be the energy posses by the particle in the field due to the force exerted on it. The stationary state one-dimensional Schrodinger's wave equation is given by the following:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\varphi + V(x)\right)\varphi = E\varphi \qquad (2.1)$$

The potential of the constant force F reads $V(x) = -Fx,$ where F represents force and x the distance moved by the particle. Hence the stationary state Schrodinger's equation for this scenario is given by the stationary state Schrodinger's equation with the required potential

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\varphi - Fx\varphi - E\varphi = 0 \qquad (2.2)$$

$$\frac{d^2\varphi}{dx^2} + \frac{2m}{\hbar^2}(Fx + E)\varphi = 0 \quad \Rightarrow \quad \frac{d^2\varphi}{dx^2} + \frac{2mF}{\hbar^2}\left(x + \frac{E}{F}\right)\varphi = 0 \qquad (2.3)$$

$$\frac{d^2\varphi}{dx^2} + \frac{2mF}{\hbar^2}(x - x_0)\varphi = 0 \qquad (2.4)$$

here $\hbar = h/2\pi$ and h is Planck's constant, m is the particle mass. At positive F the classically permitted region extends to the right from the turning point $x_0 = -E/F$. Further simplification of (2.4) is accomplished through the stationary phase transformation or change of variables:

$$y = ax + b \qquad (2.5)$$

where y has the dimensions of length and with the coefficients defined as follows:

$$a = \left(\frac{2mF}{\hbar^2}\right)^{1/3} \text{ and } b = \left(\frac{2mF}{\hbar^2}\right)^{1/3}\frac{E}{F} \qquad (2.6)$$

$$y = \left(\frac{2mF}{\hbar^2}\right)^{1/3} x + \left(\frac{2mF}{\hbar^2}\right)^{1/3}\frac{E}{F} = \left(\frac{2mF}{\hbar^2}\right)^{1/3} x - \left(\frac{2mF}{\hbar^2}\right)^{1/3} x_0 \qquad (2.7)$$

$$y = \left(\frac{2mF}{\hbar^2}\right)^{1/3}(x - x_0) \quad (x \text{ near } x_0) \qquad (2.8)$$

$$\frac{dy}{dx} = \left(\frac{2mF}{\hbar^2}\right)^{1/3} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(\frac{2mF}{\hbar^2}\right)^{1/3} \cdot \left(\frac{2mF}{\hbar^2}\right)^{1/3} = \left(\frac{2mF}{\hbar^2}\right)^{2/3} \qquad (2.9)$$

Suppose $\varphi(x) = y(x)$ then we can use the chain rule of differentiation to determine the first and second derivatives of (2.9). That is

$$\frac{d\varphi}{dx} = \frac{dy}{dx}\cdot\frac{d\varphi}{dy} \Rightarrow \frac{d}{dx}\left(\frac{d\varphi}{dx}\right) = \frac{dy}{dx}\frac{d}{dy}\left(\frac{dy}{dx}\frac{d\varphi}{dy}\right) \quad ; \quad \left(\frac{d}{dx} \to \frac{dy}{dx}\frac{d}{dy}\right) \tag{2.10}$$

$$\frac{d}{dx}\left(\frac{d\varphi}{dx}\right) = \left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)\cdot\left(\frac{d}{dy}\right)\left(\frac{d\varphi}{dy}\right) = \left(\frac{dy}{dx}\right)^2\left(\frac{d^2\varphi}{dy^2}\right) \tag{2.11}$$

$$\frac{d^2\varphi}{dx^2} = \left(\frac{dy}{dx}\right)^2 \cdot \frac{d^2\varphi}{dy^2} \Rightarrow \frac{d^2\varphi}{dx^2} = \left[\left(\frac{2mF}{\hbar^2}\right)^{1/3}\right]^2 \cdot \frac{d^2\varphi}{dy^2} \tag{2.12}$$

$$\frac{d^2\varphi}{dx^2} = \left(\frac{2mF}{\hbar^2}\right)^{2/3} \cdot \frac{d^2\varphi}{dy^2} \tag{2.13}$$

The substitution of the value of the equation in (2.13) into (2.4) will yield the following results:

$$= \left(\frac{2mF}{\hbar^2}\right)^{2/3} \cdot \frac{d^2\varphi}{dy^2} + \left(\frac{2mF}{\hbar^2}\right)(x - x_0)\varphi = 0 \tag{2.14}$$

$$\left(\frac{2mF_1}{\hbar^2}\right)^{2/3} \cdot \frac{d^2\varphi}{dy^2} + \left(\frac{2mF}{\hbar^2}\right)(x - x_0)\varphi = 0 \tag{2.15}$$

Further simplification of (2.15) is accomplished when we multiply through it by $\left(\frac{2mF}{\hbar^2}\right)^{-2/3}$.

$$\left(\frac{2mF}{\hbar^2}\right)^{-2/3}\left(\frac{2mF}{\hbar^2}\right)^{2/3} \cdot \frac{d^2\varphi}{dy^2} - \left(\frac{2mF}{\hbar^2}\right)^{-2/3}\left(\frac{2mF}{\hbar^2}\right)(x - x_0)\varphi = 0 \tag{2.16}$$

$$\frac{d^2\varphi}{dy^2} + \left(\frac{2mF}{\hbar^2}\right)^{1/3}(x - x_0)\varphi = 0 \tag{2.17}$$

Hence with the implementation of (2.8) in (2.17), then the stationary state Schrödinger's equation given by (2.1) finally reduces to the standard dimensionless eigen-value equation:

$$\frac{d^2\varphi}{dy^2} + y\,\varphi = 0 \tag{2.18}$$

Thus equation (2.18) is known as the Airy differential equations or the Stokes equation which are finite at the origin. The general solution of this equation is given in terms of Airy functions.

2.1 The Series Solution of Airy Differential Equations.

The general equation for a homogeneous second order linear differential equation is of the form

$$y'' + p(t)\,y' + q(t)\,y = 0 \tag{2.19}$$

The series solution method is used primarily, when the coefficients $p(t)$ or $q(t)$ are non-constant, that is, are given variables. An example of such a case is the Airy's equation. Thus we can recast the Airy differential equations given by (2.18) in the form given by:

$$\varphi'' + y\varphi = 0 \tag{2.20}$$

where y which is the coefficient of φ is also a variable. To find the power series solutions for this second-order linear differential equation. The general form of a power series is given by

$$\varphi(y) = \sum_{n=0}^{\infty} a_n y^n \tag{2.21}$$

Our exercise here would be to determine the right choice for the coefficients (a_n).

$$\varphi'(y) = \sum_{n=0}^{\infty} n\,a_n y^{n-1} \tag{2.22}$$

$$\varphi''(y) = \sum_{n=0}^{\infty} n(n-1)\,a_n y^{n-2} \tag{2.23}$$

By substituting equation (2.23) into the Airy differential equation given by (2.20), we get

$$\sum_{n=0}^{\infty} n(n-1)a_n y^{n-2} + y\sum_{n=0}^{\infty} a_n y^n = 0 \tag{2.24}$$

$$\sum_{n=2}^{\infty} n(n-1)a_n y^{n-2} + \sum_{n=0}^{\infty} a_n y^{n+1} = 0 \tag{2.25}$$

Our next goal is to simplify this expression such that (basically) only one summation sign remains. The obstacle we may encounter is that the powers of both sums are different, y^{n-2} for the first sum and y^{n+1} for the second sum. We can make them to be the same by shifting the index of the first sum up by 2 units and the index of the second sum down by 1 unit to obtain

$$\sum_{n=0}^{\infty} (n+2)(n+2-1)a_{n+2} y^{n+2-2} + \sum_{n=1}^{\infty} a_{n-1} y^{n-1+1} = 0 \tag{2.26}$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} y^{n} + \sum_{n=1}^{\infty} a_{n-1} y^{n} = 0 \tag{2.27}$$

Now we can run into the next problem: the second sum starts at $n = 1$, while the first sum has one more term and starts at $n = 0$. We therefore split off or sum up the zeroth term of the first part of the summation that is:

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} y^{n} = 2\bullet 1\bullet a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} y^{n} \tag{2.28}$$

Consequently. we can now combine the two sums in equation (2.28) as follows:

$$2a_2 + \sum_{n=1}^{\infty} \left\{ (n+2)(n+1)a_{n+2} y^{n} + a_{n-1} y^{n} \right\} = 0 \tag{2.29}$$

$$2a_2 + \sum_{n=1}^{\infty} \left\{ (n+2)(n+1)a_{n+2} + a_{n-1} \right\} y^{n} = 0 \tag{2.30}$$

The power series on the left is identically equal to zero, consequently all of its coefficients are equal to 0 for all $n = 1,\ 2,\ 3,\ \cdots$:

$$2a_2 = 0 \quad \Rightarrow \quad a_2 = 0 \tag{2.31}$$

Thus the remaining expression in the parenthesis of equation (2.30) will now read as follows:

$$(n+2)(n+1)a_{n+2} + a_{n-1} = 0 \tag{2.32}$$

$$a_{n+2} = -\frac{1}{(n+1)(n+2)} a_{n-1} \tag{2.33}$$

for all $n = 1,\ 2,\ 3,\ 4, \cdots$. These equations are known as the "recurrence relations" of the differential equations. The recurrence relations permit us to compute all coefficients in terms of a_0 and a_1. We already know from the $0th$ recurrence relation that $a_2 = 0$. Let us now compute a_3 by reading off the recurrence relation for $n = 1, 2, 3, \cdots 7$:

For $n = 1$: $$a_3 = -\frac{1}{(2)(3)} a_0 = -\frac{1}{6} a_0 \tag{2.34}$$

For $n = 2$: $$a_4 = -\frac{1}{(3)(4)} a_1 = -\frac{1}{12} a_1 \tag{2.35}$$

For $n = 3$: $$a_5 = -\frac{1}{(4)(5)} a_2 = 0 \quad \text{(since } a_2 = 0\text{)} \tag{2.36}$$

For $n = 4$: $$a_6 = -\frac{1}{(5)(6)} a_3 = -\frac{1}{(5)(6)}\left(-\frac{1}{(2)(3)} a_0\right) = \frac{1}{(2\bullet 3)(5\bullet 6)} a_0 = \frac{1}{180} a_0 \tag{2.37}$$

For $n = 5$: $$a_7 = -\frac{1}{(6)(7)} a_4 = -\frac{1}{(6)(7)}\left(-\frac{1}{(3)(4)} a_1\right) = \frac{1}{(3\bullet 4)(6\bullet 7)} a_1 = \frac{1}{504} a_1 \tag{2.38}$$

For $n = 6$: $$a_8 = \frac{1}{(7)(8)} a_5 = 0 \quad \text{(since } a_5 = 0\text{)} \tag{2.39}$$

For $n = 7$: $$a_9 = -\frac{1}{(8)(9)} a_6 = -\frac{1}{(8)(9)}\left(\frac{1}{(2\bullet 3)(5\bullet 6)} a_0\right) = -\frac{1}{12960} a_0 \tag{2.40}$$

To recognize the patterns evolving in the series we have to consider the following three cases:

Case I: All the terms $a_2, a_5, a_8, \cdots$ are equal to zero. We can write this in compact form as

$$a_{3n+2} = 0 \quad (2.41)$$

for all $n = 0, 1, 2, 3, \cdots$.

Case II: All the terms $a_3, a_6, a_9, \cdots$ are multiples of a_0. We can be more precise if we compactly write:

$$a_{3n} = \frac{(-1)^k}{(2.3)(5.6)(8.9)\ldots(3n-1)(3n)} a_0 \quad (2.42)$$

for all $n = 1, 2, 3, 4, \cdots$ and where $(-1)^k = \mp 1$ for $k = 1, 2, 3, \cdots$.

Case III: All the terms $a_4, a_7, a_{10}, \cdots$ are multiples of a_1. We can be more precise if we write it in the form:

$$a_{3n+1} = \frac{(-1)^k}{(3.4)(6.7)(9.10)\ldots(3n)(3n+1)} a_1 \quad (2.43)$$

for all $n = 1, 2, 3, 4, \cdots$ and where $(-1)^k = \mp 1$ for $k = 1, 2, 3, \cdots \infty$.

Thus the general form of the solutions to Airy differential equation (2.20) is therefore of the form

$$\varphi(y) = a_0 \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^k y^{3n}}{(2.3)(5.6)(8.9)\ldots(3n-1)(3n)} \right) + a_1 \left(y + \sum_{n=1}^{\infty} \frac{(-1)^k y^{3n+1}}{(3.4)(6.7)(9.10)\ldots(3n)(3n+1)} \right) \quad (2.44)$$

In practice, for example, when we expand (2.44) on account of the index n we get the following;

$$\varphi(y) = a_0 \left(1 - \frac{y^3}{6} + \frac{y^6}{180} - \ldots \right) + a_1 y \left(1 - \frac{y^3}{12} + \frac{y^6}{504} - \ldots \right) \quad (2.45)$$

Note that, as always, $\varphi(0) = a_0$ and $\varphi'(0) = a_1$. Thus it is trivial to determine a_0 and a_1 when you want to solve an initial value problem. In particular

$$\varphi_1(y) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^k y^{3n}}{(2\bullet 3)(5\bullet 6)(8\bullet 9)\ldots(3n-1)\bullet(3n)} \quad (2.46)$$

$$\varphi_2(y) = y + \sum_{n=1}^{\infty} \frac{(-1)^k y^{3n+1}}{(3\bullet 4)(6\bullet 7)(9\bullet 10)\ldots(3n)\bullet(3n+1)} \quad (2.47)$$

Again $(-1)^k = \mp 1$ for $k = 1, 2, 3 \cdots \infty$. The reader should also note that we are only calculating for the index $n = 1$ and $n = 2$ since $\varphi_1(y)$ and $\varphi_2(y)$ both go to zero as $y \to \infty$.

$$\varphi(y) = a_0 \varphi_1(y) + a_1 \varphi_2(y) \quad (2.48)$$

Thus $\varphi_1(y)$ and $\varphi_2(y)$ form a fundamental system of solutions for Airy differential Equations.

2.2 The Asymptotic Series Representation of Airy functions: $Ai(y)$ and $Bi(y)$.

In another development, we can rewrite equation (2.42) and (2.43) in terms of Gamma notation or function. In that case both equations respectively become:

$$a_{3n} = \Gamma\left(\frac{2}{3}\right) \frac{(-1)^k}{9^n\, n!\, \Gamma\left(n + \frac{2}{3}\right)} a_0 \quad \text{and} \quad a_{3n+1} = \Gamma\left(\frac{4}{3}\right) \frac{(-1)^k}{9^n\, n!\, \Gamma\left(n + \frac{4}{3}\right)} a_1 \quad (2.49)$$

$$\varphi(y) = a_0 \sum_{n=0}^{\infty} \Gamma\left(\frac{2}{3}\right) \frac{(-1)^k y^{3n}}{9^n\, n!\, \Gamma\left(n + \frac{2}{3}\right)} + a_1 \sum_{n=0}^{\infty} \Gamma\left(\frac{4}{3}\right) \frac{(-1)^k y^{3n+1}}{9^n\, n!\, \Gamma\left(n + \frac{4}{3}\right)} \quad (2.50)$$

where $(-1)^k = \mp 1$ for $k = 1, 2, 3, \cdots \infty$. That means if we expand (2.50) carefully we shall realize (2.45). Hence the general solution of the Airy equation is given in terms of Airy functions

$$\varphi(y) = c_1 Ai(y) + c_2 Bi(y) \quad (2.51)$$

The Airy functions and their derivatives have rather simple series representations at the origin:

$$Ai(y) = Ai(0)\varphi_1 + Ai'(0)\varphi_2 \quad (2.52)$$

$$Bi(y) = Bi(0)\varphi_1 + Bi'(0)\varphi_2 \quad (2.53)$$

The series given in (2.52) and (2.53) converge at the whole plane and their symbolic forms are the following: where $Ai(0) = \varphi(y) = a_0$ and $Ai'(0) = \varphi'(y) = a_1$, similarly for $Bi(y)$.

$$Ai(y) = \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)}\sum_{n=0}^{\infty}\frac{1}{9^n n!\left(\frac{2}{3}\right)}(-1)^k y^{3n} - \frac{1}{\sqrt[3]{3}\,\Gamma\left(\frac{1}{3}\right)}\sum_{n=0}^{\infty}\frac{1}{9^n n!\left(\frac{2}{3}\right)}(-1)^k y^{3n+1} \quad (2.54)$$

$$Ai'(y) = -\frac{1}{\sqrt[3]{3}\,\Gamma\left(\frac{1}{3}\right)}\sum_{n=0}^{\infty}\frac{1}{9^n n!\left(\frac{1}{3}\right)}(-1)^k y^{3n} + \frac{1}{2\cdot 3^{2/3}\Gamma\left(\frac{2}{3}\right)}\sum_{n=0}^{\infty}\frac{1}{9^n n!\left(\frac{5}{3}\right)}(-1)^k y^{3n+2} \quad (2.55)$$

$$Bi(y) = \frac{1}{\sqrt[6]{3}\,\Gamma\left(\frac{2}{3}\right)}\sum_{n=0}^{\infty}\frac{1}{9^n n!\left(\frac{2}{3}\right)}(-1)^k y^{3n} + \frac{\sqrt[6]{3}}{\Gamma\left(\frac{1}{3}\right)}\sum_{n=0}^{\infty}\frac{1}{9^n n!\left(\frac{4}{3}\right)}(-1)^k y^{3n+1} \quad (2.56)$$

$$Bi'(y) = \frac{\sqrt[6]{3}}{\Gamma\left(\frac{1}{3}\right)}\sum_{n=0}^{\infty}\frac{1}{9^n n!\left(\frac{1}{3}\right)}(-1)^k y^{3n} + \frac{1}{2\cdot\sqrt[6]{3}\,\Gamma\left(\frac{2}{3}\right)}\sum_{n=0}^{\infty}\frac{1}{9^n n!\left(\frac{5}{3}\right)}(-1)^k y^{3n+2} \quad (2.57)$$

$$Ai(0) = \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} = 0.35502\ ; \quad Ai'(0) = -\frac{1}{3^{1/3}\Gamma\left(\frac{1}{3}\right)} = -0.25881 \quad (2.58)$$

$$Bi(0) = \frac{1}{3^{1/6}\Gamma\left(\frac{2}{3}\right)} = 0.61492\ ; \quad Bi'(0) = -\frac{3^{1/6}}{\Gamma\left(\frac{1}{3}\right)} = 0.44828 \quad (2.59)$$

However, in this study, we shall restrict our operations to only the Airy functions $Ai(y)$ and $Bi(y)$ so that their derivatives can be left for later studies. Thus with (2.58) and (2.59), then equation (2.52) and (2.53) subsequently becomes on account of Airy functions as follows:

$$Ai(y) = 0.355\left(1+\sum_{n=1}^{\infty}\frac{(-1)^k y^{3n}}{(2\cdot 3)(5\cdot 6)\ldots(3n-1)(3n)}\right) - 0.2588\left(1+\sum_{n=1}^{\infty}\frac{(-1)^k y^{3n+1}}{(3\cdot 4)(6\cdot 7)\ldots(3n)(3n+1)}\right) \quad (2.60)$$

$$Bi(y) = 0.6149\left(1+\sum_{n=1}^{\infty}\frac{(-1)^k y^{3n}}{(2\cdot 3)(5\cdot 6)\ldots(3n-1)(3n)}\right) + 0.4482\left(1+\sum_{n=1}^{\infty}\frac{(-1)^k y^{3n+1}}{(3\cdot 4)(6\cdot 7)\ldots(3n)(3n+1)}\right) \quad (2.61)$$

However, without loss of dimensionality, (2.60) and (2.61) in terms of Gamma functions are:

$$Ai(y) = 0.355\sum_{n=0}^{\infty}\Gamma\left(\frac{2}{3}\right)\frac{(-1)^k y^{3n}}{9^n n!\,\Gamma\left(n+\frac{2}{3}\right)} - 0.2588\sum_{n=0}^{\infty}\Gamma\left(\frac{4}{3}\right)\frac{(-1)^k y^{3n+1}}{9^n n!\,\Gamma\left(n+\frac{4}{3}\right)} \quad (2.62)$$

$$Bi(y) = 0.6149\sum_{n=0}^{\infty}\Gamma\left(\frac{2}{3}\right)\frac{(-1)^k y^{3n}}{9^n n!\,\Gamma\left(n+\frac{2}{3}\right)} + 0.4482\sum_{n=0}^{\infty}\Gamma\left(\frac{4}{3}\right)\frac{(-1)^k y^{3n+1}}{9^n n!\,\Gamma\left(n+\frac{4}{3}\right)} \quad (2.63)$$

where we have said that $(-1)^k = \mp 1$ for odd and even values of the index $k = 1, 2, 3 \cdots \infty$. Note: when (2.60) – (2.63) is expanded on account of the index k and in ascending powers of n the resulting series equation are the same for both $Ai(y)$ and $Bi(y)$. Again in this work, we calculated for the index $n = 1$ and $n = 2$ since $Ai(y)$ and $Bi(y)$ both go to zero as $y \to \infty$.

3.0 Presentation of results.

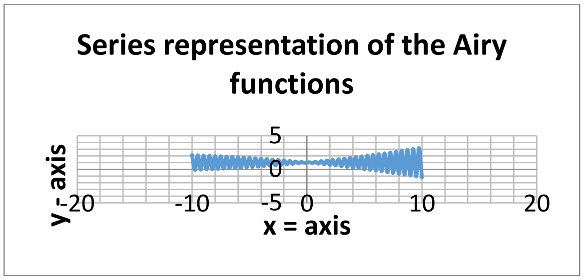

Fig. 1: shows the graph of the series equation $\varphi_1(y)$ given by (2.46) with arbitrary values of: $y = [-10, 10] = -10, -9.75, -9.5, -9, \cdots, 10$:

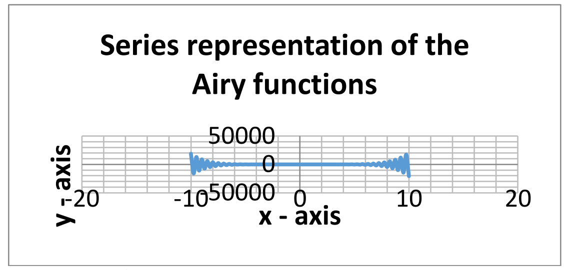

Fig. 2: shows the graph of the series equation $\varphi_2(y)$ given by (2.47) with arbitrary values of: $y = [-10, 10] = -10, -9.75, -9.5, -9, \cdots, 10$: which gave 81 possible values.

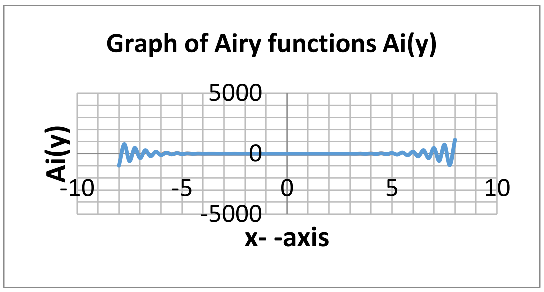

Fig. 3: shows the graph of the Airy function $Ai(y)$ given by equation (2.60) with arbitrary values of: $y = [-8, 8]$ = -8, -7.75, -7.5, -7.25, -7, ⋯, 8: which gave 65 possible values.

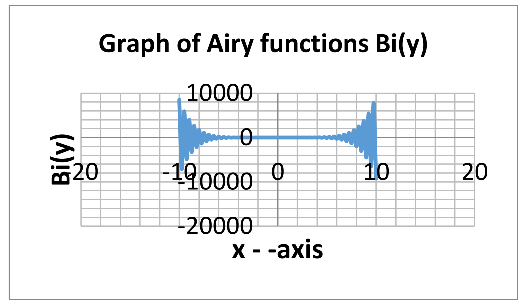

Fig. 4: shows the graph of the Airy function $Bi(y)$ given by equation (2.61) with arbitrary values of: $y = [-8, 8]$ = -8, -7.75, -7.5, -7.25, -7, ⋯, 8: which gave 65 possible values.

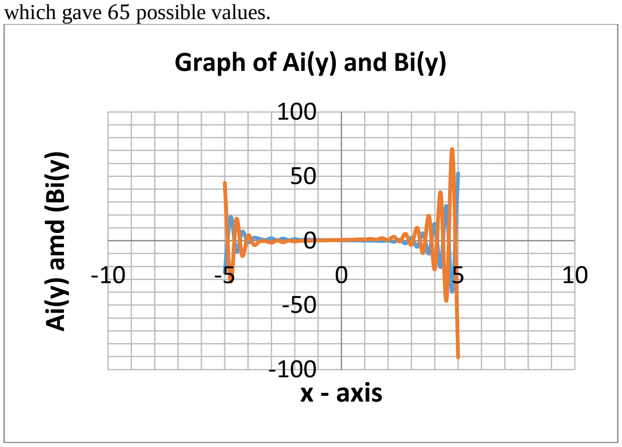

Fig. 5: shows the graph of the Airy function $Ai(y)$ and $Bi(y)$. $y = [-5, 5]$ = -5, -4.75, -4.5, -4.25, -4, ⋯, 5; which gave 41 possible values. The Airy function $Ai(y)$ is represented by the blue curve while the brown curve represent the Airy function $Bi(y)$.

4.0 Discussion of results.

It is shown in fig. 1 that the wave function $\varphi_1(y)$ of the particle motion has a regular frequency of oscillation with decreasing amplitude when y is negative. The amplitude becomes exactly zero at the classical turning point. At the classical turning point the energy posses by the particle is equal to the potential field, $E = V(x)$. The particle however tunnels across the potential barrier otherwise the classical turning point to the right side, the region where y is positive. In this region the particle also has a regular frequency, but with increasing amplitude of oscillation.

The second independent wave function $\varphi_2(y)$ describing the motion of the non relativistic particle is shown in fig.2. The spectrum of both wave functions $\varphi_1(y)$ and $\varphi_2(y)$has similar interpretation. However, the classical turning points for $\varphi_2(y)$ has a wider gap with zero amplitude and frequency of oscillation.

Figure 3, 4 and 5 shows the graph of the Airy functions $Ai(y)$ and $Bi(y)$. The Airy functions have two linearly independent solutions. The Airy function $Ai(y)$ is the solution subject to the condition $\varphi(y) \rightarrow 0$ as $y \rightarrow \infty$. The standard choice for the other is the Airy function of the second kind, denoted $Bi(y)$. It is defined as the solution with greater amplitude of oscillation than $Ai(y)$ as $y \to -\infty$ which differs in phase by $/2$ (90^0). They are both non degenerate functions and are oppositely related. When y is negative, $Ai(y)$ and $Bi(y)$ oscillate around zero with decreasing frequency and amplitude. When y is positive, $Ai(y)$ and $Bi(y)$ oscillate around zero with increasing frequency and amplitude.

5.0 Conclusion.

The classically allowed region namely the region where $V(x) < E$, the series solution given by the Airy functions are real and the approximate wave function is oscillatory. In the classically forbidden region $V(x) > E$ the solutions are growing or decaying. It is evident that both of these approximate solutions become singular near the classical turning points, where $V(x) = E$, and cannot be valid. The turning points are the points where the classical particle changes direction. The Airy functions which describes the motion of the non relativistic particle can be good both in the region $E > V(x)$ and the region $E < V(x)$ but cannot be good in between the regions close to the classical turning point $E = V(x_c)$. The various gaps at the critical turning points can be used to study energy bands or gaps in semiconductor physics.

References

[1] G. B. Airy, On the intensity of light in the neighborhood of a caustic, Trans. Camb. Phil. Soc. 6 (1838), 379 – 403.
[2] G. Wentzel. "Eine Verallgemeinerung der Quantenbedingungen f • ur die Zweckeder. Wellenmechanik" Z. Phys. ,38:518{529, 1926.
[3] H.A. Kramers. "Wellenmechanik und halbz • ahlige Quantisierung". Zeit. f. Physik, 39:828{840, 1926.
[4] L. Brillouin. "Remarques sur la mecanique ondulatoire". J. Phys. Radium, 7:353{368, 1926.
[5] H. Jeffreys. "On certain approximate solutions of linear differential equations of the second order". Proc. London Math. Soc. , 23:428{436, 1925.
[6] J. L. Dunham. "The Wentzel-Brillouin-Kramers Method of Solving the Wave Equation". Physical Review, 41:713{720, 1932.
[7] G. Wentzel, Z. Physik 38: 518 (1926)
[8] H. A. Kramers, Z. Physik 39: 828 (1926)
[9] L. Brillouin, J. Phys. Radium, 7: 353 (1926)
[10] E. Schrodinger, Z. Physik, 4: 347 (1921)
Airy functions. *Encyclopedia of Mathematics.* URL: http://www.encyclopediaofmath.org/index.php?title=Airy_functions&oldid=45054
[11] T. V. Fityo, I. O. Vakarchuk, and V. M. Tkachuk, "One-dimensional Coulomb-like problem in deformed space with minimal length," Journal of Physics A: Mathematical and General,vol.39,no.9,pp.2143–2149,2006.
[12] P. Pedram, "New approach to nonperturbative quantum mechanics with minimal length uncertainty," Physical Review D ,vol.85,no.2,ArticleID024016,2012.
[13] T. V. Fityo, I. O. Vakarchuk, and V. M. Tkachuk, "WKB approximation in deformed space with minimal length," Journal of Physics. A. Mathematical and General ,vol.39,no.2,article 379, 2006.
[14] P. Pedram, "A higher order GUP with minimal length uncertainty and maximal momentum," Physics Letters B ,vol.714,no. 2-5, pp. 317–323, 2012.
[15] C.-L. Ching, R. R. Parwani, and K. Singh, "Constraints and spectra of a deformed quantum mechanics," Physical Review D, vol. 86, no. 8, Article ID 084053, 2012.
[16] G.B. Airy, *Trans. Cambridge Philos. Soc.* , **6** (1838) pp. 379–402
[17] G.G. Stokes, *Trans. Cambridge Philos. Soc.* , **10** (1857) pp. 105–128
[18] V.A. Fok, "Tables of the Airy functions" , Moscow (1946) (In Russian)
[19] A. Segun, M. Abramowitz, "Handbook of mathematical functions" , *Appl. Math. Ser.* , **55** , Nat. Bur. Standards (1970)
[20] V.M. Babich, V.S. Buldyrev, "Asymptotic methods in the diffraction of short waves" , Moscow (1972) (In Russian) (Translation forthcoming: Springer)
[21] M.V. Fedoryuk, "The saddle-point method" , Moscow (1977) (In Russian)
[22] E.M. Lifshits, "The classical theory of fields" , Addison-Wesley (1951) (Translated from Russian)
[23] V.P. Maslov, M.V. Fedoryuk, "Quasi-classical approximation for the equations of quantum mechanics" , Reidel (1981) (Translated from Russian)
[24] A.A. Dorodnitsyn, "Asymptotic laws of distribution of the characteristic values for certain types of second-order differential equations" *Uspekhi Mat. Nauk* , **6** : 7 (1952) pp. 3–96 (In Russian)
[25] W. Wasov, "Asymptotic expansions for ordinary differential equations" , Interscience (1965)
[26] M.V. Fedoryuk, "Asymptotic methods for linear ordinary differential equations" , Moscow (1983) (In Russian)