Transactions of the Nigerian Association of Mathematical Physics
***Volume 13, (October – December, 2020 Issue*), pp117 –122**
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ANALYTICAL SOLUTION OF EULER-BERNOULLI BEAM SUBJECTED TO CONCENTRATED MOVING LOAD

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Abstract

Analytical solution of Euler-Bernoulli beam subjected to concentrated moving load was investigated in this paper. The governing equation of fourth order partial differential equation was reduced to an ordinary differential equation using series solution. The frequency of the oscillation was found and substituted back into the assumed solution which is the solution of the governing equation. Numerical result was presented and it is found that the dynamic response of the beam increases as the length of the load and length of the mass increases but decreases with an increase in the length of the beam, mass of the load and foundation constant.

***Keywords*:** Analytical, Euler-Bernoulli Beam, Concentrated, Moving Load, Subjected.

Introduction

The vibration analysis for structures is a very important field in engineering and computational mechanics. These dynamical problems are classically described by a partial differential equation associated with a set of classical boundary conditions. The analysis of free vibration of beam has been a topic of interest for well over a century [24].

The structures on which these moving loads are usually modelled are by elastic beams, plates or shells. The problem of an elastic beam under the action of the moving loads was considered by Willis [24]. He made the assumption that the mass of the beams is smaller than that of the load and obtained an approximate solution to the problem. Yoshida [25] studied the vibration of a beam subjected to moving concentrated moving using finite element method. A simply supported to a constant moving force at uniform speed was considered by Krylov [14] who used the method of expansion of the associated Eigenmodes. He assumed the mass of the load to be smaller than that of the beam. Bolotin [3] carried out a dynamic analysis of the problem involving a concentrated mass traversing a simply supported beam at a constant speed. His approach involves using Galerkin's method. The response of finite simply supported Euler-Bernoulli beam to a unit force moving at a uniform velocity was investigated by Lee [16]. The effects of this moving force on beams with and without an elastic foundation were analyzed. In all the studies discussed above, it was only the force effect of the moving loads that was taken into consideration. The moving load problem involving both the inertia effect as well as the force effects were not considered for several years. This type of dynamical problem was first considered by Kalker [12], later by Jeffcott [11] whose iterative method became divergent in some cases. Recently, Esmailzadeh and Gorashi [6] worked on the vibration analysis of beams traversed by uniform a partially distributed moving masses using analytical-numerical method. They discovered that the inertia effect of the distributed moving mass is of importance in the dynamic behaviour of the structure. The critical speeds of the moving load were also calculated for the mid-span of the beam. The length of the distributed moving mass was also found to affect the dynamic response. The effects of the speed of the moving load, the foundation stiffness and the length of the beam on the response of the beam have been studied and dynamic amplifications of deflection and stress have been evaluated. Based on the Lagrangian approach, Chang [4] analyzed the vibration of a multi-span non-uniform bridge subjected to a moving vehicle by using modified beam vibration functions as the assumed modes. The vehicle is modeled as a two-degree of freedom system. Obtained results are presented in form of dynamic amplification factors and compelled published results where applicable. Savin [23] derived analytic expressions of the dynamic amplification factor and the characteristics response spectrum for weakly damped beams with various boundary conditions subjected to point loads moving at constant speeds. The obtained coefficients are given as functions of the ratio

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of the span length to the load wavelength and the loads wavelength respectively. Pesterve*et al* [22] developed simple tools for finding the maximum deflection of a beam for any given velocity of the travelling force. It is shown that for given boundary conditions, there exists a unique response-velocity dependence function. They suggested a technique to determine this function which is based on the assumption that the maximum beam response can be adequately approximated by means of the first mode. Also, the maximum response function is calculated analytically for a simply supported beam and constructed numerically for a clamped-damped beam. Friction dampers are another common passive vibration control system which dissipate energy through friction forces. These forces are generated with moving parts by sliding over each other. The energy dissipated by a friction damper reduces the energy demand on the structure and damps the structural response. The friction damper system includes the friction unit and a structural system in order to integrate the friction unit with the structure. The structural system can be either steel braces bolted to corner regions of the open bay space in the frame or an infill wall with gaps around the edges to prevent stiffness interaction of the wall with the frame members. Friction dampers are used as sacrificial or non-sacrificial elements. Their utilization as sacrificial elements is a very common attitude in civil engineering environment. In earthquake engineering applications, some of the structural members might be sacrificed in order to prevent the collapse of entire structure. These structural members absorb and dissipate the transmitted energy through plastic deformation in specially detailed regions. Location of the friction damper and stiffness of the braces which are used in order to install dampers are the main factors that affect the design parameters of the damper. [19] Dahlberg [5] uses the modal analysis technique to investigate the influence of modal cross-spectral densities on the spectral densities of some responses of simply supported beams. The random response of damped beams was studied by Jacquot [10]. The author presents a method of vibration analysis using the response power spectral density function and mean square response of considered beam structures excited by a second stationary random process. Kukla and Skalmierski [15] dealt with the random vibration of a clamped-pinned beam. The flux of energy which is emitted by the vibrating beam was investigated. Papadimitriou et al [21] provide a methodology for optimal establishment of the number and location of sensors on randomly vibrating structures for the purpose of the response predictions at unmeasured locations in structural systems. The author referees the results of considerations to randomly vibrating beams and plates. It is well known that damping becomes important when the need to have a thorough understanding of the control and mechanical response of vibrating structures arises.The problem of determining the dynamic response of a rectangular, damped, elastic plate carrying uniform a partially moving load is investigated. The elastic plate is assumed to have uniform cross-sectional area. The effect of both rotatory inertia as well as shear deformation is assumed negligible. The moving a partially distributed load is also assumed moving at uniform velocity. A constant damping coefficient is used throughout the analysis. Viscous damping whose coefficient is assumed to be directly proportional to the mass distribution of the system is considered. [7] An asymptotic analysis of eigen frequencies of uniform beam with both structural and viscous damping coefficient has also been carried out in Hankum and Goong [8] and Huang [9]. Furthermore, Kenny [13] took up the problem of investigating the dynamic response of infinite elastic beams on elastic foundation when the beam is under the influence of a dynamic load moving with constant speed. Lie included the effects of viscous damping in the governing differential equation of motion. More recently, Oni [20] considered the problem of a harmonic time variable concentrated force moving at a uniform velocity over a Unite deep beam. The methods of integral transformations are used. In particular, the Unite Fourier transform is used for the length coordinate and the Laplace transform the time coordinate. Series solution, which converges as obtained for the deflection of simply supported beams. The analysis of the solution was carried out for various speeds of the load. Oni [20] used the Galerkin method to obtain the response to several moving masses of a non-uniform beam resting on an elastic foundation. The effects of the elastic foundation on the transverse displacement of the non-uniform beam were analyzed for both the moving mass and the associated moving force problems. Awodola T. O. [2] worked on the influence of foundation and axial force on the vibration of a simply supported thin (Bernoulli Euler) beam, resting on a uniform foundation, under the action of a variable magnitude harmonic load moving with variable velocity is investigated in the paper. The governing equation is a fourth-order partial differential equation. For the solution of this problem, in the first instance, the finite Fourier sine transformation is used to reduce the equation to a second order partial differential equation. The reduced equation is then solved using the Laplace transformation. Numerical analysis shows that the transverse deflection of the thin beam, resting on a uniform foundation, under the action of a variable magnitude harmonic load moving with variable velocity decreases as the foundation constant increases. It also shows that as the axial force increases, the transverse deflection of the thin beam decreases. Furthermore, Milormir, Stanisic M. and Hardin J. C. [18] developed a theory describing the response of a Bernoulli-Euler beam under an arbitrary number of concentrated moving masses. The theory is based on the Fourier technique and shows that, for a simply supported beam, the resonance frequency is lower with no corresponding decrease in maximum amplitude when the inertia is considered.

This paper deals with the response of Timoshenko beam subjected to uniform a partially-distributed moving loads. The main objectives of this project are as follows: To present the analysis of Timoshenko beam with an attached mass at the end x = L, but arbitrary supported at the end x = 0, to uniform a partially distributed moving load; To be able to transformed the governing equation and then solve the reduced equation; To investigate the effect of shear deformation on a a partially distributed moving load. This work will assist the practicing engineers to evaluate the dynamic response Euler-Bernoulli beam subjected to a concentrated moving load.

Formulation of the Problem

The problem of the displacement response of Simply-Supported Euler-Bernoulli beam resting on Winkler foundation carrying concentrated load is generated by the fourth order differential equation

$$EI\lambda_{xxxx}(x,t) + m\lambda_{tt}(x,t) + k\lambda(x,t) = \phi_f(x,t) \quad (1)$$

where

$$\phi_f(x,t) = M\delta(x - vt) \quad (2)$$

But

$$\xi - \frac{\epsilon}{2} = vt \quad (3)$$

Equation 2 becomes

$$\delta(x,t) = \frac{1}{\epsilon}\left[H\left(x - \xi + \frac{\epsilon}{2}\right) - H\left(x - \xi - \frac{\epsilon}{2}\right)\right] \quad (4)$$

where

$$\frac{d}{dx}H(x) = \delta(x)$$

λ = is the transverse displacement
M = is the mass of the beam

X = spatial coordinate
T = Time
E = Young's modulus
I = moment of inertia
M = Mass per unit length of the beam
Δ = Dirac delta function

The pertinent boundary conditions for the dynamical systems under consideration can be any of the following classical boundary conditions,

$$\lambda(x,t) = \lambda_{xx}(x,t) = 0, \text{ at } x = 0 \text{ or } x = l \quad (6)$$

$$\lambda_{xx}(x,t) = \lambda_{xxx}(x,t) = 0, \text{ at } x = 0 \text{ or } x = l \quad (7)$$

$$\lambda_{x}(x,t) = \lambda_{xxxx}(x,t) = 0, \text{ at } x = 0 \text{ or } x = l \quad (8)$$

Finally, the initial conditions are:

$$\lambda(x,0) = 0 \quad (9)$$

$$\lambda_t(x,0) = 0 \quad (10)$$

METHOD OF SOLUTION

Assume a solution such that the transverse vibration of the beam may be expressed in the following series form

$$\lambda(x,t) = \sum_{j=1}^{\infty} \sin\frac{j\pi x}{L} T_j(t) \quad (11)$$

Substituting equations assume, forx and fort into equation 1, we obtain the following

$$EI\left(\frac{\pi}{L}\right)^4 \sum_{j=1}^{\infty} j^4 \sin\frac{j\pi x}{L} T_j(t) + m\sum_{j=1}^{\infty}\sin\frac{j\pi x}{L}\ddot{T}_j(t) + k\sum_{j=1}^{\infty}\sin\frac{j\pi x}{L}T_j(t) = M\delta(x - vt) \quad (14)$$

Hence, the force term *f*(*x*,*t*) defined in equation 1 can be expressed as $\lambda(x,t) = \sum_{j=1}^{\infty} \sin\frac{j\pi x}{L} T_f^j(t)$

$$\sum_{j=1}^{\infty} \sin\frac{j\pi x}{L} T_f^j(t) = M\delta(x - vt)$$

$$\sum_{j=1}^{\infty} \sin\frac{j\pi x}{L} T_f^j(t) = M\delta(x - vt) = M\frac{1}{\epsilon}\left[H\left(x - \xi + \frac{\epsilon}{2}\right) - H\left(x - \xi - \frac{\epsilon}{2}\right)\right] \quad (15)$$

Multiply equation (15) by $\sin\frac{i\pi x}{L}$ to normalize the equation (15)

$$\sum_{j=1}^{\infty}\sin\frac{j\pi x}{L}\sin\frac{i\pi x}{L}T_f^j(t)=M\sin\frac{i\pi x}{L}\frac{1}{\epsilon}\left[H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right] \qquad (16)$$

Integrate **

$$\sum_{j=1}^{\infty}\int_0^L\sin\tfrac{j\pi x}{L}\sin\tfrac{i\pi x}{L}T_f^j(t)dx=\tfrac{M}{\epsilon}\int_0^L\sin\tfrac{i\pi x}{L}\left\{\left[H\ \ x-\xi+\tfrac{\epsilon}{2}\right)-H\ \ x-\xi-\tfrac{\epsilon}{2}\right)\right]\right\}dx$$

$$\sum_{j=1}^{\infty}\int_0^L\sin\tfrac{j\pi x}{L}\sin\tfrac{i\pi x}{L}T_f^j(t)dx=\tfrac{M}{\epsilon}\int_0^L\sin\tfrac{i\pi x}{L}\ \ \xi+\tfrac{\epsilon}{2}\right)-\sin\tfrac{i\pi x}{L}\ \ \xi-\tfrac{\epsilon}{2}\right)d\xi$$

∞ L

By Orthogonality condition

$$\sum_{j=1}\int_0\sin\tfrac{j\pi x}{L}\sin\tfrac{i\pi x}{L}T_f^j(t)dx=T_f^j(t\quad {}_f^j(t)=\sum_j\int\sin\tfrac{j\pi x}{L}\sin\tfrac{i\pi x}{L}T_f^j(t)dx^{\infty\,L}$$

so that T =1 0

$$=\tfrac{M}{\epsilon}\int_0^L\sin\tfrac{i\pi x}{L}\ \ \xi+\tfrac{\epsilon}{2}\right)d\xi-\tfrac{M}{\epsilon}\int_0^L\sin\tfrac{i\pi x}{L}\ \ \xi-\tfrac{\epsilon}{2}\right)d\xi$$

hence

$$T_f^j=\frac{2ML}{j\pi}\sin\left(\frac{j\pi\xi}{L}\right)\sin\left(\frac{i\pi\epsilon}{2L}\right) \qquad (17)$$

Substituting equation(4) into equation(2b)

$$\text{EI}\ \ \tfrac{\pi}{L}\right)^4\sum_{j=1}^{\infty}j^4\sin\tfrac{j\pi x}{L}T_j(t)+m\sum_{j=1}^{\infty}\sin\tfrac{j\pi x}{L}\ddot{T}_j(t)+k\sum_{j=1}^{\infty}\sin\tfrac{j\pi x}{L}T_j(t)$$

$$=\sum_{j=1}^{\infty}\sin\tfrac{j\pi x}{L}\left\{\tfrac{2ML}{j\pi}\sin\ \ \tfrac{j\pi\xi}{L}\right)\sin\ \ \tfrac{i\pi\epsilon}{2L}\right)\right\}$$

$$\sum_{j=1}^{\infty}\sin\tfrac{j\pi x}{L}\left\{EI\ \ \tfrac{\pi}{L}\right)^4T_j(t)+m\ddot{T}_j(t)+kT_j(t)\right\}$$

∞

$$=\sum_{j=1}^{\infty}\sin\tfrac{j\pi x}{L}\left\{\tfrac{2ML}{j\pi}\sin\ \ \tfrac{j\pi\xi}{L}\right)\sin\ \ \tfrac{i\pi\epsilon}{2L}\right)\right\}$$

$$\tfrac{2ML}{j\pi}\sin\ \ \tfrac{j\pi\xi}{L}\right)\sin\ \ \tfrac{i\pi\epsilon}{2L}\right)$$

But $\sum_{j=1}\sin\frac{j\pi x}{L}\ \ /$

$$\text{EI}\left(\tfrac{\pi}{L}\right)^4T_j(t)+m\ddot{T}_j(t)+kT_j(t)=0$$

$$m\ddot{T}_j(t)+T_j(t)\left\{EI\left(\frac{\pi}{L}\right)^4+k\right\}=\frac{2ML}{j\pi}\sin\left(\frac{j\pi\xi}{L}\right)\sin\left(\frac{i\pi\epsilon}{2L}\right) \qquad (18)$$

Let $\gamma=EI\left(\frac{\pi}{L}\right)^4+k$ Equation 5 becomes

$$m\ddot{T}_j(t)+\gamma T_j(t)=\frac{2ML}{j\pi}\sin\left(\frac{j\pi\xi}{L}\right)\sin\left(\frac{i\pi\epsilon}{2L}\right) \qquad (19)$$

Numerical Analysis

$$\ddot{T}_j(t)=\frac{T_{j+1}-2T_j+T_{j-1}}{h^2} \qquad (20)$$

Substituting equation(7) into equation(6),

$$\text{m}\left\{\tfrac{T_{j+1}-2T_j+T_{j-1}}{h^2}\right\}+\gamma T_j(t)=\tfrac{2ML}{j\pi}\sin\ \ \tfrac{j\pi\xi}{L}\right)\sin\ \ \tfrac{i\pi\epsilon}{2L}\right)$$

$$\text{m}(T_{j+1}-2T_j+T_{j-1})+h^2\gamma T_j(t)=\tfrac{2h^2ML}{j\pi}\sin\ \ \tfrac{j\pi\xi}{L}\right)\sin\ \ \tfrac{i\pi\epsilon}{2L}\right)$$

$$\mathrm{m}T_{j+1} - 2mT_j + mT_{j-1} + h^2\gamma T_j(t) = \frac{2h^2ML}{j\pi}\sin\ \frac{j\pi\xi}{L})\sin\ \frac{i\pi\epsilon}{2L})$$
$$\mathrm{m}T_{j+1} - (2m - h^2\gamma)T_j + mT_{j-1} = \frac{2h^2ML}{j\pi}\sin\ \frac{j\pi\xi}{L})\sin\ \frac{i\pi\epsilon}{2L})$$
$$\mathrm{m}T_{j+1} = (2m - h^2\gamma)T_j - mT_{j-1} + \frac{2h^2ML}{j\pi}\sin\ \frac{j\pi\xi}{L})\sin\ \frac{i\pi\epsilon}{2L})$$

Finally,

$$T_{j+1} = \frac{(2m - h^2\gamma)}{m}T_j - T_{j-1} + \frac{2h^2ML}{jm\pi}\sin\left(\frac{j\pi\xi}{L}\right)\sin\left(\frac{i\pi\epsilon}{2L}\right) \tag{21}$$

Substituting equation 21 into equation assume, gives the transverse displacement $\lambda(x,t)$ of the Beam.

Numerical Simulation

Beam dimension and specification:

Figure 1 shows the dynamic response of the Beam at various values of ξ. It is observed that the dynamic response of the beam increases as the length of the load increases.

Figure 2 shows the dynamic response of the beam at various values of. It is also observed that the dynamic response of the beam increases as the length of the mass increases.

Figure 3 displays the dynamic response profile for the length of the beam. It is observed that the dynamic response of the beam decreases with an increase in the length of the beam.

Figure 4 and Figure 5 shows the dynamic response of the beam at for K = 0,1,2 and K = 1,3,5 respectively. It is observed that the dynamic response of the beam decreases as the value of the foundation constant increases.

Figure 6 displays the dynamic beam's response variation profile for mass the load. It observed that the dynamic response of the beam decreases with an increase in the mass of the load.

The above results presented in this chapter confirms the results in Oni and Jimoh (2014).

Conclusion

The governing equation of fourth order partial differential equation was reduced to an ordinary differential equation using series solution. The frequency of the

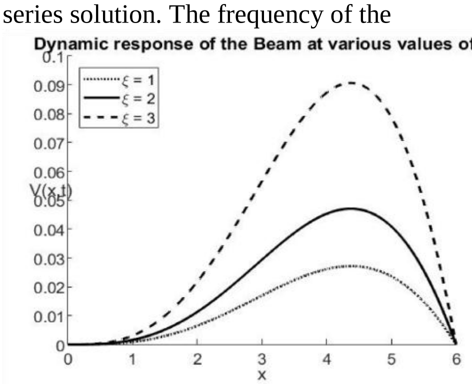


Figure 1: Dynamic response of Beam for various values of ξ

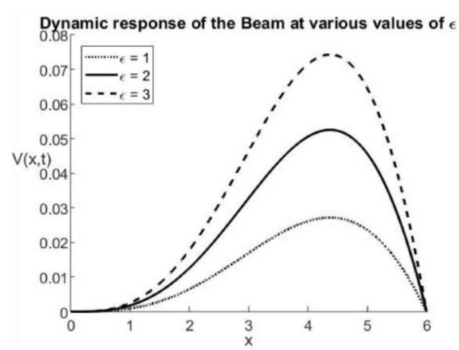


Figure 2: Dynamic response of Beam for various values of

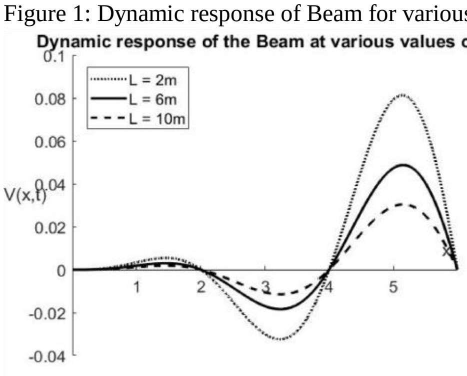


Figure 3: Dynamic response of Beam for various values of L

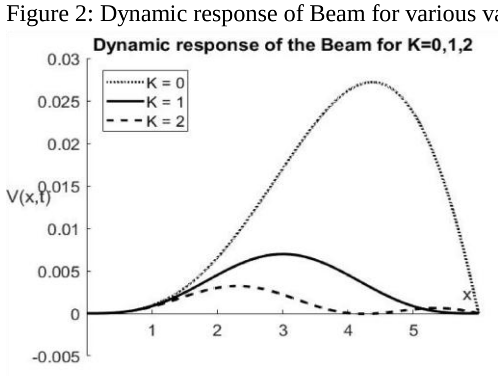
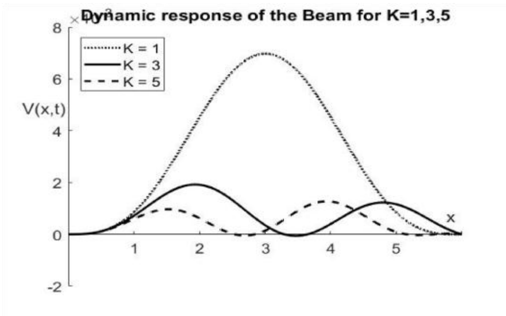


Figure 4: Dynamic response of Beam for K = 0,1,2



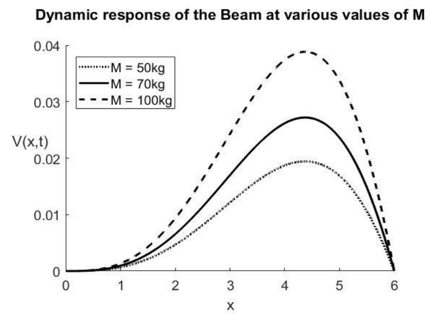


Figure 5: Dynamic response of Beam for K = 1,3,5 Figure 6: Dynamic response of Beam for various values of M oscillation was found and substituted back into the assumed series solution which is the solution of the governing equation. Numerical result was presented and plotted against x for various parameters using a computer program (MATLAB).

From the numerical results, this study concludes that the response of the beam increases as the length of the load and length of the mass increases but decreases with an increases in the length of the beam, mass of the load and foundation constant.

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