Transactions of the Nigerian Association of Mathematical Physics
***Volume 13, (October – December, 2020 Issue*), pp1 – 8**
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HEISENBERG AND ENTROPIC UNCERTAINTY MEASURES FOR THE SHIFTED STATIC SCREENED COULOMB POTENTIAL

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Abstract

The spreading of the quantum mechanical probability density for the Shifted static screened coulomb potential is studied in both the position and momentum spaces analytically and numerically, by means of local (Fisher information) and global (Shannon, Rényi, and Tsallis entropies and Onicescu information energy) measures. The trends in the variation of the information-theoretic measures with the potential screening parameter considered for this atomic model is discussed. The Shannon-entropy-based uncertainty relation (Bialynicki-Birula and Mycielski BBM inequality), Heisenberg uncertainty relation and Fisher-information-based uncertainty relation have been verified to hold for this atomic model.

Keywords: Shannon entropy; Fisher information; Rényi entropy; Tsallis entropy, Onicescu information; Schrödinger equation; Shifted Static Screened Coulomb Potential, Fourier transform.

1. Introduction

Quantum information theory has in the last few years provided an interesting and increasing applications in several disciplines ranging from physics, chemistry, biology, medicine, computer science, neural network, image recognition to linguistics and social sciences, among many others [1–6]. This is due to the factthat, information theory of quantum mechanical system is the strongest support of the modern quantum communications, computation and the density functional methods which is the basis theory for several technological developments [7, 8]. This trending research area has led to numerous researcher studying both the global (Shannon entropy, Rényi entropy, Tsallis entropy, Onicescu information energy) and local(Fisher information) information-theoretic measures for different quantum mechanical systems[9–24].These information-theoretic quantities compliment and differently quantify the probability densityρ(r)of the quantum mechanical system. They are preferred theoretical statistical tools than the celebrated standard deviation or variance, due to the fact that, they do not make reference to some specific point of the corresponding Hilbert space [25].

The Shannon entropy is defined in the position and momentum spaces as [12–15, 26]:

$$S_r = -\int_b^a \rho(\mathrm{r}) \ln \rho(\mathrm{r}) \mathrm{dr}\,, \qquad S_p = -\int_b^a \rho(\mathrm{p}) \ln \rho(\mathrm{p}) \mathrm{dp}\,, \tag{1}$$

where S_r is the position space Shannon entropy, S_p is the momentum space Shannon entropy,$\rho(\mathrm{r}) = |\varphi(r)|^2$and $\rho(\mathrm{p}) = |\varphi(p)|^2$ are the probability densities in the position and momentum spaces, respectively[13, 15, 27]. The Fisher information in the position-space $I_r$is given as [28, 29]:

$$I_r = \int \frac{[\rho\prime(\mathrm{r})]^2}{\rho(\mathrm{r})} dr = 4\,[\varphi'(r)]^2 \mathrm{dr} = 4 < \mathrm{p}^2 > \tag{2}$$

and the corresponding quantity for the momentum space Fisher information $I_p$is defined as

$$I_p = \int \frac{[\rho\prime(\mathrm{p})]^2}{\rho(\mathrm{p})} dr = 4\,[\vartheta'(p)]^2 \mathrm{dp} = 4 < \mathrm{r}^2 >. \tag{3}$$

The Rényi entropy is a generalization of the Shannon entropy. It is a single parameter family of the entropic measures which share several relevant properties with the Shannon entropy. Rényi entropy is defined as [30]:

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$$R_q[p_n] = \frac{1}{1-q}\ln\left[\int(\rho(\mathrm{r}))^q\,dr\right] = \frac{1}{1-q}\ln W_q[p_n]\,, \quad q > 0,\ q \neq 1\,, \tag{4}$$

the corresponding Rényi entropy for the momentum space coordinate is given as:

$$R_q[\gamma_n] = \frac{1}{1-q}\ln\left[\int(\gamma(\mathrm{p}))^q\,dp\right] = \frac{1}{1-q}. \tag{5}$$

In the limit $q \to 1$, the Rényi entropy reduces to Shannon entropy [13, 30]. The Onicescu information energy was introduced by Onicescu in an attempt to define a better measure of dispersion distribution than that of Shannon entropy [31]. Onicescu information is defined as [31]:

$$E[p_n] = \int \rho^2(\mathrm{r})\,dr = \int(\rho(\mathrm{r}))^q\,(\mathrm{r})dr = W_q[p_n]\,, \quad q = 2 \tag{6}$$

and the corresponding Onicescu energy for the momentum space coordinate is given

$$E[\gamma_n] = \int \gamma^2(\mathrm{p})\,dp = \int(\gamma(\mathrm{p}))^q\,(\mathrm{p})dp = W_q[\gamma_n]\,, \quad q = 2\,. \tag{7}$$

The greater the information energy, the more concentrated is the probability distribution and the smaller the information content. Onicescu energy product can then be determined as$E_{p\gamma}= E_p E_\gamma$. Tsallis entropy is one of the global measures for the spreading of density. In the limit $q \to 1$,the Tsallis entropy also reduces to Shannon entropy. It is defined in the position and momentum space coordinates, respectively as [32]:

$$T_q[p_n] = \frac{1}{q-1}\left(1 - \left[\int(\rho(\mathrm{r}))^q\,dr\right]\right) = \frac{1}{q-1}\left(1 - W_q[p_n]\right)\,, \quad q > 0,\ q \neq 1\,, \tag{8}$$

and

$$T_q[\gamma_n] = \frac{1}{q-1}\left(1 - \left[\int(\gamma(\mathrm{p}))^q\,dp\right]\right) = \frac{1}{q-1}\left(1 - W_q[\gamma_n]\right)\,, \quad q > 0,\ q \neq 1\,, \tag{9}$$

where$W_q[p_n]$ is the entropic moments.

The Static screened coulomb potential is also known as the Yukawa potential, it was proposed by Hideaki Yukawa in 1935 [33]. Rogers *et al*. (1970) proposed a shifted Yukawa potential, called the shifted static screened coulomb potential (SSSCP) [34]. This was done in an attempt to describe the nucleon-nucleoninteraction arising from the pion-exchange mechanism in nuclear physics [33, 35, 36]. Our aim in this article is to study the Heisenberg and entropic measures for the Shifted static screened coulomb Potential.

This paper is structured as follows: In the next Section, the normalised wave function and the probability density for the static screened coulomb potential is obtained in terms of the Jacobi polynomials. Section 3 contains of the evaluation of the Shannon entropy. In Section 4, the Fisher information is presented for this atomic model. The Rényi entropy, Tsallis entropy and Onicescu Information energy is presented in Section 5 with the numerical results and discussion. Our concluding remark is given in Section 6.

2. Calculation of the Wave Function for the Shifted Static Screened Coulomb Potential

The Shifted Static Screened Coulomb Potential (SSSCP) is given as [34]

$$\mathrm{V(r)} = \frac{\mathrm{Ze}^2}{r}e^{-\frac{r}{D}} - \frac{e^2}{D}, \tag{10}$$

Where r is the inter-particle distance, Z is the atomic number, Dis the screening parameter which characterizes the range of the interaction, for some values of $a = 1/D$. The radial Schrödinger Equation for the Shifted static screened coulomb potential given as [34]:

$$-\frac{d^2R_{n,l}(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left[E + \left(\frac{\mathrm{Ze}^2}{r}e^{-\frac{r}{D}}\right) - \frac{\hbar^2\ell(\ell+1)}{2\mu r^2}\right]R_{n,l}(r) = 0. \tag{11}$$

On solving equation (11), the wave function and energy eigenvalues for the shifted static screened coulomb potential are respectively expressed as:

$$R_{n\ell}^{\mathrm{SSSCP}}(\mathrm{r}) = N_{n\ell}s^{\zeta}(1-s)^{\ell+1}P_n^{(2\zeta,2\ell+1)}(1 - 2\mathrm{s}), \quad \mathrm{s} = e^{-2ar}, \mathrm{a} = 1/D \tag{12}$$

and

$$\mathrm{E}= -\frac{2\hbar^2\mathrm{a}^2}{\mu}\left[\frac{(n+1)^2+\ell^2+2\ell+2n\ell-\frac{\mu\mathrm{Ze}^2}{\mathrm{a}\hbar^2})}{2(n+\ell+1)}\right]^2 - \mathrm{a}e^2, \tag{13}$$

where$\zeta = \sqrt{-\frac{\mu E}{2\mathrm{a}^2\hbar^2} - \frac{\mathrm{ae}^2}{2}}$,n is the principal quantum number and $\ell$is the orbital quantum number. Thecorresponding probability density is obtained by squaring the wave function (12), which gives

$$\rho(\mathrm{r}) = N_{n\ell}^2 e^{-4\mathrm{ar}\zeta}(1-e^{-2\mathrm{ar}})^{2\ell+2}\left[P_n^{(2\zeta,2\ell+1)}(1 - 2e^{-2\mathrm{ar}})\right]^2 = N_{n\ell}^2 s^{2\zeta}(1-s)^{2\ell+2}\left[P_n^{(2\zeta,2\ell+1)}(1 - 2s)\right]^2. \tag{14}$$

The normalization constant for SSSCP is obtained as [35]:

$$N_{n\ell}^2 = \left[\int_0^\infty \left|R_{n,l}(r)\right|^2 dr\right]^{-1} = \left[\int_0^1 \left|R_{n,l}(s)\right|^2 \frac{ds}{2as}\right]^{-1}, \tag{15}$$

The normalised wave function $R_n^{\mathrm{SSSCP}}(r)$ for two low lying states n= 0, 1, are obtained as:

$$R_0^{\mathrm{SSSCP}}(r) = \sqrt{\frac{2a\Gamma(2\ell+2\zeta+3)}{\Gamma(2\ell+3)\Gamma(2\zeta)}}(1-e^{-2ar})^{\ell+1}e^{-2ar\zeta} \tag{16}$$

and

$$R_1^{\mathrm{SSSCP}}(r) = \frac{N_1}{2}(1-e^{-2ar})^{\ell+1}e^{-2ar\zeta}(1-e^{-2ar})(2\ell+2\zeta+3)-2\ell+2\zeta-1) \tag{17}$$

with

$$N_1 = \sqrt{\frac{2a(\ell+\zeta+2)\Gamma(2\ell+2\zeta+3)}{(\ell+2)(2\ell+1)\Gamma(2\ell+3)\Gamma(2\zeta)}}. \tag{18}$$

To obtain the corresponding normalized wave function in the momentum-space, the Fourier transform of$R_n^{\mathrm{SSSCP}}(p)$ is calculated. Two low lying states *n* = 0, 1, for the momentum-space wave functions are evaluated as [37–39]

$$R_0^{\mathrm{SSSCP}}(p) = \frac{1}{\sqrt{2\pi}}\int_0^\infty R_0(r)e^{-ipr}\,dr = \sqrt{\frac{a\Gamma(2\ell+2\zeta+3)}{\pi\Gamma(2\ell+3)\Gamma(2\zeta)}}\left[\frac{\Gamma(\ell+2)\Gamma(\frac{ip}{2a}+\zeta)}{2a\Gamma(\ell+\frac{ip}{2a}+\zeta+2)}\right] \tag{19}$$

And+

$$R+^{\mathrm{SSSC+P}}_{\cdot\ 1}(p) = \frac{1}{\sqrt{2\pi}}\int_0^\infty R_1(r)e^{-ipr}\,dr = \sqrt{\frac{a\Gamma(2\ell+2\zeta+3)}{\pi\Gamma(2\ell+3)\Gamma(2\zeta)}}\left[\frac{\Gamma(\ell+2)(a(\ell+2\zeta+2)-i(\ell+2)p)\Gamma(\frac{ip}{2a}+\zeta)}{2a^2\Gamma(\ell+\frac{ip}{2a}+\zeta+3)}\right]. \tag{20}$$

3. Shannon entropy for the Shifted static screened coulomb potential(SSSCP)

The Shannon entropy for the shifted static screen coulomb potential in the position space is obtained using equations (1) to have [40, 41:

$$S_n^{\mathrm{SSSCP}}(s) = -\frac{\mathrm{N}_{n\ell}^2}{2a}\int_0^1 s^{2\zeta-1}(1-s)^{2\ell+2}\left[P_n^{(2\zeta,2\ell+1)}(s)\right]^2 \times \ln\left(\mathrm{N}_{n\ell}^2\, s^{-2\zeta-1}(1-s)^{2\ell+2}\left[P_n^{(2\zeta,2\ell+1)}(s)\right]^2\right). \tag{21}$$

The Shannon entropy is further simplified as

$$\begin{aligned} S_n^{\mathrm{SSSCP}}(s) &= -\ln \mathrm{N}_{n\ell}^2 \\ &\quad -\frac{\mathrm{N}_{n\ell}^2}{2a}\int_0^1 s^{2\zeta-1}(1-s)^{2\ell+2}\left[P_n^{(2\zeta,2\ell+1)}(s)\right]^2 \ln[s^{2\zeta-1}(1-s)^{2\ell+2}]ds - \frac{\mathrm{N}_{n\ell}^2}{2a}\int_0^1 s^{2\zeta-1}(1 \\ &\quad - s)^{2\ell+2}\left[P_n^{(2\zeta,2\ell+1)}(s)\right]^2 \ln\left[P_n^{(2\zeta,2\ell+1)}(s)\right]^2 ds \\ &= -\ln \mathrm{N}_{n\ell}^2 + \frac{\mathrm{N}_{n\ell}^2}{2a}\left(\mathrm{E}\left[P_n^{(2\zeta,2\ell+2)}\right] + \mathrm{I}\left[P_n^{(2\zeta,2\ell+2)}\right]\right), \quad s = e^{-2ar}, \end{aligned} \tag{22}$$

where the entropic integrals are simplified as [40, 42]:

$$\begin{aligned} \mathrm{I}\left[P_n^{(2\zeta,2\ell+2)}\right] &= -\int_0^1 s^{2\zeta-1}(1-s)^{2\ell+2}\left[P_n^{(2\zeta,2\ell+1)}(s)\right]^2 \ln[s^{2\zeta-1}(1-s)^{2\ell+2}]ds, \\ &= -2\zeta\varphi(n+2\zeta+1) - (2\ell+1)\varphi(n+2\ell+2) + (2\zeta+2\ell+1) \times \left[-\ln 2 + \frac{1}{2n+2\zeta+2\ell+2} + 2\varphi(2n+2\zeta+2\ell+2) - \right. \\ &\quad \left. \varphi(n+2\zeta+2\ell+2)\right] \end{aligned} \tag{23}$$

and

$$\begin{aligned} \mathrm{E}\left[P_n^{(2\zeta,2\ell+2)}\right] &= \int_0^1 s^{2\zeta-1}(1-s)^{2\ell+2}\left[P_n^{(2\zeta,2\ell+1)}(s)\right]^2 \ln\left[P_n^{(2\zeta,2\ell+1)}(s)\right]^2 ds \\ &= \ln 2 - 1 - (2\zeta+2\ell+1)\ln 2 + \sigma(1). \end{aligned} \tag{24}$$

Table 1: Numerical Results for Shannon Entropy for the SSSCP in the ground and first excited eigenstates for various values of a, with ℓ= 0,Ze^2 = 1,m=minimum value= $1+\ln\pi$.

$a=\frac{1}{D}$	$S_r(n=0)$	$S_p(n=0)$	S_r+S_p	m	$S_r(n=0)$	$S_r(n=0)$	S_r+S_p	m
0.01	1.1545146682950742	1.2241047561777427	2.37861	2.1447	2.2353192854241186	0.5299565550455032	2.76529	2.1447
0.05	1.1565178973937518	1.2225022557066834	2.37902	2.1447	2.2595261111834720	0.5036987063463891	2.76323	2.1447
0.10	1.1628166578022892	1.217464472229951	2.38028	2.1447	2.3439136251831556	0.4127795959292214	2.75669	2.1447
0.20	1.1886128361996330	1.1968458863311586	2.38546	2.1447	2.9532258799425060	-0.215506474850895	2.73772	2.1447
0.30	1.2338716976107890	1.1607073988854362	2.39001	2.1447	2.8652443000540155	-0.124944433482795	2.74030	2.1447
0.40	1.3025165343858425	1.1059267759860578	2.40845	2.1447	1.9577886146143102	0.7814699074685885	2.73926	2.1447
0.50	1.4013877112383273	1.0269468496794247	2.42834	2.1447	1.5267636347562081	1.2154796668234935	2.74224	2.1447

We observed that some of the results in Table (1) have negative Shannon entropies. The system'sentropy could be negative for some potential parameter a, indicating the fact that, the probability densitiesfor these entropies are highly localized. The Bialynicki-Birula-Mycielski (BBM) inequality ($S_T = S_r + S_p > (1 + \ln\pi) > 2.1447$) showsto hold for SSSCP. From Table 1, the position space Shannon entropy$S_r$abruptly increased and then decreased with the potential parameter a while the behaviour of themomentum space entropies $S_P$is the reverse. In general, $S_r$increases with energy levels which meansthat, when the energy of the system increases, the system becomes more unstable and the disorder occursaccordingly.

4. Fisher Information For Shifted Static Screen Coulomb Potential

From equation (2), the Fisher information for SSSCP is written as [28]:

$$\mathrm{I}_n^{\mathrm{SSSCP}}(r) = 4\int_0^\infty |R'_{n.l}(r)|^2\, dr = 8a\frac{1}{s}\int_0^1 |R'_{n.l}(s)|^2\, ds \;= 16a\int_{-1}^1\left(\frac{2}{1-r}\right)|R'_{n.l}(r)|^2 dr, \quad s = e^{-2ar},\ r = 1-2s. \quad (25)$$

Substituting equation (12) into equation (25) gives the following expression for the position space Fisher information as

$$\mathrm{I}_n^{\mathrm{SSSCP}}(r) = 32a^2\mathrm{N}_{n\ell}^2\int_{-1}^1\left(\frac{2}{1-r}\right)\left\{(\ell+1)\left(\frac{1+r}{2}\right)^{\ell}\left(\frac{1-r}{2}\right)^{\zeta+1}P_n^{(2\zeta,2\ell+1)}(r) + \left(\frac{1+r}{2}\right)^{\ell+1}\left(\frac{1-r}{2}\right)^{\zeta+1}(2\ell+n+2\zeta+2)P_{n-1}^{(2\zeta+1,2\ell+2)}(r) - \zeta\left(\frac{1+r}{2}\right)^{\ell+1}\left(\frac{1-r}{2}\right)^{\zeta}P_n^{(2\zeta,2\ell+1)}\right\}^2 dr. \quad (26)$$

For n-state, the analytical evaluation of the integral in equation (26) is very difficult, hence, few low-lying normalised states n= 0, 1 are studied. Then, the Fisher information for SSSCP are obtained as follows:

For n= 0, equation (26) becomes

$$\mathrm{I}_0^{\mathrm{SSSCP}}(\mathrm{r}) = \frac{16\mathrm{a}\,\ell(\ell+1)\zeta\,\Gamma(2\ell)\,\Gamma(2\zeta)}{\Gamma(2(\ell+\zeta+1))}(\mathrm{N}_0^{\mathrm{SSSCP}})^2, \quad (27)$$

also, for n = 1, we have:

$$\mathrm{I}_0^{\mathrm{SSSCP}}(\mathrm{r}) = \frac{16\mathrm{a}\,\ell(\ell+1)(3\ell+2)\Gamma(2\ell)\Gamma(2\zeta+2)}{\Gamma(2\ell+2\zeta+3)}(\mathrm{N}_1^{\mathrm{SSSCP}})^2. \quad (28)$$

To obtain the momentum coordinate Fisher information, the Fourier transform of the spatial coordinate wave function is evaluated and then, substituted into equation (3) in a similar way like theposition space Fisher information.The following equations are used to evaluate the expectation values$\langle r\rangle_n$, $\langle p\rangle_n$, $\langle r^2\rangle_n$, $\langle p^2\rangle_n$:

$$\langle r\rangle_n = \int_0^\infty \mathrm{R}_{n,l}^{\mathrm{SSSCP}}\, r\, \mathrm{R}_{n,l}^{\mathrm{SSSCP}} dr, \langle r^2\rangle_n = , \quad (29)$$

$$\langle p^2\rangle_n = \int_0^\infty \mathrm{R}_{n,l}^{\mathrm{SSSCP}}\, p^2\, \mathrm{R}_{n,l}^{\mathrm{SSSCP}} dr, \langle p\rangle_n = 0. \quad (30)$$

Table 2: Numerical Results of the Fisher-information-based uncertainty relation for the SSSCP in the ground eigenstates for various values of a, with ℓ= 0, Ze^2 = 1,min=minimum value.

A	$\mathbf{I_r}$	$\mathbf{I_P}$	$\boldsymbol{I_r I_P}$**>36.0**	**min(**$\boldsymbol{I_r I_P}$**)**
0.1	3.9599999999866253	12.182634425058495	48.2431	36.0
0.2	3.8399999999843963	12.76388888888889	49.0133	36.0
0.3	3.6400000000400510	13.858471199130527	50.4449	36.0
0.4	3.3599999999600385	15.718820861677990	52.8152	36.0
0.5	2.9999999998399210	18.888888888888886	56.6667	36.0

Table 3: Numerical Results of the Heisenberg Uncertainty Relation for the SSSCP in the ground eigenstates for various values of a, with ℓ= 0, Ze^2 = 1,min=minimumvalue.

$\boldsymbol{a}$	$\langle \boldsymbol{r^2}\rangle$	$\langle \boldsymbol{r}\rangle$	**Δ (r)**	$\langle \boldsymbol{p^2}\rangle$	**Δ (p)**	$\Delta r\Delta p \geq \frac{\hbar}{2}$	$(\Delta \mathbf{r})^2(\Delta \mathbf{p})^2$	**min** $\{(\Delta \boldsymbol{r})^2(\Delta \boldsymbol{p})^2\}$
0.1	3.045658606	1.5001000100014608	0.89182877	0.99	0.994987437	0.887358428	0.787404980	0.25
0.2	3.190972222	1.5122482032594320	0.95082995	0.96	0.979795897	0.931619284	0.904118071	0.25
0.3	3.464617800	1.5461345329009326	1.03638111	0.91	0.953939201	0.988644569	0.977418086	0.25
0.4	3.929705215	1.6063170704723975	1.16165859	0.84	0.916515139	1.064677686	1.133538576	0.25
0.5	4.722222222	1.7020675561966576	1.35099528	0.75	0.866025404	1.169996236	1.368891193	0.25

Table 4: Numerical Results of the Fisher-information-based uncertainty relation for the SSSCP in the first excited eigenstates for various values of a, with ℓ= 0, Ze^2 = 1,min = minimum value.

A	I_r	I_P	$I_r I_P$>36.0	min($I_r I_P$)
0.1	0.8399999999797840	199.48242630385607	167.5649	36.0
0.2	0.3599999999999616	532.46460065507700	191.6874	36.0
0.3	0.5200000000000309	416.24583383649070	216.4479	36.0
0.4	2.2799999999937812	78.100452564747230	178.0691	36.0
0.5	4.9999999995979270	34.558888888888890	172.7945	36.0

Table 5: Numerical Results of the Heisenberg Uncertainty Relation for the SSSCP in the first excited eigenstates for various values of a, with ℓ= 0, Ze^2 = 1,min =minimum value.

a	$\langle r^2\rangle$	$\langle r\rangle$	Δ (r)	$\langle p^2\rangle$	Δ (p)	$\Delta r\Delta p \geq \frac{\hbar}{2}$	$(\Delta \mathbf{r})^2(\Delta \mathbf{p})^2$	min $\{(\Delta r)^2(\Delta p)^2\}$
0.1	49.87060658	6.5059523809524790	2.746486882	0.21	0.458257569	1.258598403	1.584069944	0.25
0.2	133.1161502	10.126984126984127	5.528140976	0.09	0.300000000	1.658442293	2.750430855	0.25
0.3	104.0614585	8.7239010989010670	5.287249576	0.13	0.360555128	1.906344945	3.634151057	0.25
0.4	19.52511314	3.9557985873775277	1.968951670	0.57	0.754983444	1.486525912	2.209759287	0.25
0.5	8.639722222	2.6583333333333323	1.254187431	1.25	1.118033989	1.402224176	1.572986113	0.25

The results in Tables (2) and (4) indicate that, the position-space Fisher information I_r(n =0) decreases with potential parameter a, the momentum-space Fisher information $I_p$increases with thepotential parameter a. For the Fisher information, the higher the values of $I_r$and I_p, the more localizedis the probability density, the smaller the uncertainty and the higher the accuracy is in predicting thelocalization of the particle in this system, while the reverse occur for lower values of $I_r$and I_p. FromTables (3) and (5), our observations include the squeezed state for some of the values. A state is definedto be squeezed if $(\Delta r)^2$<0.5 or $(\Delta p)^2$<0.5, where $(\Delta x)^2$=$\langle x^2\rangle - \langle x\rangle^2$, x = r or p [45]. Our results show a squeezed phenomenon for the states with n = 0, 1 for some values of the potential parameter a, and the least value obtained is greater than the minimum value $(\Delta r)^2(\Delta p)^2$> 0.25. We also found that,$\Delta r\Delta p$of the ground and first excited states have values$\geq \frac{\hbar}{2}$, which verifies that Heisenberg UncertaintyRelation holds for SSSCP.

5. Renyi Entropy, Tsallis entropy and Onicescu information energy for the Shifted Static Screened Coulomb Potential

The Rényi entropy from equation (4) is expressed as

$$R_q[p_{n,\alpha,\beta(r)}] = \frac{1}{1-q}\ln W_q\left[p_{n,\alpha,\beta}\right], \qquad (31)$$

where the entropic moment $W_q\left[p_{n,\alpha,\beta}\right]$ is expressed as [13, 14, 46, 47]:

$$W_q[p_n] = \int_0^\infty [p_{n,\alpha,\beta(r)}]^q dr = \frac{\mathrm{N}_{\mathrm{n}\ell}^{2\mathrm{q}}}{4\mathrm{a}}\int_{-1}^{1}\left(\frac{1-r}{2}\right)^{2\zeta q-1}\left(\frac{1+r}{2}\right)^{2\ell q+2q}\left[P_n^{(2\zeta,2\ell+1)}(r)\right]^{2q} dr. \qquad (32)$$

Thus, the Rényi entropy for shifted static screened coulomb potential is finally obtained as:

$$\mathrm{R}_q^{\mathrm{SSSCP}}[p_n] = \frac{1}{1-q}\ln\left[\frac{\mathrm{N}_{\mathrm{n}\ell}^{2\mathrm{q}}}{4\mathrm{a}}\mathrm{c}_{\tilde{0}}(0,2\mathrm{q},\mathrm{n},2\zeta,2\ell+1,2\zeta q,2\ell q+2q)\right]. \qquad (33)$$

Tsallis entropy can be expressed in equation (8) as

$$T_q[p_n] = \frac{1}{q-1}\left(1 - W_q\left[p_{n,\alpha,\beta}\right]\right), \qquad (34)$$

where$W_q\left[p_{n,\alpha,\beta}\right]$ is as given in equation (32). The Tsallis entropy for Shifted Static Screened Coulomb potential is expressed as follows:

$$\mathrm{T}_q^{\mathrm{SSSCP}}[p_n] = \frac{1}{q-1}\left[1 - \frac{\mathrm{N}_{\mathrm{n}\ell}^{2\mathrm{q}}}{4\mathrm{a}}\mathrm{c}_{\tilde{0}}(0,2\mathrm{q},\mathrm{n},2\zeta,2\ell+1,2\zeta q,2\ell q+2q)\right]. \qquad (35)$$

Onicescu information energy for the Shifted Static Screened Coulomb Potential is expressed from equation (6) as

$$E[p_n] = \int \rho^2(\mathrm{r})\, dr, \qquad (36)$$

which is the same for $W_2[p_n]$ of equation (32), when q=2. Therefore, $W_2[p_n]$ is simplified as

$$W_2[p_n] = \frac{\mathrm{N}_{\mathrm{n}\ell}^{4}}{4\mathrm{a}}\mathrm{c}_{\tilde{0}}(0,4,\mathrm{n},2\zeta,2\ell+1,2\zeta q,2\ell-2q). \qquad (37)$$

Hence, the Onicescu information energy for the Shifted Static Screened Coulomb potential is written as

$$\mathrm{E}_q^{\mathrm{SSSCP}}[p_n] = \frac{\mathrm{N}_{\mathrm{n}\ell}^{4}}{4\mathrm{a}}\mathrm{c}_{\tilde{0}}(0,4,\mathrm{n},2\zeta,2\ell+1,2\zeta q,2\ell q+2q). \qquad (38)$$

Table 6: Numerical Results for Renyi Entropy for the SSSCP in the ground and first excited eigenstates for various values of a, with ℓ= 0, Ze^2 = 1.

$a=\frac{1}{D}$	$R_2^{SSSCP}[r](n=0)$	$R_2^{SSSCP}[r](n=1)$	$R_2^{SSSCP}[p](n=0)$	$R_2^{SSSCP}[p](n=1)$
0.01	0.9809042576995242	2.0495980804387880	33.312719086058536	34.827667726454110
0.05	0.9827071878358964	2.0724472331563684	35.827498589637834	36.626402044460555
0.10	0.9883764586470799	2.1519370446338920	38.607110203748796	36.713429601161480
0.20	1.0116009116784714	2.7235195787239220	38.728750194295310	32.706685981837950
0.30	1.0523829453603950	2.6217264004074490	37.404593012578690	35.562077172125170
0.40	1.1143606456361705	1.7406973242909816	38.387351025850776	36.934915836970000
0.50	1.2039728043262514	1.3173014896351150	37.058882266595740	37.030834660426440

From the numerical results in Table (6), the values of Rényi entropies in the position coordinate $R_2^{SSSCP}(n)$ for n = 0, 1 first increases and then decreases with increasing potential parameter a. Rényientropies in the momentum coordinate $R_2^{SSSCP}(n)$ for n = 0, 1 have similar behaviour.

Table 7: Numerical Results of Tsallis Entropies in the ground and first excited eigenstates for various values of a for the SSSCP, with q = 2, ℓ= 0, Ze^2 = 1.

$a=\frac{1}{D}$	$T_2^{SSSCP}[r](n=0)$	$T_2^{SSSCP}[r](n=1)$
0.01	0.6250281257031313	0.8712133449369647
0.05	0.6257035647279438	0.8741226467716846
0.10	0.6278195488721696	0.8837412588477404
0.20	0.6363636363636332	0.9343566894263735
0.30	0.6508951406649769	0.9273227156340695
0.40	0.6718749999999742	0.8246019513574563
0.50	0.7000000000000945	0.7321428571434400

In Table (7), the values of the position Tsallis entropies $T_2^{SSSCP}(r)$ for n = 0 increases with increasing potential parameter a while for n = 1, $T_2^{SSSCP}(n)$ first increases and then decreases with increasingpotential parameter a.

Table 8: Numerical Results of Onicescu information energy in the ground and first excited eigenstates for various values of a for the SSSCP, with q = 2, ℓ= 0, Ze^2 = 1.

$a=\frac{1}{D}$	$E_2^{SSSCP}[r](n=0)$	$E_2^{SSSCP}[r](n=1)$
0.01	0.37497187429686873	0.12878665506303535
0.05	0.37429643527205625	0.12587735322831542
0.10	0.37218045112783044	0.11625874115225960
0.20	0.36363636363636676	0.06564331057362646
0.30	0.34910485933502310	0.07267728436593052
0.40	0.32812500000002570	0.17539804864254366
0.50	0.29999999999990540	0.26785714285656004

The values of the position Onicescu information energy $E_2^{SSSCP}(2)$ for n = 0 decreases with increasing potentialparameter a. While for n = 1, $E_2^{SSSCP}(2)$ first decreases and then increases with increasingpotential parameter a. In the case of the Onicescu information energy, the greater the information energy,the more concentrated is the probability distribution and the smaller the information content of thesystem.

6. Concluding remarks

In this work, we have studied the information-theoretic measures which include: Shannon entropy, Fisher information, Rényi entropy, Tsallis entropy and Onicescu information energy in the position r and themomentum p eigenstates for the Shifted Static Screened Coulomb Potential in both the position andmomentum spaces. The wave function for this atomic model has been obtained in terms of the Jacobipolynomials by finding their bound state solutions and there after squared to obtain the probabilitydensity. Then, the Shannon entropy was calculated analytically by determining the entropic integralsasymptotically using the I_p-method. Some numerical results of the Shannon entropy was observed to benegative, which means that the probability densities for these entropies are highly localized. Also, $S_r$hadvalues that abruptly increased and then decreased exponentially with the potential parameter a while forS_P, the reverse occurred. The Fisher information was also evaluated analytically for the ground and firstexcited states. The Rényi entropy, Tsallis entropy and Onicescu information energy for q = 2 have alsobeen obtained analytically, by making use of the linearization formula of Srivastava-Dauost function forJacobi Polynomials in terms of the multivariate Lauricella hypergeometric function. The numerical resultsand the variations of the information-theoretic measures for this atomic potential in the ground and firstexcited states with various values of the potential parameter aare displayed. Characteristic features ofthe position and momentum probability densities are illustrated. The Shannon-entropy-based uncertaintyrelation, that is, Bialynicki-Birula-Mycielski(BBM) inequality ($S_T= S_r+S_p>D(1+\ln \pi) = 2.1447$) andFisher-information-based uncertainty relation ($I_{rp}= I_r I_P>36.0$) which is a stronger version of Heisenberguncertainty principle has been verified to hold for this atomic model.

Acknowledgments
The authors would like to thank the organisers of the Conference in honour of Prof.J. O. A. Idiodi on his 70*th* birthday. Isonguyo, C. N. and Oyun, O. S. wish to thank the Department of Physics, University of Ilorin, Nigeria for hospitality during their Postgraduate studies.

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