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AN ANALYSIS OF NONLINEAR PDE ARISING IN ELASTO-PLASTIC FLOW

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Abstract

Lie symmetry analysis is performed on a class of non-linear partial differential equations (PDEs), which describes the longitudinal motion of an elasto-plastic bar and anti-plane shearing deformation. All the geometric vector fields of the equation are obtained. Interestingly, it is shown that this equation admits nonlocal type symmetry. The symmetry reductions and some new exact explicit solutions are also presented.

***Keywords*:** Lie Symmetry, Symmetry reduction, Invariant solutions

1 Introduction

Nonlinear partial differential equations (NLPDEs) [1] are globally used to describe complex phenomena in several fields of sciences. especially in physical sciences. Therefore, solving nonlinear problems plays a vital role in nonlinear sciences. In line with this direction, many effective methods for finding exact solutions of NLPDEs have been established and developed in the past few decades. Among these methods, the Lie symmetry method, also called Lie group method, isone of the most powerful methods used in finding solutions of NLPDEs. The fundamental basis of this method is that whena differential equation is invariant under a Lie group of transformations [2–4], a reduction transformation occurs. For PDEs
with two independent variables, a single group reduction transforms the PDEs into ordinary differential equations (ODEs),which are generally easier solve to obtain the exact or numerical solutions. In the recent past years, there have been significant developments in symmetry methodsfor differential equations as clearly revealed by the number of research papers, books and new symbolic software devoted tothe subject.

In the present paper, we consider a generalized fourth order nonlinear PDE arising in elasto-plastic flow[5]

$$u_{tt} - 2\beta u_{xxt} + \alpha u_{xxxx} - \gamma(u_x^n)_x = 0, \qquad (1)$$

where α,β,γ are constants and n > 0.

The Lie symmetries and symmetry solutions of the wave equations are derived.

2 Preliminaries

Definition 2.1.

A $k^{th} -$ order (k $\geq$ 1) system E^σ of s partial differential equations of n independent variables $x^i,\ i = 1,2,\dots,n$ and m-dependent variables $u^\alpha:\ \alpha = 1,2,\dots,m$ is defined by;

$$E^\sigma\left(x^i, u^\alpha, u_{(1)}, \cdots, u_{(k)}\right) = 0, \quad \sigma = 1,\cdots,s \qquad (2)$$

where $u_{(1)},\dots,u_{(k)}$ denote the collection of all first, second, ..., kth-order partial derivatives.

Definition 2.2.

The Euler-Lagrangian operator is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}, \qquad \alpha = 1,2,\dots,\mathrm{m} \qquad (3)$$

where $D_i = \frac{\partial}{\partial x^i} + u_i^\alpha\frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha\frac{\partial}{\partial u_j^\alpha} + \cdots, \qquad i = 1,2,\dots,\mathrm{n} \qquad (4)$

is the total derivative operator with respect to x^i

Definition 2.3.

The Euler-Lagrangian equations, associated with (2) are the equations

$$\frac{\delta L}{\delta u^\alpha} = 0, \qquad \alpha = 1,2,\dots,\mathrm{m} \qquad (5)$$

where L is referred to as a Lagrangian of (2).

Definition 2.4.

A Lie Backlund operator X is defined by

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$$X = \varepsilon\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\delta}{\delta u^\alpha} + \sum_{s\geq 1}\varsigma^\alpha_{i_1\cdots i_s}\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}, \qquad \alpha = 1,2,\ldots,m \qquad (6)$$

where $\varsigma^\alpha_{i_1\cdots i_s}$ are given as

$$\varsigma^\alpha_i = D_i(\eta^\alpha) - \varsigma^\alpha_{i_1\cdots i_s}D_i\varepsilon^j, \qquad \varsigma^\alpha_{i_1\cdots i_s} = D_{i_s}(\varsigma^\alpha_{i_1\cdots i_s}) - u^\alpha_{ji_1\cdots i_{s-1}}D_{i_s}(\varepsilon^j), \qquad s \geq 1. \quad (7)$$

Definition 2.5.

The Lie point symmetry of equation (2) is a generator X of the form (6) that satisfies

$$X^{[k]}F_{|F=0} = 0, \qquad (8)$$

where $X^{[k]}$ is the kth prolongation of X i.e.,

$$X^{[k]} = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} + \varsigma^\alpha_i(x,u,u_{(1)})\frac{\partial}{\partial u^\alpha_i} + \cdots + \varsigma^\alpha_{i1\ldots ik}(x,u,u(k))\frac{\partial}{\partial u^\alpha_{i1\ldots ik}}. \quad (9)$$

Definition 2.6.

A Lie Backlund operator X of the form (6) is called a Noether symmetry generator associated with a Lagrangian L of (5) if there exists a vector $B = (B^1, B^2, \cdots, B^n)$
such that

$$XL + LD_i(\varepsilon^i) = D_i(B^i), \qquad (10)$$

where X isprolonged to the degree of L [1]. If the vector B is identically zero, then X is a strict Noether symmetry [6]. For each Noether symmetry generator X associated with a given Lagrangian L corresponding to the Euler-Lagrange differential equations, a conserved quantity is obtained [7] using the equation

$$T^i = B^i - N^i\text{L}, \qquad i = 1,2,\ldots,n \qquad (11)$$

3 Lie point symmetries

In this section, the Lie point symmetry generators admitted by (1) are presented. The Lie symmetries is formed by the set of vector fields of the form

$$X = \xi^t(t,x,u)\frac{\partial}{\partial t} + \xi^x(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \qquad (12)$$

The operator X satisfies the Lie symmetry condition [1]

$$X^{[4]}[u_{tt} - 2\beta u_{xxt} + \alpha u_{xxxx} - \gamma(u_x^n)_x]|_{(1)} = 0, \qquad (13)$$

where $X^{[4]}$is the fourth prolongation of the operator X and can be computed from (9).

Expansion and separation of (15) with respect to the powers of different derivatives of u yields the following over determined system in the unknown coefficients ξ^t, ξ^x and η:

$$\eta_{tu} = 0, \eta_{uu} = 0, -\eta_{xu} = 0, -\eta_{xxxx} = \frac{1}{\alpha}(-\gamma\eta_u(u_x^n)_x + 2\gamma\xi^t(u_x^n)_x),$$
$$+2\beta\eta_{xxt} - \eta_{tt}, \xi^t_u = 0, \xi^t_x = 0, \xi^t_{tt} = 0, \xi^x_t = 0, \xi^x_u = 0, \xi^x_x = \tfrac{1}{2}\xi^t_t. \qquad (14)$$

Solving the over determined system (14) for $\xi^t(t,x,u), \xi^x(t,x,u)$ and $\eta(t,x,u)$we obtain the Lie symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}, \quad X_4 = u\frac{\partial}{\partial u}, \quad X_5 = \frac{1}{6}x^3\frac{\partial}{\partial u}, \quad X_6 = x\frac{\partial}{\partial u},$$
$$X_7 = \tfrac{1}{2}x^2\frac{\partial}{\partial u}, X_8 = 2\gamma\iint(u_x^n)_x\, dtdt \qquad (15)$$

$X_1 - X_7$ are called Lie point symmetries while X_8 is referred to as nonlocal symmetry due to the presence of the integral in the infinitesimal.

Special cases

(i)$\alpha = 0, \beta = 0, n = 1, \gamma \neq 0$

This case gives rise to the linear wave equation

$$u_{tt} - \gamma u_{xx} \qquad (16)$$

Equation (18) admits eight Lie point symmetries given by

$$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x}, X_3 = u\frac{\partial}{\partial u}, X_4 = \frac{\partial}{\partial u}, X_5 = x\frac{\partial}{\partial u}, X_6 = tx\frac{\partial}{\partial u},$$
$$X_7 = t\frac{\partial}{\partial u}, X_8 = 2x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \qquad (17)$$

as well as an infinite symmetry $X_9 = F_1(t,x)\frac{\partial}{\partial u}$, where $F_1(t,x)$is a solution of equation (16)and hence called the solution symmetry. This symmetry always comes up as a result of the linearity of the equation in question.

(ii) $\beta = 0, \alpha\gamma \neq 0$, n = 3

In this case equation (1) reduces to a well-known PDE, the modified Boussinesq equation given by [7]

$$u_{tt} + \alpha u_{xxxx} - \gamma(u_x^3)_x = 0. \qquad (18)$$

which was presented in the famous Fermi-Pasta-Ulam problem. The symmetries are

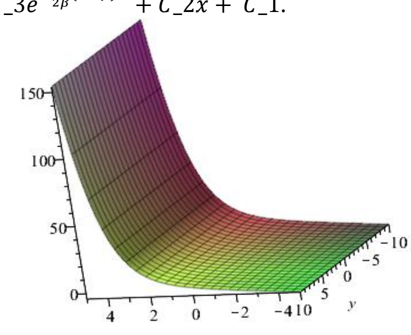
$$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x}, X_3 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}, X_4 = \frac{\partial}{\partial u}, X_5 = t\frac{\partial}{\partial u}. \qquad (19)$$

Equation (18) is used to investigate the behavior of systems which are primarily linear but a nonlinearity is introduced as a perturbation. It also arises in other physical applications[8]. In [9], three types of symmetry reductions of equation (20) were derived and it was shown thatthe equation is unintegrable. The soliton solutions of some special cases of equation (18) were obtained in [8 10,11,12,13]by various techniques.

(iii) $\alpha = 0, \beta\gamma \neq 0$, n = 1
This gives rise to
$$u_{tt} - 2\beta u_{xxt} - \gamma u_{xx} . \quad (20)$$
The Lie symmetries are
$$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x}, X_3 = t\frac{\partial}{\partial u}, X_4 = \frac{\partial}{\partial u}, X_5 = u\frac{\partial}{\partial u}, X_6 = tx\frac{\partial}{\partial u}, X_7 = x\frac{\partial}{\partial u}. \quad (21)$$

4 Symmetry reduction and Exact/Invariant Solution
One of the main aims for finding the symmetries of differential equations is to use them to reduce the differential equations which could be solved to obtain exact solutions.
In this section, we will make use of the symmetries obtained in Section 3 to reducethe wave equations and obtain exact solutions where possible. In particular, we use the translation generators in t variable given as $X_1 = \frac{\partial}{\partial t}$
The similarity variables and the similarity solutions of the equations can be obtained by solving characteristic equation given as
$$\frac{dt}{\xi^t} = \frac{dx}{\xi^x} = \frac{du}{\eta} \quad (22)$$
The general solution of these equations involves two constants, one becomes independent variabler and other plays the role of new dependent variable w.

(i)$\alpha = 0, \beta = 0, n = 1, \gamma \neq 0$
Using the generator, $X_1 = \frac{\partial}{\partial t}$on solving equation (22) we obtain the invariants
$$r = x, w(r) = u. \quad (23)$$
Substituting (23) into equation (16) leads to the reduced second orderlinear ODE
$$w'' - \gamma w'' = 0 \quad (24)$$
whose general solution in terms of the original variable u is
$u(x) = (1-\gamma)C_1 x + C_2,$ where $C_1, C_2,$ are constants.
(ii) $\beta = 0, \alpha\gamma \neq 0$, n = 3
Similar approach reduces (18) to
$$\alpha w'''' - 3\gamma w'' w'^2 = 0. \quad (25)$$
The general solution of (25) is an elliptic function given as,
$$u(x) = \frac{2}{3}\sqrt{2}\left(\sqrt{\alpha}\left(\frac{\gamma}{C_1}\right)^{-\frac{1}{4}} EllipticF\left(N\left(\frac{\gamma}{C_1}\right)^{\frac{1}{4}}, I\right) + x + C_2\right)^{\frac{3}{2}} + C_3.$$
where $C_1, C_2, C_3$are constants.
(iii) $\alpha = 0, \beta\gamma \neq 0$, n = 1
With the same generator X_1, (20) reduces to the following third order nonlinear ordinary differential equation in x variable
$$w'' - 2\beta w''' - \gamma w'' = 0. \quad (26)$$
The solution of (26) is
$$C\_3 e^{-\frac{1}{2\beta}(1-\gamma)x} + C\_2x + C\_1. \quad (27)$$



Graphical representation of solution (29) for γ>β,
C_1 = C_2 = C_3 = 1

5 Discussion and Conclusion

In this paper, the invariance properties ofa class of non-linear PDEs, found in elasto-plastic flow were presented by using the Lie symmetry analysis. All the geometric vector fields of the non-linear PDEs were obtained. We discovered that the analyzed equation admits nonlocal type and solution symmetries as a result of the linearity of the equation when n = 1.

The symmetry reductions were performed and the invariant solutions of the equations were presented.

The results obtained here can be used in many important space times to facilitate the solutions of the wave equations in these space times.

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