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AN EFFICIENT HYBRID METHOD OF DIRECT COMPUTATIONAL AND HOMOTOPY ANALYSIS METHOD FOR SOLVING FREDHOLM INTEGRAL EQUATIONS

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Abstract

In this paper, we proposed a combination of direct computational method with homotopy analysis method, namely DHAM, in order to find approximate solutions of Fredholm integral equations. We present convergence analysis of the proposed method of DHAM for solving Fredholm integral equations to the exact solution. illustrative examples were given to show applicability and efficiency of the proposed method of DHAM in solving Fredholm integral equations. The result obtained by DHAM is in line with the theoretical finding.

Keywords: Fredholm integral equation, direct computational method, homotopy analysis method

1. Introduction

Integral equations arise naturally in physics, chemistry, biology and engineering [4]. Most physical phenomena are governed by functional equations such as differential equations (DE), difference equations, integral equation (IE) and other functional equations. More details about the sources and origins of integral equations can be found in [1-3]. The exact solution of integral equations is some time difficult to find especially the nonlinear ones. Different approaches were made by researchers to find the exact or approximate solution of IEs. In [4].Sinc collocation method based on the double exponential transformation was applied to solve Voltera integral equation. A modified homotopy perturbation method was applied to solve DE and IE [5,6]. Guzeliya construct solution of Voltera and Abel integral equations by a generalized power series [7]. Ababneh used Picard approximation method to solve Voltera IE by using of self-canceling noise terms approach which is proposed by Wazwaz [8]. In [9], Ahmet obtained numerical solution of Fredholm IE using weighted mean value theorem. In [10], review of different numerical methods for solving both linear and nonlinear Fredholm integral equations was discussed. In [11], Laplace transform and Taylor's series was used to solve Voltera IE of convolution type. Their method was shown to require much less computational work than traditional methods.

Since the introduction of homotopy analysis method (HAM) [12,13] many authors applied the method to solve integral equations. In [14] HAM was applied to both Fredholm and Voltera integral equations with numerical examples given to show the performance and reliability of the method. Convergence of homotopy HAM was studied in [15] with examples to illustrate the validity for the conditions presented in the method. For other applications of HAM see [16-18]. S. L. Bichi et.al [19] developed the direct-homotopy analysis method (DHAM) for solving Fredholm integro-differential equations.

In this paper, we consider integral equation of the form:

$$\Phi(x) = f(x) + \Upsilon \int_a^b k(x,t)\Phi(t)dt, \tag{1}$$

where$k(x,t) = g(x)s(t)$i.e$k(x,t)$which is separable and $f(x)$ are known functions, $a, b$and $\Upsilon$are constant and $\Phi(x)$is the unknown function to be determine.

Description of Homotopy Analysis method

Consider

$$\mathrm{N}[\Phi(x)] = 0 \tag{2}$$

where$N$is a nonlinear operator, $\Phi(x)$ is the unknown function of independent variable x. For simplicity, we ignore all initial or boundary conditions. Liao [12] constructed the so called zero-order deformation equation as follows:

$$(1-q)L[\psi(x;q) - \Phi_0(x)] = hqH(x)N[\psi(x;q)], \tag{3}$$

where$q \in [0,1]$is the embedding parameter, $h \neq 0$is an auxiliary parameter, $H(x) \neq 0$ is an auxiliary function, $L$is a linear operator and $\Phi_0(x)$ is the initial guess of $\Phi(x)$. When $q = 0$and $q = 1$ it holds that:

$$\psi(x;0) = \Phi_0(x) \text{and} \psi(x;1) = \Phi(x) \tag{4}$$

respectively. Thus, from (4) as $q$increases from 0 to 1, $\psi(x;q)$ varies from initial guest$\Phi_0(x)$ to solution$\Phi(x)$. According to taylor's theorem,$\psi(x;q)$ can be expanded in power of $q$as follows:

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$$\psi(x;q) = \Phi_0(x) + \sum_{m=1}^{\infty} \Phi_m(x) q^m, \qquad (5)$$

where

$$\Phi_m(x) = \frac{1}{m!} \frac{\partial^m \psi(x;q)}{\partial q^m} | q = 0. \qquad (6)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter and the auxiliary function are properly chosen, the series (5) converges at $q = 1$. So, we obtain

$$\Phi(x) = \Phi_0(x) + \sum_{m=1}^{\infty} \Phi_m(x). \qquad (7)$$

Define a vector

$$\overrightarrow{\Phi_n} = \left\{ \Phi_0(x), \Phi_1(x), \Phi_2, \ldots, \Phi_n(x), \right\}.$$

Differentiating (3) $m$times with respect to the embedding parameter $q$and setting $q = 0$ then divide through by $m!$ we obtain the mthorder deformation equation as:

$$L\left[\Phi_m(x) - \chi_m \Phi_{m-1}(x)\right] = hH(x) R_m(\Phi_{m-1}), \qquad (8)$$

where

$$R_m\left(\Phi_{m-1}\right) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\psi(x;q)]}{\partial q^{m-1}} | q = 0, \qquad (9)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}.$$

2. Direct-homotopy Analysis Method

In this section we construct the new method of direct homotopy analysis method (DHAM)as follows:
Consider the Fredholm integral equation:

$$\Phi(x) = f(x) + \Upsilon \int_a^b k(x,t) \Phi(t) dt, \qquad (10)$$

where$f(x)$ and $k(x,t)$ are known functions with $k(x,t) = g(x)s(t)$. We define the nonlinear operator as:

$$N[\Phi(x)] = \Phi(x) - f(x) - \Upsilon g(x) \int_a^b s(t) \Phi(t) dt. \qquad (11)$$

The corresponding mth-order deformation equation is:

$$L\left[\Phi_m(x) - \chi_m \Phi_{m-1}(x)\right] = hH(x) R_m(\Phi_{m-1}), \qquad (12)$$

where

$$R_m\left(\Phi_{m-1}\right) = \Phi_{m-1}(x) - f(x)\left(1 - \chi_m\right) - \Upsilon g(x) \int_a^b s(t) \Phi_{m-1}(t) dt. \qquad (13)$$

Choosing the auxiliary linear operator $L[\Phi(x)] = \Phi(x)$, we obtain

$$\Phi_1(x) = hH(x)[\Phi_0(x) - f(x) - \Upsilon g(x) \int_a^b s(t)[\Phi_0(t) dt.], \qquad (14)$$

and

$$\Phi_m(x) = \Phi_{m-1}(x) + hH(x)\left[\Phi_{m-1}(x) - \Upsilon g(x) \int_a^b s(t) \Phi_{m-1}(t) dt\right], \qquad (15)$$

for

$$m = 2,3,4, \ldots . \qquad (16)$$

With proper choice of the initial guess $\Phi_0(x)$, the auxiliary function $H(x)$ and the auxiliary parameter $h$the series

$$\Phi_0(x) + \sum_{m=1}^{\infty} \Phi_m(x), \qquad (17)$$

converges to the exact solution. Therefore the solution is:

$$\Phi(x) = \Phi_0(x) + \sum_{m=1}^{\infty} \Phi_m(x). \qquad (18)$$

3. Convergence analysis

Theorem 3.1. Given that the series (17) is convergent, with $\Phi_0(x)$ as initial guess and $\Phi_m(x)$obtained from equation (12) and equation (13). Then the convergence is to the exact solution ofequation (10).

Proof of Theorem 3.1. Let

$$\Phi(x) = \sum_{m=0}^{\infty} \Phi_m(x)$$

be a convergent series, then it is true that

$$\lim_{n\to\infty} \Phi_m(x) = 0.$$

By computing

$$\sum_{m=1}^{n}\left[\Phi_m(x) - \chi_m \Phi_{m-1}(x)\right] = \Phi_1(x) + \left(\Phi_2(x) - \Phi_1(x)\right) + \cdots$$

$$+(\Phi_n(x) - \Phi_{n-1}(x)) = \Phi_n(x). \quad (19)$$

As $n \to \infty$, we obtain

$$\sum_{m=1}^{\infty} \Phi_m(x) = \lim_{n\to\infty} \Phi_m(x) = 0. \quad (20)$$

By the choice of our linear operator L

$$\sum_{m=1}^{\infty} L\left[\Phi_m(x) - \chi_m \Phi_{m-1}(x)\right] = L\sum_{m=1}^{\infty}\left[\Phi_m(x) - \chi_m \Phi_{m-1}(x)\right] = 0 \quad (21)$$

Using equation (8) given as

$$L\left[\Phi_m(x) - \chi_m \Phi_{m-1}(x)\right] = hH(x)R_m\left(\Phi_{m-1}\right). \quad (22)$$

Thus, we have from equation (21) and equation (22) that

$$\sum_{m=1}^{\infty} L\left[\Phi_m(x) - \chi_m \Phi_{m-1}(x)\right] = \sum_{m=1}^{\infty} hH(x)R_m\left(\Phi_{m-1}\right) \quad (23)$$

$$= hH(x)\sum_{m=1}^{\infty} R_m\left(\Phi_{m-1}\right). \quad (24)$$

$$= 0. \quad (25)$$

Since $h \neq 0$and $H(x) \neq 0$we have

$$\sum_{m=1}^{\infty} R_m\left(\Phi_{m-1}\right) = 0 \quad (26)$$

from equation (13) and equation (26)

$$\sum_{m=1}^{\infty} R_{m-1}\left(\Phi_{m-1}\right) = \sum_{m=1}^{\infty}[\,\Phi_{m-1}(x) - f(x)\left(1-\chi_m\right)$$

$$-\Upsilon g(x)\int_a^b s(t)\Phi_{m-1}(t)dt]$$

$$= \sum_{m=1}^{\infty} \Phi_{m-1}(x) - f(x) - \Upsilon g(x)\int_a^b s(t)\sum_{m=1}^{\infty} \Phi_{m-1}(t)dt$$

$$= \sum_{m=0}^{\infty} \Phi_m(x) - f(x) - \Upsilon g(x)\int_a^b s(t)\sum_{m=0}^{\infty} \Phi_m(t)\,dt$$

$$= 0 \quad , \quad (27)$$

implies

$$\Phi(x) = f(x) + \Upsilon\int_a^b k(x,t)\Phi(t)dt. \quad (28)$$

Therefore, $\Phi(x)$ is the exact solution of equation (10).

4. Numerical results

In this section we present some numerical examples that uses DHAM to solve Fredholm integral equations (10).Throughout the following example we make choices of the auxiliary parameter $h = -1$and auxiliary function $H(x) = 1$.The result we obtained from DHAM is compared with exact solution and absolute error is shown in the given tables.

Example 4.1 Consider integral equation:

$$\Phi(x) = 1 + \frac{1}{4}\int_0^{\frac{\pi}{2}} cosx\Phi(t)dt,$$

with exact solution $\Phi(x) = 1 + \frac{\pi}{6}\cos x$. Choosing $\Phi_0(x) = 1$ as initial guess, we obtain from (13) and (14) that

$$\Phi_1(x) = -\Phi_0(x) + 1 + \frac{1}{4}\int_0^{\frac{\pi}{2}} cosx\Phi_0(t)dt,$$

$$\Phi_m(x) = \frac{1}{4} cosx \int_0^{\frac{\pi}{2}} \Phi_{m-1}(t)dt.$$

Table 1: The exact and approximate solution of example 4.1

m	x	Exact solution	DHAM (18)	Absolute error
	0.0	1.5235987756	1.5235987756	4.54150E_19
	0.2	1.5131616602	1.5131616602	4.45096E_19
	0.4	1.4822664087	1.4822664087	4.18299E_19
	0.7	1.4004704328	1.4004704328	3.47353E_19
30	0.9	1.3254742183	1.3254742183	2.82304E_19
	1.0	1.2829016258	1.2829016258	2.45378E_19

Table 1, provides the exact solution example 4.1 and its approximate solution obtained by DHAM (18). The errors obtained indicated that DHAM is efficient

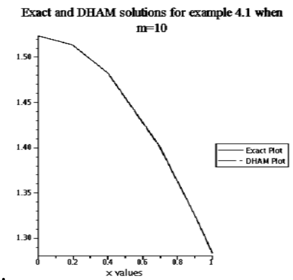

Figure 1: Exact and DHAM solutions for Example 4.1

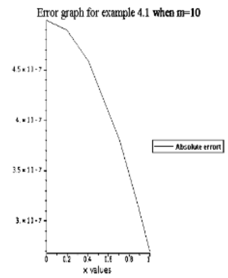

Figure 2: Errors for Example 4.1

Example 4.2 Consider integral equation:

$$\Phi(x) = sinx + \frac{1}{2}\int_0^{\frac{\pi}{2}} cosx\Phi(t)dt,$$

with exact solution $\Phi(x) = sinx + \cos x$ .Choosing $\Phi_0(x) = x$ as initial guess from (13) and (14). Weobtain:

$$\Phi_1(x) = -\Phi_0(x) + sinx + \frac{1}{2}\int_0^{\frac{\pi}{2}} cosx\Phi_0(t)dt,$$

$$\Phi_m(x) = \frac{1}{2} cosx \int_0^{\frac{\pi}{2}} \Phi_{m-1}(t)dt.$$

Table 2: The exact and approximate solution of example 4.2

m	x	Exact solution	DHAM (18)	Absolute error
	0.0	1.0000000000	0.9999999993	7.13671E_10
	0.1	1.0948375819	1.0948375812	7.10107E_10
30	0.3	1.2508566958	1.2508566951	6.81796E_10
	0.5	1.3570081004	1.3570080999	6.26306E_10
	0.7	1.4090598745	1.4090598740	5.45846E_10
	1.0	1.3817732907	1.3817732903	3.85598_10

In Table 2, the exact solution of problem in example 4.2 is compared with approximate solution obtained by DHAM (18). The errors obtained indicated that DHAM is accurate.

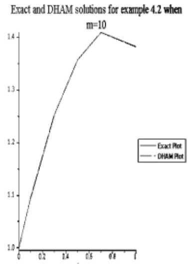

Figure 3: Exact and DHAM solutions for Example 4.2

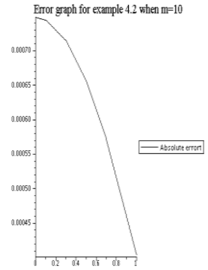

Figure 4: Errors for Example 4.2

Example 4.3 Consider integral equation:

$$\Phi(x) = \frac{9x^2}{10} + \frac{1}{2}\int_0^1 x^2t^2\,\Phi(t)dt,$$

with exact solution $\Phi(x) = x^2$ Choosing $\Phi_0(x) = x$ as initial guess from (13) and (14). Weobtain:

$$\Phi_1(x) = -\Phi_0(x) + \frac{9x^2}{10} + \frac{x^2}{2}\int_0^1 t^2\,\Phi_0(t)dt,$$

$$\Phi_m(x) = \frac{x^2}{2}\int_0^1 t^2\,\Phi_{m-1}(t)dt.$$

Table 3: The exact and approximate solution of example 4.3

m	x	Exact solution	DHAM (18)	Absolute error
30	0.0	0.0000000000	0.0000000000	0
	0.2	0.0400000000	0.0400000000	0
	0.4	0.1600000000	0.1600000000	0
	0.7	0.4900000000	0.4900000000	0
	0.9	0.8100000000	0.8100000000	0
	1.0	1.0000000000	1.0000000000	0

In Table 3 the exact solution of problem in example 4.3 is compared with approximate solution of DHAM (18). The errors obtained indicate the accuracy of the result obtained by DHAM.

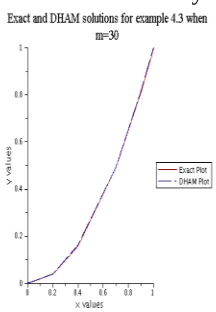

Figure 5: Exact and DHAM solutions for Example 4.3

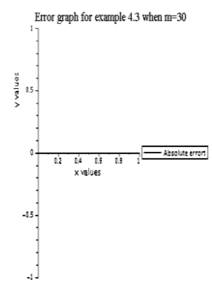

Figure 6: Errors for Example 4.3

Table 4: Comparison between ADM; HPM and DHAM obtained from example 4.1

x	ADM	HPM	DHAM
0.1	5.0877E_4	5.0877E_4	1.1890E_4
0.2	5.0113E_4	5.0113E_4	1.1711E_4
0.3	4.8848E_4	4.8848E_4	1.1416E_4
0.4	4.7096E_4	4.7096E_4	1.1006E_4
0.5	4.4873E_4	4.4873E_4	1.0486E_4
0.6	4.2201E_4	4.2201E_4	9.8625E_5
0.7	3.9108E_4	3.9108E_4	9.1396E_5
0.8	3.5624E_4	3.5624E_4	8.3254E_5
0.9	3.1784E_4	3.1784E_4	7.4280E_5

In Table 4, the absolute error obtained with ADM and HPM is compared with absolute error obtained by DHAM (each by taking first four terms of the series) problem in example 4.1.

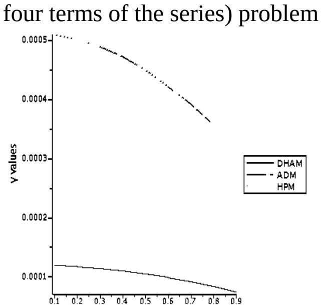

Figure 7: Comparison of the absolute errors obtained by ADM, HPM and DHAM in Table 3

5. Conclusion

In this paper, the approximate solution of integral equations with separable kernel was obtained by applying a combine methods of direct computational and homotopy analysis methods. We observed from the tables in the numerical results that the absolute errorsobtained are becoming better by taking more terms of the series equation (17).This is because the series (17) converges to the exact solution as proved in the convergence analysis. Comparison was also made with ADM and HPM which shows that our method of DHAM is efficient.

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