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OSCILLATION CRITERIA FORA FORCED SUPERLINEARCONFORMABLE FRACTIONAL DIFFERENTIAL EQUATION WITH DAMPING TERM

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Abstract

In this paper we establish some new oscillation criteriafor the solution of a forced superlinear conformable fractional differential equationwith damping term by using the averaging functions method.Our results provide extensions and improvmentto some existing ones. Some examples are also given to show the relevance of our results.

Keywords: Oscillation; Forced; Superlinear, Damping, Conformable fractional differential equation

1. Introduction

The fractional calculus [1, 2, 3] has attracted many researchers since the last two centuries. The impact of this fractional calculus on both pure and applied branches of sciences and engineering gained substantial increase during the last two decades. Also, research on oscillation theory as part of the qualitative theory of differential equations has been developing rapidly in the last decades, particularly on the oscillatory behaviour of integer order differential equations [4-6, 7]. Furtherextensions have been done on oscillation of fractional differential equations using Riemann-Liouville, Caputo and modified Riemann-Liouville [8-10, 11-13, 14]. However, since the introduction of conformable fractional derivatives, not many researchers have worked on the oscillation of the solution of conformable equations. These include :
[15]worked on oscillatory properties of a class of conformable fractional generalised Lienard equations

$$T_\alpha(r(t)T_\alpha(x(t))) + f(x(t))(T_\alpha(x(t)))^2 + g(x(t)) = 0 \quad t \ge t_0 \tag{1}$$

where T_α denotes the conformable fractional derivative w.r.t α , $0 < \alpha \le 1$.
Also, [16] established Kamenev Type oscillatory criteria for linear conformable fractional differential equations

$$(p(t)y^\alpha(t))^\alpha + q(t)y(t) = 0 \quad t \ge t_0 \tag{2}$$

where $p \in C([t_0,\infty),(0,\infty))$, $q \in C([t_0,\infty),\mathrm{R})$, $0 < \alpha \le 1$ and q might change signs.
In [17], the oscillation of solutions to the generalized forced nonlinear conformable fractional differential equation of the form

$$T_\alpha[a(t)\psi(x(t))T_\alpha x(t)] + P(t,x(t),T_\alpha x(t)) = Q(t,x(t),T_\alpha x(t)) \quad t \ge t_0, \tag{3}$$

were cosidered where T_α denotes the operator called conformable fractional derivative of order α with respect to variable t, C^α denotes continuous function with fractional derivative of order α , $a \in C^\alpha[[t_0,\infty),\mathrm{R}]$ and $P,Q \in \cdot C^\alpha[[t_0,\infty)\times \mathrm{R}^2,\mathrm{R}]$.
In this paper, we establish the oscillation of solutions to a forced superlinear fractional differential equation with damping term

$$D_t^\alpha[a(t)\sigma(x'(t))D_t^\alpha x(t)] + p(t)D_t^\alpha x(t) + g(t)f(x(t)) = Q(t,x(t),D_t^\alpha x(t)) \tag{4}$$

where

a,p,g : are continuous functions on the interval (t_0,∞)

σ, f : are continuous functions on the real line R, with $f(x) > 0$, $\forall\ x \in \mathrm{R}$.

Q: Is a continuous functions on $[t_0,\infty)\times\mathrm{R}^2$, with

$$\frac{Q(t,x(t),D_t^\alpha x(t))}{f(x(t))} \le q(t), \quad \forall\ t \in [t_0,\infty),\ q(t) \in ([t_0,\infty),\mathrm{R}) \ \ and \ \ x \ne 0.$$

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2. Preliminaries

For the purpose of this paper, we use the definition of fractional derivative of order $\alpha \in (0,1]$ by R. Khalil [18].

Definition 1 [19,18] *Given a function* $f:[0,\infty)\to \mathrm{R}$. *Then the "conformable fractional derivative" of* f *of order* α *is defined by*

$$D_\alpha(f)(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \quad \forall t>0,\ \alpha\in(0,1)$$

If f is α-differentiable in some $(0,a)$, $a>0$, and $\lim_{t\to 0^+} f^\alpha(t)$ exists, then define

$$f^\alpha(0) = \lim_{t\to 0^+} f^\alpha(t)$$

Definition 2 *A solution* $x(t)$ *of* (4) *is said to be oscillatory if it has infinite number of zeros, otherwise it is said to be nonoscillatory. The equation is said to be oscillatory if all its solutions are oscillatory.*

Some properties of the conformable fractional derivative of order $\alpha\in(0,1]$ which will be useful in this work are summarises below. For all $a,b,p\in\Re$, we have

$$\left.\begin{aligned} D^\alpha(af+bg) &= aD_\alpha(f)+bD_\alpha(g)\\ D^\alpha(t^p) &= pt^{p-\alpha}\\ D^\alpha(\lambda) &= 0\\ D^\alpha(fg) &= fD_\alpha(g)+gD_\alpha(f).\\ D^\alpha\left(\frac{f}{g}\right) &= \frac{gD_\alpha(f)-fD_\alpha(g)}{g^2}\\ D^\alpha(f)(t) &= t^{1-\alpha}\frac{df}{dt}(t)\end{aligned}\right\} \tag{5}$$

We refer the readers who are not familiar with conformable fractional derivatives to see [18,19] for details.

3. Main Results

In this section, we establish different oscillatory conditions for equation (4). Here, we let

$$xf(x)>0,\quad f'(x)\ge 0 \ \ and \ \ 0<f^2(x)\le\phi \ \ for \ \ x\ne 0 \tag{6}$$

$$\int^{\infty}\frac{du}{f(u)}<\infty \quad and \quad \int\frac{du}{f(u)}<\infty \tag{7}$$

$$\int^{\infty}\frac{\sqrt{f'(u)}}{f(u)}du<\infty \quad and \quad \int\frac{\sqrt{f'(u)}}{f(u)}du<\infty \tag{8}$$

$$\min\{\sup_{u>0}\sqrt{f'(u)}\int_u^\infty\frac{\sqrt{f'(z)}}{f(z)}dz,\quad \sup_{u<0}\sqrt{f'(u)}\int_u^{-\infty}\frac{\sqrt{f'(z)}}{f(z)}dz\}>0 \tag{9}$$

$$\Phi(t):=p(t)\rho(t)-k_2t^{1-\alpha}a(t)\rho'(t)\ge 0 \quad and \quad \Phi'(t)\le 0 \quad t\ge 0 \tag{10}$$

$$\varphi(t)=\frac{1}{t^{1-\alpha}a(t)\rho(t)} \tag{11}$$

Theorem 1

Suppose equation (6) – (11) *hold and there exists a differentiable function* $\rho:[t_0,\infty)\to(0,\infty)$ *such that*

$$-\infty<\limsup_{t\to\infty}\int_{t_0}^t\frac{\rho(u)r(u)}{u^{1-\alpha}}du<\infty \tag{12}$$

and

$$\int^{\infty}\varphi(s)ds=\infty, \tag{13}$$

Then, for any integers $\beta,\gamma>1$, equation (4) is oscillatory if

$$\limsup_{t\to\infty}\frac{1}{(t-\tau)^{\beta+\gamma}}\int_\tau^t[(t-u)^{\beta+\gamma}\frac{\rho(u)r(u)}{u^{1-\alpha}}+\frac{(t-u)^{\beta+\gamma-2}}{4V(u)}$$

$$\times[\Phi(u)\varphi(u)(t-u)+(\beta+\gamma)]]^2du=\infty \tag{14}$$

where

$$V(t)=\frac{b\varphi(t)}{\int_{t_0}^{t}\varphi(s)ds}$$

Proof. Let $x(t)$ be a non oscillatory solution of equation (4), without loss of generality we assume that $x(t)>0$ for $t\geq t_0>0$. Let W be defined by

$$W(t)=\rho(t)\frac{a(t)\sigma(x'(t))D_t^{\alpha}x(t)}{f(x(t))}$$

$$D_t^{\alpha}W(t)=D_t^{\alpha}[\rho(t)\frac{a(t)\sigma(x'(t))D_t^{\alpha}x(t)}{f(x(t))}]$$

$$=\frac{a(t)t^{2(1-\alpha)}\sigma(x'(t))\rho'(t)x'(t)}{f(x(t))}+\rho(t)\frac{D_t^{\alpha}[a(t)\sigma(x'(t))D_t^{\alpha}x(t)]}{f(x(t))}-\frac{W^2(t)f'(x(t))}{a(t)\rho(t)\sigma(x'(t))} \qquad (15)$$

From equation (4),

$$\frac{D_t^{\alpha}(a(t)\sigma(x'(t))D_t^{\alpha}x(t))}{f(x(t))}=\frac{Q(t,x(t),D_t^{\alpha}x(t))}{f(x(t))}-p(t)\frac{D_t^{\alpha}x(t)}{f(x(t))}-g(t)$$

$$\leq -r(t)-p(t)\frac{t^{1-\alpha}x'(t)}{f(x(t))} \qquad (16)$$

where $r(t)=g(t)-q(t)$.

Substituting (16) into (15), we have

$$D_t^{\alpha}W(t)\leq -\rho(t)r(t)-\rho(t)\frac{p(t)t^{1-\alpha}x'(t)}{f(x(t))}+\frac{k_2a(t)t^{2(1-\alpha)}\rho'(t)x'(t)}{f(x(t))}$$

$$-\frac{W^2(t)f'(x(t))}{k_2a(t)\rho(t)}$$

$$W'(t)\leq -\frac{\rho(t)r(t)}{t^{1-\alpha}}-\Phi(t)\frac{x'(t)}{f(x(t))}-\frac{W^2(t)f'(x(t))}{t^{1-\alpha}k_2a(t)\rho(t)} \qquad (17)$$

integrating inequality (17) w.r.t ds, we have

$$W(t)\leq W(t_0)-\int_{t_0}^{t}\frac{\rho(s)r(s)}{s^{1-\alpha}}ds-\int_{t_0}^{t}\Phi(s)\frac{x'(s)}{f(x(s))}ds-\int_{t_0}^{t}\frac{W^2(s)f'(x(s))}{k_2s^{1-\alpha}a(s)\rho(s)}ds$$

$$W(t)\leq W(t_0)-\int_{t_0}^{t}\frac{\rho(s)r(s)}{s^{1-\alpha}}ds-\int_{t_0}^{t}\Phi(s)\frac{x'(s)}{f(x(s))}ds-\frac{1}{k_2}\int_{t_0}^{t}W^2(s)f'(x(s))\varphi(s)ds \qquad (18)$$

In what follows, we consider the following two cases.

CASE 1 The integral

$$\int_{t_0}^{t}W^2(s)f'(x(s))\varphi(s)ds \quad \textit{is} \quad \textit{finite}$$

There exists a positive constant M_1 so that

$$\int_{t_0}^{t}W^2(s)f'(x(s))\varphi(s)ds\leq M_1 \quad for \quad t\geq t_0.$$

Considering (8) and then using the schwartz inequality, for $t\geq t_0$, we have

$$\left|\int_{t_0}^{t}\frac{x'(s)\sqrt{f'(x(s))}}{f(x(s))}ds\right|^2=\left|\int_{t_0}^{t}(\sqrt{\varphi(s)}W(s)\sqrt{f'(x(s))})ds\right|^2$$

$$\leq(\int_{t_0}^{t}\sqrt{\varphi(s)}ds)^2(\int_{t_0}^{t}(\sqrt{\varphi(s)}W(s)\sqrt{f'(x(s))})ds)^2$$

$$\leq(\int_{t_0}^{t}\sqrt{\varphi(s)}ds)^2(\int_{t_0}^{t}\varphi(s)W^2(s)f'(x(s))ds)$$

$$\leq M_1\int_{t_0}^{t}\varphi(s)ds \qquad (19)$$

Also from (9), we let

$$\sqrt{f'(x(t))}\int_{x(t)}^{\infty}\frac{\sqrt{f'(u)}}{f(u)}du\geq B \quad for \quad t\geq t_0 \qquad (20)$$

where B is a positive constant. Next, we put

$$B_1 = \int_{x(t_0)}^{\infty} \frac{\sqrt{f'(u)}}{f(u)} du > 0$$

Therefore, from (20) we have

$$f'(x(t)) \ge B^2 [\int_{x(t)}^{\infty} \frac{\sqrt{f'(u)}}{f(u)} du]^{-2}$$

$$= B^2 [B_1 - \int_{x(t_0)}^{x(t)} \frac{\sqrt{f'(u)}}{f(u)} du]^{-2}$$

$$\ge B^2 [B_1 + |\int_{t_0}^{t} \frac{x'(s)\sqrt{f'(x(s))}}{f(x(s))} ds|]^{-2}$$

$$f'(x(t)) \ge B^2 [B_1 + (M_1 \int_{t_0}^{t} \varphi(s) ds)^{1/2}]^{-2}$$

There exists a positive constant b (depending on the constants B, B_1 *and* M_1), so that

$$f'(x(t)) \ge b(\int_{t_0}^{t} \varphi(s) ds)^{-1} \quad for \quad t \ge T^* > t_0 \tag{21}$$

using (21) and the definition of $W(t)$ in equation (17), we obtain

$$W'(t) \le -\frac{\rho(t) r(t)}{t^{1-\alpha}} - \frac{1}{k_2} \Phi(t)\varphi(t) W(t) - \frac{b\varphi(t) W^2(t)}{k_2 \int_{t_0}^{t} \varphi(s) ds} \quad for \quad t \ge T^*$$

$$= -\frac{\rho(t) r(t)}{t^{1-\alpha}} - \frac{1}{k_2} [\Phi(t)\varphi(t) W(t) + V(t) W^2(t)] \quad for \quad t \ge T \tag{22}$$

If we multiply both sides of the inequality (22) by $(t-u)^{\beta+\gamma}$ and integrate from τ to t we have

$$\int_{\tau}^{t} (t-u)^{\beta+\gamma} W'(u) du \le -\int_{\tau}^{t} (t-u)^{\beta+\gamma} \frac{\rho(u) r(u)}{u^{1-\alpha}} du$$

$$-\frac{1}{k_2} \int_{\tau}^{t} (t-u)^{\beta+\gamma} [\Phi(u)\varphi(u) W(u) + V(u) W^2(u)] du$$

$$\int_{\tau}^{t} (t-u)^{\beta+\gamma} \frac{\rho(u) r(u)}{u^{1-\alpha}} du \le (t-\tau)^{\beta+\gamma} W(\tau) - (\beta+\gamma) \int_{\tau}^{t} (t-u)^{\beta+\gamma-1} W(u) du$$

$$-\frac{1}{k_2} \int_{\tau}^{t} (t-u)^{\beta+\gamma} V(u) W^2(u) du - \frac{1}{k_2} \int_{\tau}^{t} (t-u)^{\beta+\gamma} \Phi(u)\varphi(u) W(u) du$$

simplifying, we have

$$\int_{\tau}^{t} [(t-u)^{(\beta+\gamma)} \frac{\rho(u) r(u)}{u^{1-\alpha}} + \frac{(t-u)^{\beta+\gamma-2} ((\beta+\gamma) + (t-u)\Phi(u)\varphi(u))^2}{4k_2 V(u)}] du$$

$$\le (t-\tau)^{(\beta+\gamma)} W(\tau) \quad for \quad t \ge \tau \tag{23}$$

dividing (23) by $(t-\tau)^{\beta+\gamma}$ and taking the upper limit as $t \to \infty$, we obtain

$$\limsup_{t\to\infty} \frac{1}{(t-\tau)^{\beta+\gamma}} \int_{\tau}^{t} [(t-u)^{\beta+\gamma} \frac{\rho(u) r(u)}{u^{1-\alpha}} + \frac{(t-u)^{\beta+\gamma-2}}{4k_2 V(u)} [\Phi(u)\varphi(u)(t-u) + (\beta+\gamma)]^2] du$$

$$\le W(\tau) < \infty$$

This contradicts (14) Next, we show the second case.

CASE 2 The integral

$$\int_{t_0}^{t} \varphi(s) W^2(s) f'(x(s)) ds \quad is \ infinite$$

By (12), it follows from (18) that for some positive constant L

$$-W(t) \ge L + \frac{1}{k_2} \int_{t_0}^{t} \varphi(s) W^2(s) f'(x(s)) ds \tag{24}$$

where

$$L = -W(t_0) + \int_{t_0}^{t} \frac{\rho(s) r(s)}{s^{1-\alpha}} ds$$

We choose a $T^* \geq t_0$ so that

$$\theta \equiv L + \frac{1}{k_2}\int_{t_0}^{T^*} \varphi(s) W^2(s) f'(x(s)) ds > 1$$

Then (24) ensures that W is negative on $[T^*, \infty)$. Now, multiply (24) by $\frac{f'(x(t))}{f(x(t))}$, we have

$$-\frac{(\varphi(t)W^2(t) f'(x(t)))/k_2}{L + \frac{1}{k_2}\int_{t_0}^{t} \varphi(s) W^2(s) f'(x(s)) ds} \geq -\frac{x'(t) f'(x(t))}{f(x(t))}$$

since

$$\frac{1}{f(x(t))} = \frac{W(t)\varphi(t)}{k_2 x'(t)}$$

integrate both sides of the inequality above w.r.t. ds from t to T^*, we have

$$-\ln[L + \frac{1}{k_2}\int_{t_0}^{\tau} \varphi(s) W^2(s) f'(x(s)) ds]_{T^*}^{t} \geq \ln[f(x(s))]_{T^*}^{t}$$

$$\ln[\frac{L + \frac{1}{k_2}\int_{t_0}^{t} \varphi(s) W^2(s) f'(x(s)) ds}{L + \frac{1}{k_2}\int_{t_0}^{T^*} \varphi(s) W^2(s) f'(x(s)) ds}]^{-1} \geq \ln[\frac{f(x(t))}{f(x(T^*))}]$$

$$\frac{1}{L + \frac{1}{k_2}\int_{t_0}^{t} \varphi(s) W^2(s) f'(x(s)) ds} \geq \frac{f(x(t))}{\theta f(x(T^*))}$$

From (24) and the above inequality, we deduce that

$$-W(t) \geq L + \frac{1}{k_2}\int_{t_0}^{t} \varphi(s) W^2(s) f'(x(s)) ds \geq \frac{1}{L + \frac{1}{k_2}\int_{t_0}^{t} \varphi(s) W^2(s) f'(x(s)) ds} \geq \frac{f(x(t))}{\theta f(x(T^*))}$$

So

$$W(t) \leq -\frac{f(x(t))}{k_2 \theta f(x(T^*))}$$

which implies that

$$x'(t) \leq -\frac{\phi}{k_2 \theta f(x(T^*))}\varphi(t)$$

integrate both sides from T^* to t

$$x(t) \leq x(T^*) - \frac{\phi}{k_2 \theta f(x(T^*))}\int_{T^*}^{t} \varphi(s) ds \quad for \quad t \geq T^*$$

which, in view of (13), leads to contradiction

i.e $\lim_{t\to\infty} x(t) = -\infty$

This complete the proof.

Theorem 2 *Suppose equations* (10) *and* $\varphi(t)$ *in* (11) *hold with* (6) *and* (12) *respectively replace with*

$$xf(x) > 0 \quad and \quad 0 < f'(x(t)) \leq k_3 \quad for \quad x \neq 0 \tag{25}$$

and

$$-\infty < \int_{t_0}^{t} \frac{x'(\tau)}{\varphi(\tau) f(x(\tau))} ds \leq L_1 \tag{26}$$

for constants k_3 and L_1. If there exists a positive continuously differentiable function ρ defined in Theorem 3 above, then equation (4) is oscillatory if

$$\limsup_{t\to\infty} \int_{t_0}^{t}\int_{t_0}^{\tau}[\frac{\rho(u) r(u)}{u^{1-\alpha}} - \frac{\Phi^2(u)\varphi(u)}{4k_2^2 k}] du d\tau = \infty \tag{27}$$

Proof. On the contrary, we assume that equation (4) has a non-oscillatory solution $x(t)$. Without loss of generality, we assume that $x(t) > 0$ for $t \geq t_0 > 0$. Following the proof of Theorem 3, we obtain equation (17) i.e

$$W'(t) \leq -\frac{\rho(t)r(t)}{t^{1-\alpha}} - \Phi(t)\frac{x'(t)}{f(x(t))} - \frac{f'(x(t))W^2(t)}{k_2 t^{1-\alpha} a(t)\rho(t)} \quad for \quad t \geq T$$

$$W'(t) \leq -\frac{\rho(t)r(t)}{t^{1-\alpha}} - \frac{1}{k_2}\Phi(t)\varphi(t)W(t) - \frac{f'(x(t))W^2(t)}{k_2 t^{1-\alpha} a(t)\rho(t)}$$

using (25) in inequality above, it implies that

$$\frac{\rho(t)r(t)}{t^{1-\alpha}} \leq -W'(t) - \frac{1}{k_2}\Phi(t)\varphi(t)W(t) - k\varphi(t)W^2(t) \quad for \quad t \geq T$$

where $k = k_3/k_2$

$$\frac{\rho(t)r(t)}{t^{1-\alpha}} \leq -W'(t) - \varphi(t)[\frac{1}{k_2}\Phi(t)W(t) + kW^2(t)] \tag{28}$$

integrate (28), we have

$$\int_{t_0}^{t}\frac{\rho(u)r(u)}{u^{1-\alpha}}du \leq -\int_{t_0}^{t}W'(u)du - \int_{t_0}^{t}\varphi(u)[\frac{1}{k_2}\Phi(u)W(u) + kW^2(u)]du$$

$$\int_{t_0}^{t}\frac{\rho(u)r(u)}{u^{1-\alpha}}du \leq -W(t) + W(t_0) - \int_{t_0}^{t}\varphi(u)[\frac{1}{k_2}\Phi(u)W(u) + kW^2(u)]du$$

simplifying the above inequality gives

$$\int_{t_0}^{t}[\frac{\rho(u)r(u)}{u^{1-\alpha}} - \frac{\varphi(u)\Phi^2(u)}{4k_2^2 k}]du \leq W(t_0) - \frac{k_2 x'(t)}{\varphi(t)f(x(t))} \tag{29}$$

integrate (29), we have

$$\int_{t_0}^{t}\int_{t_0}^{\tau}[\frac{\rho(u)r(u)}{u^{1-\alpha}} - \frac{\varphi(u)\Phi^2(u)}{4k_2^2 k}]dud\tau \leq W(t_0)\int_{t_0}^{t}d\tau - \int_{t_0}^{t}\frac{k_2 x'(\tau)}{\varphi(\tau)f(x(\tau))}d\tau$$

$$\leq W(t_0)(t-t_0) - k_2 L_1 \tag{30}$$

dividing (30) by $(t-t_0)$ and taking the upper limit as $t \to \infty$, we arrive at

$$\limsup_{t\to\infty}\frac{1}{(t-t_0)}\int_{t_0}^{t}\int_{t_0}^{\tau}[\frac{\rho(u)r(u)}{u^{1-\alpha}} - \frac{\varphi(u)\Phi^2(u)}{4k_2^2 k}]dud\tau \leq \limsup_{t\to\infty}\frac{1}{(t-t_0)}[W(t_0)(t-t_0) - k_2 L_1] < \infty$$

this contradicts (27) which complete the proof.

Example 1. Consider the nonlinear forced fractional differential equation

$$D_t^{\alpha}[(t\exp(t^{1/2}))^{-1}\sigma(x'(t))D_t^{\alpha}x(t)] + 4t^{-2/7}x^2 sgn x(t) = -\frac{t^{-4/5}(2x(t) + D_t^{\alpha}x(t))^2}{1+x^2} \tag{31}$$

From (31), we deduce that

$$\left.\begin{aligned} &a(t) = (t\exp(t^{1/2}))^{-1}, \quad p(t) = 0, \quad f(x(t)) = x^3(t);\ f'(x(t)) = 3x^2 \\ &Q(t,x,D_t^{\alpha}x(t)) = \frac{t^{-4/5}(2x(t) + D_t^{\alpha}x(t))^2}{1+x^2} \\ &g(t) = 4t^{-2/7} \end{aligned}\right\} \tag{32}$$

Let

$$\left.\begin{aligned} &\rho(t) = 1,\ t_0 = 2,\ \alpha = 1/4, \\ &\beta = 5/2, \gamma = 3/2 \end{aligned}\right\} \tag{33}$$

Also,

$$Q(t,x,D_t^{\alpha}x(t)) = -\frac{t^{-4/5}(2x(t) + D_t^{\alpha}x(t))^2}{1+x^2}$$
$$\leq t^{-4/5} = q(t) \tag{34}$$

and

$$r(t) = g(t) - q(t) = 4t^{-2/7} - t^{-4/5} \tag{35}$$

$$xf(x) = x \times x^3 = x^4 > 0 \quad f^2(x) = x^6(t) > 0 \quad and \quad f'(x) = 3x^2 > 0 \quad \forall x \neq 0$$

$$\int^{\infty} \frac{\sqrt{f'(u)}}{f(u)} du = \int \frac{(3x^2)1/2}{x^3} dx = 0.87 < \infty$$

Without loss of generality, equation (6) to (9) hold.

Substitute (32), (33) and (35) into (12), we have

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{\rho(u)r(u)}{u^{1-\alpha}} du = \limsup_{t\to\infty} \int_{2}^{t} (4u^{-29/28} - u^{-31/20}) du = 108.02$$

This shows that equation (12) hold i.e $-\infty < 108.02 < \infty$

Also, we substitute (32) - (35) into the left hand side of equation (14), we have

$$\limsup_{t\to\infty} \frac{1}{(t-2)^4} \int_{2}^{t} [(t-u)^4 (4u^{-29/28} - u^{-31/20})] du$$

$$= \limsup_{t\to\infty} \frac{1}{(t-2)^4} [0.81t^{4.96} + 96.06t^4 - 109.83t^{3.96} + 8.24t^{3.45}$$

$$-27.36t^3 + 11.31t^2 + 33.16t - 21.93]$$

$$= \limsup_{t\to\infty} \frac{1}{(1-2/t)^4} [0.81t^{0.96} + 96.06 - 109.83/t^{0.04} + 8.24/t^{0.55}$$

$$-27.36/t + 11.31/t^2 + 33.16/t^3 - 21.93/t^4] = \infty$$

This shows that (14) holds, hence equation (31) is oscillatory.

Example 2. Consider the nonlinear forced fractional differential equation

$$D_t^{\alpha}[\exp(2t)\sigma(x'(t))D_t^{\alpha}x(t)] + 2t^{3/10}D_t^{\alpha}x(t) + t^3x^2 = -\frac{t^{3/5}(2x(t) + D_t^{\alpha}x(t))^2}{x^2} \tag{36}$$

From (36), we deduce that

$$\left.\begin{aligned} &a(t) = \exp(2t), \quad p(t) = 2t^{3/10}, \quad f(x(t)) = x^2(t); \ f'(x(t)) = 2x(t) \\ &Q(t, x, D_t^{\alpha}x(t)) = -\frac{t^{3/5}(2x(t) + D_t^{\alpha}x(t))^2}{x^2} \\ &g(t) = t^3 \end{aligned}\right\} \tag{37}$$

Let

$$\left.\begin{aligned} &\rho(t) = 1, \ t_0 = 2, \ \alpha = 2/5, \\ &k_2 = 5, k = 3 \end{aligned}\right\} \tag{38}$$

Also,

$$Q(t, x, D_t^{\alpha}x(t)) = -\frac{t^{3/5}(2x(t) + D_t^{\alpha}x(t))^2}{x^2}$$

$$\leq 4t^{1/2} = q(t) \tag{39}$$

and

$$r(t) = g(t) - q(t) = t^3 - 4t^{1/2} \tag{40}$$

Substitute (37) - (40) into (27), we have

$$\limsup_{t\to\infty} \int_{t_0}^{t}\int_{t_0}^{\tau} [\frac{\rho(u)r(u)}{u^{1-\alpha}} - \frac{\Phi^2(u)\varphi(u)}{4k_2^2k}] du d\tau = \limsup_{t\to\infty} \int_{t_0}^{t}\int_{t_0}^{\tau} [u^{12/5} - 4u^{-1/10} - 1/75\exp(-2u)] du d\tau$$

$$= \limsup_{t\to\infty} \int_{t_0}^{t} [0.29\tau^{17/5} - 4.4\tau^{9/10} + 0.01\exp(-2\tau) + 5.14] d\tau$$

$$= \limsup_{t\to\infty} [0.07t^{22/5} - 2.32t^{19/10} - 0.005\exp(-2t) + 5.14t - 3.08] = \infty$$

This shows that (27) holds, hence equation (36) is oscillatory.

Conclusions

In this article, we have established some new oscillation results for a forced superlinear conformable fractional differential equations with dampping term. This extends and also improves on some existing results in the literature [15-17]. Since the new results are derived, we provided two examples to illustrate the relevance of the results obtained.

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