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NUMERICAL SOLUTIONS OF COUPLED VISCOUS BURGERS' EQUATION BY CUBIC SPLINE COLLOCATION METHOD

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Abstract

In this paper we numerically solved the coupled viscous Burgers' equation with appropriate initial and boundary conditions. We use the Cubic B-Spline collocation method on the space integration and applied the Crank-Nicolson method for the temporal variable. This method proved to be unconditionally stable when we applied the Von Neumann stability algorithm. We provided some numerical examples to validate the scheme; our solutions were also compared with existing ones and it proved to possess faster convergence to the exact solutions.

1. Introduction

Burgers' equation is one of the most common nonlinear and time dependent partial differential equation (PDEs) in fluid mechanics.

Burgers' equation normally describes various kinds of phenomena such as mathematical model of turbulence and the approximate theory of flow through a shock wave travelling in a viscous fluid [1]. In literature, there abound numerical methods that have been implemented for approximating solution of Burgers' equation. There are authors who use numerical techniques based on finite difference, [2], finite element [3] and boundary element [4] methods in attempting to solve Burgers equations. In [5], a parameter uniform implicit difference scheme for solving time-dependent Burgers' equation was used.

In this paper, we are concerned with numerical solution of coupled viscous Burgers' equation derived in [1] to study the model of poly dispersive sedimentation. This is a simple model of sedimentation of two kinds of particles in fluid suspension or colloids under the influence of gravity.

The exact solution of coupled Burgers' equation has been obtained by Kaya [6] using Adomian decomposition method and soliton. There are a lot of recent studies that treated the viscous Burgers' equation, viz-a-viz Harmonics differential quadrature finite differences coupled approach [7].

This paper seeks to find numerical solution for the coupled viscous Burgers' equation given by:

$$u_t - u_{xx} + \eta u u_x + \alpha(uv)_x = 0 \qquad x\epsilon[a,b], t\epsilon[0,T] \qquad (1)$$

$$v_t - v_{xx} + \eta v v_x + \beta(uv)_x = 0 \qquad x\epsilon[a,b], t\epsilon[0,T] \qquad (2)$$

With initial condition

$$u(x,0) = \varphi_1(x), \qquad v(x,0) = \varphi_2(x)$$

And boundary conditions

$$u(a,t) = f_1(a,t), \qquad u(b,t) = f_2(b,t)$$

$$v(a,t) = g_1(a,t), \qquad v(b,t) = g_2(b,t)$$

Where η a real constant is while α and β are arbitrary constants depending on the parameters.

The treatment for this equation will involve the use of cubic B-spline collocation method on space integration and finite difference Crank-Nicolson method on the time integration.

The paper is organized as follows: in section 1 we introduce the general Burgers' equation model and its applications; section 2 highlights the basis function for cubic B-spline and their derivatives at the nodal point. Section 3 was devoted to computing approximate solutions in terms of cubic spline function and some time dependent parameter. We test the stability of our scheme using Von Neumann method in section 4; the paper is concluded with two numerical examples in section 5.

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2. Cubic Spline Bases

To construct numerical solution we consider the nodal points (x_i, t_j) defined on the region $[a, b] \times [0, T]$ where
$a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$
$x_{i+1} - x_i = h, \quad 0 < t_0 < t_1 < \cdots < t$
$t_{i+1} - t_i = \Delta t$
The following represents the cubic B-Spline basis functions at their nodal points:

$$B_m(x) = \frac{1}{h^3}\begin{cases} (x - x_{m-2})^3 & x \in [x_{m-2}, x_{m-1}) \\ (x - x_{m-2})^3 - 4(x - x_{m-1})^3 & x \in [x_{m-1}, x_m) \\ (x_{m+2} - x)^3 - 4(x_{m+1} - x)^3 & x \in [x_m, x_{m+1}) \\ (x_{m+2} - x)^3 & x \in [x_{m+1}, x_{m+2}) \\ 0 & otherwise \end{cases} \quad (3)$$

Using this B-Spline basis function, the values of $B_m(x)$ and their derivatives could be obtained from nodal points thus:
At x_{m-2} nodal point we have

$$\frac{1}{h^3}[(x_{m-2} - x_{m-2})^3] = 0 = B_m(x_{m-2})$$

At x_{m-1} nodal point we have

$$B_m(x_{m-1}) = \frac{1}{h^3}[(x_{m-1} - x_{m-2})^3 - 4(x_{m-1} - x_{m-1})^3] = \frac{1}{h^3} \times h^3 - 4(0) = 1$$

At x_m nodal point we have

$$B_m(x_m) = \frac{1}{h^3}[(x_{m+2} - x_m)^3 - 4(x_{m+1} - x_m)^3]$$

$$= \frac{1}{h^3}[[(x_{m+2} - x_{m+1}) - (x_{m+1} - x_m)]^3 - 4(x_{m+1} - x_m)^3]$$

$$\frac{1}{h^3}(2h)^3 - 4h^3 = \frac{1}{h^3} \times 4h^3 = 4$$

at x_{m+1} nodal point we have

$$B_m(x_{m+1}) = \frac{1}{h^3}[(x_{m+2} - x_{m+1})^3] = \frac{1}{h^3} \times h^3 = 1$$

$$B_m(x_{m+2}) = 0$$

1^{st} Derivatives of cubic spline at the nodal point:

$$B'_m(x_{m-2}) = \frac{1}{h^3}[3(x_{m-2} - x_{m-2})^2] = 0$$

$$B'_m(x_{m-1}) = \frac{1}{h^3}[3(x_{m-1} - x_{m-2})^2 - 12(x_{m-1} - x_{m-1})^2]$$

$$= \frac{1}{h^3} \times 3(h^2) = \frac{3}{h}$$

$$B'_m(x_m) = \frac{1}{h^3}[3(x_{m+2} - x_m)^2 - 12(x_{m+1} - x_m)^2]$$

$$= \frac{1}{h^3} \times [3(2h)^2 - 12h^2] = \frac{1}{h^3} \times 0 = 0$$

$$B'_m(x_{m+1}) = \frac{1}{h^3}[3(x_{m+2} - x_{m+1})^2] = \frac{1}{h^3} \times 3h^2 = \frac{3}{h}$$

$$B'_m(x_{m+2}) = 0$$

2^{nd} Derivatives at Nodal Points

$$B''_m(x_{m-2}) = 0$$

$$B''_m(x_{m-1}) = \frac{6}{h^3}[(x_{m-1} - x_{m-2}) - 24(x_{m-1} - x_{m-1})] = \frac{6}{h^2}$$

$$B''_m(x_m) = \frac{1}{h^3}[6(2h) - 24h] = \frac{1}{h^3}(-12) = -\frac{12}{h^2}$$

$$B''_m(x_{m+1}) = \frac{1}{h^3}[6h] = \frac{6}{h^2}$$

$$B''_m(x_{m+2}) = 0.$$

3. The Solution of Coupled Viscous Burgers' Equation

Here we discretize the time derivative of equation (1) and (2) using finite difference scheme of the Crank Nicolson scheme to obtain

$$\frac{u^{n+1}-u^n}{\Delta t}-\frac{[u_{xx}^{n+1}+u_{xx}^n]}{2}+\eta\frac{[(uu_x)^{n+1}+(uu_x)^n]}{2}+\alpha\frac{[((uv)_x)^{n+1}+((uv)_x)^n]}{2}=0 \quad (4)$$

$$\frac{v^{n+1}-v^n}{\Delta t}-\frac{[v_{xx}^{n+1}+v_{xx}^n]}{2}+\eta\frac{[(vv_x)^{n+1}+(vv_x)^n]}{2}+\beta\frac{[((uv)_x)^{n+1}+((uv)_x)^n]}{2}=0 \quad (5)$$

The nonlinearities in (2) and (3) are linearized thus:

$$(uu_x)^{n+1}=u^{n+1}u_x^n+u^nu_x^{n+1}-(uu_x)^n \quad (6)$$

$(uv)_x^{n+1}+(uv)_x^n$ Is computed thus:

$(uv)_x=vu_x+uv_x$

So that

$$[(uv_x)^{n+1}+(uv_x)^n]+[(vu_x)^{n+1}+(vu_x)^n]=(uv)_x^{n+1}+(uv)_x^n \quad (7)$$

From (5) we substitute into (7), the like terms cancel thus:

$$(uv)_x^{n+1}+(uv)_x^n=(u^{n+1}v_x^n)+(u^nv_x^{n+1})-(uv_x)^n+(uv_x)^n+(v^{n+1}u_x^n)$$
$$+(v^nu_x^{n+1})-(vu_x)^n+(vu_x)^n$$
$$=u^{n+1}v_x^n+u^nv_x^{n+1}+v^{n+1}u_x^n+v^nu_x^{n+1}\ . \quad (8)$$

Now, the following equation represents a simplified version of (4) for what follows:

$$\frac{u^{n+1}-u^n}{\Delta t}-\frac{[u_{xx}^{n+1}+u_{xx}^n]}{2}+\eta\frac{[u^{n+1}u_x^n+u^nu_x^{n+1}]}{2}+\alpha\frac{[u^{n+1}v_x^n+u^nv_x^{n+1}+v^{n+1}u_x^n+v^nu_x^{n+1}]}{2}=0$$

$$\frac{v^{n+1}-v^n}{\Delta t}-\frac{[v_{xx}^{n+1}+v_{xx}^n]}{2}+\eta\frac{[v^{n+1}v_x^n+v^nv_x^{n+1}]}{2}+\beta\frac{[u^{n+1}v_x^n+u^nv_x^{n+1}+v^{n+1}u_x^n+v^nu_x^{n+1}]}{2}=0.$$

Using Cubic B-Spline basis function$B_m(x)$ and the time dependent parameters $\delta_m(t)$ and $\sigma_m(t)$for $u(x,t)$ and $v(x,t)$ we obtain approximate solutions thus:

$$v_m(x,t)=\sum_{m=-1}^{N+1}\sigma_m(t)B_m(x),\qquad u_m(x,t)=\sum_{m=-1}^{N+1}\delta_m(t)B_m(x). \quad (9)$$

We use the Cubic B-Spline function $B_m(x)$ to deduce the approximate solution $u(x)$ and $v(x)$ in terms of time parameter of $u(x,t)$ and $v(x,t)$ respectively.

We compute

$$u_m=\sum_{M=m-2}^{m+2}\delta_M(t)B_M(x);$$

by taking into cognizance the property that $B_{m-i}(x)=B(x_{m-1})$ so that

$$u_m=\delta_{m-2}(t)B_m(x_{m-2})+\delta_{m-1}(t)B_m(x_{m-1})+\delta_m(t)B_m(x_m)+\delta_{m+1}(t)B_m(x_{m+1})+\delta_{m+2}(t)B_m(x_{m+2})$$
$$=\delta_{m-1}+4\delta_m+\delta_{m+1}$$

$$u_m'=\sum_{M=m-2}^{m+2}\delta_M(t)B_M'(x)$$
$$=\delta_{m-2}(t)B_m'(x_{m-2})+\delta_{m-1}(t)B_m'(x_{m-1})+\delta_m(t)B_m'(x_m)+\delta_{m+1}(t)B_m'(x_{m+1})$$
$$+\delta_{m+2}(t)B_m'(x_{m+2})=0+\frac{3}{h}\delta_{m-1}+0+\frac{3}{h}\delta_{m+1}(t)+0$$
$$=\frac{3}{h}[\delta_{m-1}(t)+\delta_{m+1}(t)]$$

$$u_m''=\sum_{m=m-2}^{m+2}\delta_m(t)B_m''(x)$$
$$=\delta_{m-2}(t)B_m''(x_{m-2})+\delta_{m-1}(t)B_m''(x_{m-1})+\delta_m(t)B_m''(x_m)+\delta_{m+1}(t)B_m''(x_{m+1})$$
$$+\delta_{m+2}(t)B_m''(x_{m+2})$$
$$=0+\frac{6}{h^2}\delta_{m-1}(t)-\frac{12}{h^2}\delta_m(t)+\frac{6}{h^2}\delta_{m+1}=\frac{6}{h^2}(\delta_{m-1}-2\delta_m+\delta_{m+1}).$$

Similarly,

$$v_m=\sigma_{m-1}(t)+4\sigma_m(t)+\sigma_{m+1}(t)$$
$$v_m'=\frac{3}{h}(\sigma_{m+1}-\sigma_{m-1})$$

$$v_m'' = \frac{6}{h^2}(\sigma_{m-1} - 2\sigma_m + \sigma_{m+1})$$

Next, we plug in the approximate solutions and their derivatives into (4) and (5) thus:

$$\Rightarrow \delta_{m-1} + 4\delta_m + \delta_{m+1} - u^n - \frac{\Delta t}{2}\left[\frac{6}{h^2}(\delta_{m-1} - 2\delta_m + \delta_{m+1}) + u_{xx}^n)\right]$$
$$+ \frac{\eta\Delta t}{2}\left[(\delta_{m-1} - 4\delta_m + \delta_{m+1})u_x^n + \frac{3}{h}(\delta_{m+1} - \delta_{m-1})u^n\right]$$
$$+ \frac{\alpha\Delta t}{2}\left[(\delta_{m-1} + 4\delta_m + \delta_{m+1})v_x^n + \frac{3}{h}(\sigma_{m+1} - \sigma_{m-1})u^n + (\sigma_{m-1} + 4\sigma_m + \sigma_{m+1})u_x^n\right.$$
$$\left. + \frac{3}{h}(\delta_{m-1} + \delta_{m+1})v^n\right] = 0$$

$u^n + \frac{\Delta t}{2}u_{xx}^n = (\delta_{m-1} + 4\delta_m + \delta_{m+1})\left(1 + \frac{\eta\Delta t}{2}u_x^n + \frac{\alpha\Delta t}{2}v_x^n\right) - \frac{3\Delta t}{h^2}(\delta_{m-1} - 2\delta_m + \delta_{m+1}) + \left(\frac{3\alpha\Delta t}{2h}v^n + \frac{3\eta\Delta t}{2h}u^n\right)(\delta_{m+1} - \delta_{m-1}) + \frac{3\alpha\Delta t}{2h}(\sigma_{m+1} - \sigma_{m-1})u^n + \frac{\alpha\Delta t}{2}u_x^n(\sigma_{m-1} + 4\sigma_m + \sigma_{m+1})$.

Let $a_1 = 1 + \frac{\Delta t}{2}(\eta u_x^n + \alpha v_x^n)$, $a_2 = -\frac{3\Delta t}{h^2}$, $a_3 = \frac{3\Delta t}{2h}(\alpha v^n + \eta u^n)$, $a_4 = \frac{3\Delta t}{2h}(\alpha u^n)$, $a_5 = \frac{\Delta t}{2}(\alpha u_x^n)$

Similarly, from (2) we substitute the approximate solution $v_m = \sum_{M=m-2}^{m+2}\sigma_M(t)B_M$ of $v(t,x)$ thus:

$$(\sigma_{m-1} + 4\sigma_m + \sigma_m) - v^n$$
$$= \frac{\Delta t}{2}\left[\frac{6}{h^2}(\sigma_{m-1} - 2\sigma_m + \sigma_{m+1}) - v_{xx}^n\right]$$
$$+ \frac{\eta\Delta t}{2}\left[\left[(\sigma_{m-1} - 2\sigma_m + \sigma_{m+1})v_x^n + \frac{3}{h}(\sigma_{m+1} - \sigma_{m-1})v^n\right]\right.$$
$$+ \frac{\beta\Delta t}{2}\left[(\delta_{m-1} + 4\delta_m + \delta_{m+1})v_x^n + \frac{3}{h}(\sigma_{m+1} - \sigma_{m-1})u^n + (\sigma_{m-1} + 4\sigma_m + \sigma_{m+1})u_x^n\right.$$
$$\left.\left. + \frac{3}{h}(\delta_{m+1} - \delta_{m-1})v^n\right]\right] = 0$$

$$v^n + \frac{\Delta t}{2}v_{xx}^n = \left(1 + \frac{\Delta t}{2}(\eta v_x^n + \beta u_x^n)\right)(\sigma_{m-1} + 4\sigma_m + \sigma_{m+1}) - \frac{3\Delta t}{h^2}(\sigma_{m-1} - 2\sigma_m + \sigma_{m+1})$$
$$+ \frac{3\Delta t}{2h}(\sigma_{m+1} - \sigma_{m-1})(\eta v^n + \beta u^n) + \frac{3\Delta t}{2h}(\beta v^n)(\delta_{m+1} - \delta_{m-1}) + \frac{\Delta t}{2}(\beta v_x^n)(\delta_{m-1} + 4\delta_m + \delta_{m+1})$$

Here,

$$a_6 = \left(1 + \frac{\Delta t}{2}(\eta v_x^n + \beta u_x^n)\right)$$
$$a_7 = -\frac{3\Delta t}{h^2}$$
$$a_8 = \frac{3\Delta t}{2h}(\eta v^n + \beta u^n)$$
$$a_9 = \frac{3\Delta t}{2h}(\beta v^n)$$
$$a_{10} = \frac{\Delta t}{2}(\beta v_x^n)$$

We write the difference equations in the following way:

$$a_1(\delta_{m-1} + 4\delta_m + \delta_{m+1}) + a_2(\delta_{m-1} - 2\delta_m + \delta_{m+1}) + a_3(\delta_{m+1} - \delta_{m-1}) + a_4(\sigma_{m+1} - \sigma_{m-1})$$
$$+ a_5(\sigma_{m-1} + 4\sigma_m + \sigma_{m+1}) = u^n + \frac{\Delta t}{2}u_{xx}^n \qquad (10)$$
$$a_6(\sigma_{m-1} + 4\sigma_m + \sigma_{m+1}) + a_7(\sigma_{m-1} - 2\sigma_m + \sigma_{m+1}) + a_8(\sigma_{m+1} - \sigma_{m-1}) + a_9(\delta_{m+1} - \delta_{m-1})$$
$$+ a_{10}(\sigma_{m-1} + 4\sigma_m + \sigma_{m+1}) = v^n + \frac{\Delta t}{2}v_{xx}^n \qquad (11)$$

$m = 0, \dots, N$

It is noteworthy that system equation (10) and (9) have 2(N+3) unknown i.e. $\delta_{-1}, \delta_0, \delta_1, \dots, \delta_{N+1}$ and $\sigma_{-1}, \sigma_0, \sigma_1, \dots, \sigma_{N+1}$ and $(N+1) \times 2$ equations which constitutes an over determined system of $2(N+3) \times 2(N+1)$; to make the system solvable, we reduce the unknowns by imposing boundary conditions such that we could deal with square matrix system dimension. Here, we eliminate boundary terms$\delta_{-1}, \delta_{N+1}$ and $\sigma_{-1}, \sigma_{N+1}$. This leaves us with the require system$(2N+2) \times (2N+2)$ matrix which is a bi-triagonal that could be solved by modified Thomas Algorithm.

4. Stability of the scheme

We use Von Neumann stability algorithm to check the stability of the scheme. First, we linearize the nonlinear terms vv_x *and* $(uv)_x$ by considering both$u$ *and* v as local constants μ_1 *and* $\mu_2$respectively. We substitute into (4) and (5) and obtain the following equation with the variable δ_m:

$$\mu_1 = (\delta_{m-1} + 4\delta_m + \delta_{m+1}) - \frac{3\Delta t}{h^2}(\delta_{m-1} - 2\delta_m + \delta_{m+1}) + \left(\frac{3\alpha\Delta t}{2h}\mu_2 + \frac{3\eta\Delta t}{2h}\mu_1\right)(\delta_{m-1} + \delta_{m+1}) + \frac{3\alpha\Delta t}{2h}(\sigma_{m-1} + \sigma_{m+1})\mu_1 \quad (12)$$

And

$$\mu_2 = (\sigma_{m-1} + 4\sigma_m + \sigma_{m+1}) - \frac{3\Delta t}{h^2}(\sigma_{m-1} - 2\sigma_m + \sigma_{m+1}) + \frac{3\Delta t}{2h}(\sigma_{m+1} - \sigma_{m-1})(\eta\mu_2 + \beta\mu_1) + \frac{3\Delta t}{2h}(\beta\mu_2)(\delta_{m-1} + \delta_{m+1}) \quad (13)$$

From (12) we obtain:

$$(1-\frac{3\Delta t}{h^2} + \frac{3\Delta t}{2h}(\alpha\mu_2 + \eta\mu_1)).\delta_{m-1}^{n+1} + (4+\frac{6\Delta t}{h^2}).\ \delta_m^{n+1} + (1-\frac{3\Delta t}{h^2} + \frac{3\Delta t}{2h}(\alpha\mu_2 + \eta\mu_1)).\delta_{m+1}^{n+1} + \frac{3\Delta t}{2h}(\alpha\mu_2).(\sigma_{m+1}^{n+1} - \sigma_{m-1}^{n+1}) = (1+\frac{3\Delta t}{h^2} - \frac{3\Delta t}{2h}(\alpha\mu_2 + \eta\mu_1)).\delta_{m-1}^{n} + (4-\frac{6\Delta t}{h^2}).\ \delta_m^{n} + (1+\frac{3\Delta t}{h^2} - \frac{3\Delta t}{2h}(\alpha\mu_2 + \eta\mu_1)).\delta_{m+1}^{n} - \frac{3\Delta t}{2h}(\alpha\mu_2).(\sigma_{m+1}^{n} - \sigma_{m-1}^{n}) \quad (14)$$

The equation (14) summarizes and translates to:

$$w_1\delta_{m-1}^{n+1} + w_2\delta_m^{n+1} + w_3\delta_{m+1}^{n+1} + (\sigma_{m+1}^{n+1} - \sigma_{m-1}^{n+1})w = w_4\delta_{m-1}^{n} + w_5\delta_m^{n} + w_6\delta_{m+1}^{n} + (\sigma_{m+1}^{n} - \sigma_{m-1}^{n})w \quad (15)$$

Where $w, w_1, \dots, w_6$ represent the coefficients of $(\sigma_{m+1} - \sigma_{m+1}), \delta_{m-1}, \delta_m, \delta_{m+1}$ in (14).

We bother not ourselves to summarize equation in v in form of (14) because of symmetry; so it is enough to show stability of the entire scheme via (14).

Now, by Von Neumann stability scheme, we aver that the solution of the discrete scheme (10)-(11) approximates the exact solution $u(x,t)$ of the Coupled Viscuous Burgers' equation (1)-(2). The round off error $\epsilon_{i,j}^n$ due to approximation and defined by:

$\epsilon_{i,j}^n = |u^{exact} - u^{num}|$.

Since the exact solution must satisfy the discretized equation, the error too must satisfy the discretized equation. Here we assume that the numerical solution too must also satisfy the discretized equation, but we admit that this is only possible in machine precision. We may now reformulate (10)-(11) in terms of their error terms, i.e. replacing $u_{i,j}^n$ with $\epsilon_{i,j}^n$. Obviously, the error and numerical solution have the same growth or decay rate with respect to time. For linear differential equations with periodic boundary condition, the spatial variation of error may be expanded in terms of their Fourier series thus:

$$\in(x) = \sum_{m=1}^{M} A_m e^{ik_m x};$$

[8] gives more exposition.

We refer to equation (15) and let $\delta_m^n = A\xi^n e^{imh\varphi}$ and $\sigma_m^n = B\xi^n e^{imh\varphi}$; A and B are the Harmonic amplitudes, φ *is the mode number* and h is the element size, $i = \sqrt{-1}$. Upon doing the above substitution, we get:

$|X_2 + iY|\xi^{n+1} = |X_1 - iY|\xi^n$,

so, $H = \frac{|X_1 - iY|\xi^n}{|X_2 + iY|\xi^{n+1}}$ (16)

where

$X_2 = 1 - 4K_2 sin^2\left({}^{\varphi h}/_2\right) + 2R_2 cos^2\left({}^{\varphi h}/_2\right)$,

$X_1 = 1 + 4K_2 sin^2\left({}^{\varphi h}/_2\right) + 2R_2 cos^2\left({}^{\varphi h}/_2\right)$, Y=$6dK_1\sin(\varphi h)$.

$K_1 = \frac{3}{2h}\Delta t$, $K_2 = \frac{3}{2h^2}\nu\Delta t$, d is a local constant.

We find that |H|<1, implying unconditional stability of our scheme by the Von Neumann stability theorem.

Next, we may now return to our scheme to compute the numerical solutions of the coupled viscous Burger' equation.

5. Numerical Experiment, Result and Simulation

We perform numerical experiments in order to gain insight into the performance of the current scheme. Here we will provide L_∞ *and* L_2 *errors*; this is obtained through the following formula:

$$L_\infty = \max_m\{|u_m - U_m|\},\ L_2 = \sqrt{\sum_{m=0}^{N}|u_m - U_m|^2} \Big/ \sqrt{\sum_{m=0}^{N}|u_m|^2}$$

where, u_m *is the exact solution and* U_m *is the numerical solution.*

This is standard and follows by the definition of L_∞ and L_2 $norms$.
For our experiment, we consider a coupled viscous Burgers' equation (1) and (2) with $\alpha = \beta = 1$ and $\eta = -2$; this reduce equations (1) and (2) to the following:
$u_t - u_{xx} - 2uu_x + (uv)_x = 0,\ v_t - v_{xx} - 2vv_x + (uv)_x = 0$ with the initial condition given by:$u(x,0) = v(x,0) = \sin(x)$, $boundary\ condition\ is\ sourced\ from\ the\ exact\ solution\ u(x,t)$.
We adopt the exact solution of coupled viscous Burgers' equation of [6] given by:
$u(x,t) = v(x,t) = e^{-t}\sin(x)$. The numerical solutions for this has been obtained by considering the domain $x \in [-\pi,\pi]$ with $\Delta t = \frac{1}{1000}$. The solution is as given in the table below with their number of partitions at different time steps. Because of symmetry in the initial and boundary conditions, results are presented only for u(x,t). The order of convergence is calculated through the formula:
$R = \frac{\log(Error(N_1)/Error(N_2))}{\log(N_2/N_1)}$.

$\boldsymbol{Table\ 1}$: $\boldsymbol{L_\infty\ and\ L_2\ errors\ for\ different\ time\ steps\ of\ the\ solution\ u(x,t)}$

T	N=200			N=400		Rashid (2009) for N=200 at t=1		
	L_2	L_∞		L_2	L_∞	$L_2 L_\infty$		
.1	8.2×10^{-6}	7.5×10^{-6}		2.1×10^{-6}	1.9×10^{-6}		Nil	Nil
.5	2.5×10^{-5}	4.1×10^{-5}		1.0×10^{-5}	6.2×10^{-6}		Nil	Nil
1	3.0×10^{-5}	8.2×10^{-5}		2.0×10^{-5}	7.6×10^{-6}		2.9×10^{-5}	1.2×10^{-5}

$\boldsymbol{Table2}$: $\boldsymbol{Order\ of\ Convergence\ of\ the\ Numerical\ Solutions\ to\ the\ Exact\ Solution\ u(x,t)}$

t=0.1				t=0.5		
N	L_∞	Ratio	Order of convergence	L_∞	Ratio	Order of Convergence
32	2.9×10^{-4}	Nil	Nil	9.748×10^{-4}	4.001	2.005
64	7.3×10^{-5}	4.000	2.001	2.436×10^{-4}	4.000	2.001
128	1.8×10^{-5}	3.999	1.999	6.090×10^{-5}	4.000	2.001
256	4.5×10^{-5}	3.995	1.998	1.522×10^{-5}	4.000	2.001
512	1.1×10^{-6}	3.981	1.993	1.805×10^{-5}	4.001	2.002

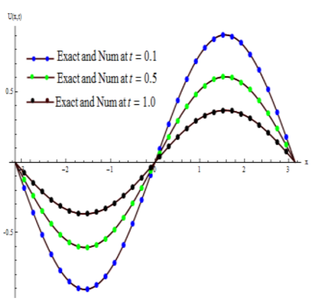

Fig. 1: Comparison between Numerical and Analytic Results

6. Conclusion

It is observed in this paper that due to time truncation error of the derivative term, the accuracy of the solutions reduces as time increases. However, the advantage of the collocation method used in this paper is that the method works well for large class of linear and nonlinear PDEs. We have presented our solutions graphically at different time steps and make some comparisons with the exact solution. The L_∞ and L_2 $norms$ of the error in numerical computations has been done and the order of convergence of the solution u(t,x) found.

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