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POST- EINSTEIN PRECESSION EQUATION IN THE GRAVITATIONAL FIELD OF A SPHERICAL STAR

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Abstract

An extended Riemannian metric tensor in a spherical gravitational field is used to deduce planetary equations of motion. The obtained planetary equations of motion are solved and the precession equation in the gravitational field of a spherical star is obtained. The precession equation reduces to the corresponding pure Newtonian equation in the order of c^0. It contains post Newtonian and Einstein terms in the order of c^{-2}. The consequences of these results are that it can be applied to study the steady change in the orientation of the axis of a rotation of the earth.

***Keywords*:** Riemannian, Metric Tensor, Planetary, Gravitational Field, Precession.

1.0 Introduction

At the end of the 19th century it was observed that Newton's dynamical theory of gravitation (NDTG) could not explain the anomalous orbital precession of the orbit of the planet as well as the gravitational red shift by the sun [1- 2].

In 1905, Einstein published his geometrical theory of gravitation (EGTG). This theory successfully explains the anomalous orbital precession of the orbit of the planet as well as the gravitational red shift by the sun [1- 2].

The first exact solution of Einstein's geometrical field equation was constructed in 1916 by Karl Schwarchild. It is the metric due to spherically symmetric body. The well known Schwarchild's metric tensors in gravitational field of static homogeneous spherical massive body of mass M situated in empty space is given explicitly by [1- 6]

$$g_{00} = 1 - \frac{GM}{c^2 r} \tag{1}$$

$$g_{11} = -\left(1 - \frac{GM}{c^2 r}\right)^{-1} \tag{2}$$

$$g_{22} = -r^2 \tag{3}$$

$$g_{33} = -r^2 \sin^2 \theta \tag{4}$$

$$g_{\mu\nu} = 0;\ \text{otherwise} \tag{5}$$

where, $r > R$ is the radius of the static spherical mass, G is the universal gravitational constant and c is the speed of light in vacuum and $f(r)$is an arbitrary function determined by the distribution.

Despite the famous test of Einstein's theory of General Relativity (GR), all the authorities in Relativity and Physics in general have continued to raise objections against the mathematical difficulty of Einstein's theory of Relativity [1- 4].

In 2009, an entirely new approach to the search for the metric tensor of space time with the influence and interaction of gravitational fields was introduced [1, 4, 9]. In this approach, the metric tensor is a fundamental quantity of nature and can be obtained through extensions of the Euclidean metric tensor.

In this paper, we introduce a new approach to derive the precession equation in the gravitational field of spherical polar coordinates using an extended Riemannian metric tensor.

2.0 Theoretical Analysis

An extended Riemannian metric tensor in the gravitational field exterior to a spherical mass is given by [9]

$$g_{00}(r,\theta,\phi,x^0) = -\left\{1 + \frac{2}{c^2} f(r,\theta,\phi,x^0)\right\} \tag{6}$$

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$g_{11}(r,\theta,\phi,x^0) = \left\{1 - \frac{2}{c^2} f(r,\theta,\phi,x^0)\right\}^{-1}$ (7)
$g_{22}(r,\theta,\phi,x^0) = r^2$ (8)
$g_{33}(r,\theta,\phi,x^0) = r^2 sin^2\theta$ (9)
$g_{\mu\nu} = 0$; otherwise (10)
where, f is an extended Riemannian gravitational scalar potential exterior to the spherical body and (ct, r, θ, ϕ) is the space- time coordinate with $(x^0 = ct)$.
The extended Riemannian gravitational scalar potential exterior and interior to spherical astrophysical bodies has been obtained. The extended Riemannian gravitational scalar potential exterior to a spherical astrophysical body (f) is shown to be given explicitly as [8]

$$f = \frac{k}{r}\left\{1 - \frac{k}{c^2 R}\right\} - \frac{k^2}{c^2 r^2} + \cdots \quad (11)$$

where $k = GM$, G is the universal gravitational constant, M is the mass of the planets, c is the speed of light R is the radius of the spherical astrophysical body and $r > R$ for the exterior field.
The general expression for the line element in the gravitational field of spherical massive body is given explicitly from the extended metric tensor by [1- 3]

$$c^2 d\tau^2 = -c^2\left\{1 - \frac{2}{c^2} f\left(r,\theta,\phi,x^{/0}\right)\right\} dt^2$$
$$-\left\{1 - \frac{2}{c^2} f(r,\theta,\phi,x^0)\right\}^{-1} dr^2 - r^2 d\phi^2 - r^2 sin^2\theta d\phi^2 \quad (12)$$

Substituting equation (6) into (7) we obtain

$$\mathrm{c}^2 \mathrm{d}\tau^2 = \mathrm{c}^2\left\{1 - \frac{2}{c^2}\left[\frac{k}{r}\left\{1 - \frac{k}{c^2 R}\right\} - \frac{k^2}{c^2 r^2}\right]\right\} dt^2$$
$$-\left\{1 - \frac{2}{c^2}\left[\frac{k}{r}\left\{1 - \frac{k}{c^2 R}\right\} - \frac{k^2}{c^2 r^2}\right]\right\}^{-1} dr^2 - r^2 d\phi^2 - r^2 sin^2\theta d\phi^2 \quad (13)$$

Now, consider the motion of a particle whose motion is confined to the equatorial plane of the spherical astrophysical body (e.g. Sun, planet or comet), then from geometry considerations in spherical polar coordinates,

$$\theta = \frac{\pi}{2}$$

Then, equation (13) becomes

$$c^2 d\tau^2 = c^2\left\{1 - \frac{2}{c^2}\left[\frac{k}{r}\left\{1 - \frac{k}{c^2 R}\right\} - \frac{k^2}{c^2 r^2}\right]\right\} dt^2$$
$$-\left\{1 - \frac{2}{c^2}\left[\frac{k}{r}\left\{1 - \frac{k}{c^2 R}\right\} - \frac{k^2}{c^2 r^2}\right]\right\}^{-1} dr^2 - r^2 d\phi^2 \quad (14)$$

Dividing all through equation (14) by $d\tau^2$ and designating proper time differentiation by dot, reduces equation (14) to

$$c^2 = c^2\left\{1 - \frac{2}{c^2}\left[\frac{k}{r}\left\{1 - \frac{k}{c^2 R}\right\} - \frac{k^2}{c^2 r^2}\right]\right\} \dot{t}^2$$
$$-\left\{1 - \frac{2}{c^2}\left[\frac{k}{r}\left\{1 - \frac{k}{c^2 R}\right\} - \frac{k^2}{c^2 r^2}\right]\right\}^{-1} \dot{r}^2 - r^2\dot{\phi}^2 \qquad = 0 \quad (15)$$

Considering a clock at rest in this gravitational field, it can be shown that

$$\dot{t} = \left\{1 - \frac{2}{c^2}\left[\frac{k}{r}\left\{1 - \frac{k}{c^2 R}\right\} - \frac{k^2}{c^2 r^2}\right]\right\}^{-1} \quad (16)$$

Also, the pure azimuthal speed can be deduced as

$$\dot{\phi} = \frac{l}{r^2} \quad (17)$$

where, l is a constant of motion, which physically corresponds to the angular momentum per unit mass.
Substituting equations (16) and (17) into equation (15) and simplifying we obtain

$$\dot{r}^2 = \frac{2k}{r} - \frac{l^2}{r^2} + \frac{2l^2 k}{c^2 r^3} - \frac{2k^2}{c^2 Rr} - \frac{k^2}{r^2} - \frac{l^2 k^2}{c^2 r^4} - \frac{2l^2 k^2}{c^4 r^3 R} \quad (18)$$

Assuming that powers of c^{-4} and above are negligible, equation (18) reduces to

$$\dot{r}^2 = \frac{2k}{r} - \frac{l^2}{r^2} + \frac{2l^2 k}{c^2 r^3} - \frac{2k^2}{c^2 Rr} - \frac{k^2}{r^2} - \frac{l^2 k^2}{c^2 r^4} \quad (19)$$

Differentiating equation (19) with respect to proper time yields

$$\ddot{r} = -\frac{2k}{r^2} + \frac{2l^2}{r^3} - \frac{6l^2 k}{c^2 r^4} + \frac{2k^2}{c^2 Rr^2} + \frac{2k^2}{r^3} + \frac{4l^2 k^2}{c^2 r^5} \quad (20)$$

Equations (19) and (20) are respectively the radial speed and radial acceleration of a particle in the equatorial plane. In the order of c^0 these results reduce to the corresponding pure Newtonian calculations [1 - 3]

$$\dot{r}^2 = \frac{2k}{r} - \frac{l^2}{r^2}, \quad \ddot{r} = -\frac{2k}{r} + \frac{l^2}{r^3} + \frac{k^2}{r^3}$$

Let us transform the motion in terms of ϕ and let u be a new coordinate defined by

$$u(\phi) = \frac{1}{r(\phi)}.$$

Then

$$\dot{r} = -l\frac{du}{d\phi} \text{ and } \ddot{r} = -l^2 u^2 \frac{d^2 u}{d^2\phi}$$

Thus equation (20) can be written as

$$\frac{d^2u}{d^2\phi}+2\left(1+\frac{k^2}{c^2}\right)u=\frac{2k}{l^2}+\frac{6k}{c^2}u^2+\frac{2k^2}{c^2l^2R}+\frac{4k^2}{c^2}u^3 \qquad (21)$$

Equation (21) is an extended Riemannian planetary equation of motion for a spherical astrophysical body and it contains post Newtonian and post Einstein terms.

We shall seek the analytical solution of equation (21) using Taylor series expansion and as a first approximation, the term with u^3 will be neglected.

Suppose

$$u(\phi)=\sum_{n=1}^{\infty}A_n exp\{ni(\omega\phi+\alpha)\} \qquad (22)$$

where, A_n, ω and α are arbitrary constants.

Equation (22) can be expanded to give

$$u(\phi)=\sum_{n=1}^{\infty}A_n exp\{ni(\omega\phi+\alpha)\}=A_0+A_1 exp\{i(\omega\phi+\alpha)\}$$
$$+A_2 exp\{2i(\omega\phi+\alpha)\}+A_3 exp\{3i(\omega\phi+\alpha)\}+\cdots \qquad (23)$$

Differentiating equation (23) and substituting in equation (16) gives

$$-\omega^2A_1exp\{i(\omega\phi+\alpha)\}-4\omega^2A_1exp\{2i(\omega\phi+\alpha)\}+9\omega^2A_1exp\{3i(\omega\phi+\alpha)\}$$
$$+2\left(1+\frac{k^2}{c^2}\right)[A_0+A_1exp\{i(\omega\phi+\alpha)\}+A_2exp\{2i(\omega\phi+\alpha)\}+\cdots]=$$
$$\frac{2k}{l^2}-\frac{2k^2}{c^2l^2R}+\frac{6k}{c^2}[A_0^2+A_0A_1exp\{i(\omega\phi+\alpha)\}+2A_0A_2exp\{2i(\omega\phi+\alpha)\}+\cdots] \qquad (24)$$

Equating constant terms of equation (24), we get

$$\frac{3k}{c^2}A_0^2-\left(1+\frac{k^2}{c^2}\right)A_0+\frac{2k}{l^2}-\frac{2k^2}{c^2l^2R}=0 \qquad (25)$$

Equation (25) is quadratic in A_0 and thus

$$A_0=\frac{c^2}{6k}\left(1+\frac{k^2}{l^2}\right)\pm\frac{c^2}{6k}\left(1+\frac{k^2}{l^2}\right)\left[1-\frac{12k}{c^2l^2}\left(\frac{2k}{l^2}-\frac{2k^2}{c^2l^2R}\right)\left(1+\frac{k^2}{l^2}\right)^{-2}\right]^{\frac{1}{2}}$$

or approximately

$$A_0=\frac{c^2}{6k}\left(1+\frac{k^2}{l^2}\right)\pm\frac{c^2}{6k}\left(1+\frac{k^2}{l^2}\right)\left[1-\frac{12k}{c^2l^2}\left(\frac{2k}{l^2}-\frac{2k^2}{c^2l^2R}\right)\left(1-\frac{2k^2}{l^2}\right)\right]^{\frac{1}{2}} \qquad (26)$$

Equating the coefficients of first order exponential terms in equation (19) gives

$$\omega^2=\left(1+\frac{K^2}{l^2}\right)-\frac{6k}{c^2}A_0 \qquad (27)$$

Substituting equation (26) into (27) and simplifying yields

$$\omega^2=\pm\left(1+\frac{k^2}{l^2}\right)\left[1-\frac{12k}{c^2}\left(\frac{2k}{l^2}-\frac{2k^2}{c^2l^2R}\right)\left(1-\frac{2k^2}{l^2}\right)\right]^{\frac{1}{2}}$$

Taking the square root of both sides

$$\omega=\left\{\pm\left(1+\frac{k^2}{l^2}\right)\left[1-\frac{12k}{c^2}\left(\frac{2k}{l^2}-\frac{2k^2}{c^2l^2R}\right)\left(1-\frac{2k^2}{l^2}\right)\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}} \qquad (28)$$

We have two roots of ω which are given explicitly as

$$\omega_\alpha=\left\{\left(1+\frac{k^2}{l^2}\right)\left[1-\frac{12}{c^2}\left(\frac{2k}{l^2}-\frac{2k^2}{c^2l^2R}\right)\left(1-\frac{2k^2}{l^2}\right)\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}} \qquad (29)$$

and

$$\omega_\beta=\left\{-\left(1+\frac{k^2}{l^2}\right)\left[1-\frac{12}{c^2}\left(\frac{2k}{l^2}-\frac{2k^2}{c^2l^2R}\right)\left(1-\frac{2k^2}{l^2}\right)\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}} \qquad (30)$$

Equation (30) is mathematical sound but physically of no significance as it yields a complex solution. Hence, we consider equation (29) as our physical expression for angular velocity.

Substituting equation (29) into (23) and simplifying, we get

$$u(\phi)=\frac{c^2}{6k}\left(1+\frac{k^2}{l^2}\right)\pm\frac{c^2}{6k}\left(1+\frac{k^2}{l^2}\right)\left[1-\frac{12k}{c^2}\left(\frac{2k}{l^2}-\frac{2k^2}{c^2l^2R}\right)\left(1-\frac{2k^2}{l^2}\right)\right]^{\frac{1}{2}}$$
$$+\ A_1exp\{i(\omega_\alpha\phi+\alpha)\}+\cdots \qquad (31)$$

The perihelion displacement angle Δ is known to be given by

$$\Delta=2\pi(\omega_\alpha^{-1}-1) \qquad (32)$$

Substituting equation (29) into (32) and simplifying we get

$$\Delta = 2\pi\left\{\left(1+\frac{k^2}{l^2}\right)^{-\frac{1}{2}}\left[1-\frac{12k}{c^2}\left(\frac{2k}{l^2}-\frac{2k^2}{c^2l^2R}\right)\left(1-\frac{2k^2}{l^2}\right)\right]^{-\frac{1}{4}}\right\}-2\pi \qquad (33)$$

Expanding terms with fractional indices and neglecting higher terms reduces equation (33) to

$$\Delta = 2\pi\left\{\left(1-\frac{k^2}{2l^2}\right)\left[1+\frac{3k}{c^2}\left(\frac{2k}{l^2}-\frac{2k^2}{c^2l^2R}\right)\left(1-\frac{2k^2}{l^2}\right)\right]\right\}-2\pi \qquad (34)$$

Simplifying equation (34) and neglecting powers of c^{-4}, we obtain

$$\Delta = \frac{6\pi k^2}{c^2l^2}-\frac{5\pi k^2}{l^2}-\frac{15\pi k^4}{c^2l^4}+\frac{4\pi k^4}{l^4}+\frac{12\pi k^6}{c^2l^4} \qquad (35)$$

Equation (30) is the precession equation in the gravitational field of a spherical star.

CONCLUSION

We have in this paper shown how to derive the planetary equation of motion and precession equation in the gravitational field of a spherical star using an extended Riemannian metric tensor. The planetary equation of motion and precession equation in the gravitational field of a spherical star is found to be equations (21) and (35) respectively. It is interesting and instructive to note that these equations reduces to the corresponding pure Newtonian equations in the order of c^0 and to the order of c^{-2} it contains additional correction terms which are not found in Einstein equation and [10]. These results point to the fact that the extended Riemannian metric tensor introduced in this research can be used to effectively obtain the precession equation with post-Einstein correction terms. These additional correctional terms are open up for theoretical development and experimental investigations and applications. This study can be applied to derive the gravitational spectral shift by the Sun and extend Robert Walker's metric tensor.

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