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ENERGY EIGENVALUES AND EIGENFUNCTIONS OF THE ECKART POTENTIAL SOLVED BY EXACT QUANTIZATION RULES

Eyube E.S.,[1] Rawen B.O.[2] and Hussaini M.[3]

[1]Department of Physics, School of Physical Sciences, Modibbo Adama University of Technology, Yola, Nigeria
[2]Directorate of Basic and Remedial Studies, Abubakar Tafawa Balewa University (ATBU), Bauchi, Nigeria
[3]Department of Physics, School of Science, Umar Suleiman College of Education, Gashua, Nigeria

Abstract

We have solved the radial Schrödinger equation with the Eckart potential using the techniques of exact quantization rule and obtained closed form expressions for the energy eigenvalues and normalized radial eigenfunctions, the results we obtained are in perfect agreement with the energy eigenvalues of the Eckart potential solved by other methods in the literature, as a special case, we have also deduced the energy eigenvalues of the Hulthén potential from our formula which also turns out to be in total agreement with results in the literature.

Keywords: Eigenvalues, Eigenfunctions, Exact quantization, Eckart potential, quantum correction, Schrödinger equation

1.0 Introduction

The exact solutions of wave equation, both relativistic and nonrelativistic occupy prominent position in the field of quantum mechanics because from such solutions, vital information regarding the system being studied can be retrieved [1-3]. The potential energy function used to solve the Schrödinger equation dictates the solution type, exact or inexact solution [4]. The harmonic oscillator and coulomb potentials are known to give exact solutions for all quantum states, $n\ell$, where n is the principal quantum number and ℓ is the angular momentum quantum number [5], few other potential functions give exact solutions for the special case of s-wave ($\ell = 0$)[6]. Most of the other known potentials have no exact solutions with the Schrödinger equation for any quantum state, customarily, analytic solutions with these class of potentials are obtained by applying approximation scheme [7-8] on the spin-orbit centrifugal term of the Schrödinger equation. Researchers have employed various solution methods to solve the Schrödinger equation, some of the solution techniques include: Nikiforov-Uvarov [9] method, Romanovski polynomial [2, 10-11] method, Laplace transform [3] method, path integral [12] method, exact quantization rule [1, 5, 8, 13-16] method and ansatz solution [17-19] method. The Eckart potential is an exponential-type potential with varying applications in chemical, solid state and nuclear physics, this important potential has been studied in the literature, the s-wave solution of the Schrödinger equation with Eckart potential has been reported [3, 16], Wei *et al*. [18] have obtained ℓ-wave scattering state solution of the Schrödinger equation with Eckart potential by ansatz solution method, Taskin and Kocal [17], by employing an improved approximation scheme have studied the *solution of Schrödinger equation with Eckart potential, also by ansatz solution method*. Encouraged by the works of Qiang *et al*. [1] in solving the Schrödinger equation with Hulthén potential by exact quantization rule, we are motivated to *solve the radial Schrödinger equation with Eckart potential from the approach of exact quantization rule*, which to the best of our knowledge is not in the literature, our result will be compared with existing results in the literature, obtained by other methods

2.0 Theoretical Approach

2.1 Review of the Concepts of Exact Quantization Rule

Here we give an outline of the methods of exact quantization rule, the complete detail is given by Ma and Xu [13]. The exact quantization rule was proposed to solve the one-dimensional Schrödinger equation given as:

Correspondence Author: Eyube E.S., Email: edwineyubes@mautech.edu.ng, Tel: +2348036870057

$$\frac{d^2\psi_{n\ell}(x)}{dx^2}+k_{n\ell}^2(x)\psi_{n\ell}(x)=0\,. \qquad (1)$$

where

$$k_{n\ell}(x)=\sqrt{\frac{2\mu}{\hbar^2}\left\{E_{n\ell}-V_{eff}(x)\right\}}\,. \qquad (2)$$

with μ as the reduced mass of the two interacting particles, $E_{n\ell}$ is the energy eigenvalue $k_{n\ell}$ (x) is the momentum of the system, V_{eff} (x) is the effective potential energy function which is a piecewise continuous real function of x. Eq. (1) can be reduced to the well-known Riccati nonlinear differential equation given by:

$$\phi_{n\ell}'(x)+k_{n\ell}^2(x)+\phi_{n\ell}^2(x)=0\,. \qquad (3)$$

where $\phi_{n\ell}(x)=\psi_{n\ell}'(x)/\psi_{n\ell}(x)$ is the logarithmic derivative of the wavefunction $\psi_{n\ell}$ (x) is known as the phase angle. Due to Sturm-Liouville theorem, $\phi_{n\ell}$ (x) decreases monotonically with respect to x between two turning points determined by the equation $E_{n\ell} \geq V_{eff}$ (x). Specifically, x increases across a node of the wavefunction $\psi_{n\ell}$ (x), where $E_{n\ell} \geq V_{eff}(x)$, $\phi_{n\ell}$ (x) decreases to $-\infty$ and jumps to $+\infty$ and then decreases again. By carefully studying the one-dimensional Schrödinger equation, Ma and Xu [13] proposed an Exact Quantization Rule givenby:

$$\int_{x_{nA}}^{x_{nB}}k_{n\ell}(x)dx=N\pi+\int_{x_{nA}}^{x_{nB}}\phi_{n\ell}(x)\left[\frac{dk_{n\ell}(x)}{dx}\right]\left[\frac{d\phi_{n\ell}(x)}{dx}\right]^{-1}dx\,. \qquad (4)$$

where x_{nA} and x_{nB} are two turning points determined by $E_{n\ell} = V_{eff}(x)$ and $x_{nA}< x_{nB}$. N is the number of nodes of $\phi_{n\ell}$ (x) in the neighborhood of $E_{n\ell} \geq V_{eff}(x)$ and it is larger by one than the number of nodes n of the wavefunction $\psi_{n\ell}$ (x), clearly, N = n + 1. The first term, Nπ, relates to the contribution from the nodes of the wave function, and the second term is called the quantum correction. Ma and Xu [13] have found that the quantum correction is independent of the number of nodes for the exactly solvable systems, therefore, it can be evaluated for the ground state (n = 0), the second term in Eq. (4) can thus be represented by:

$$Q_c=\int_{x_{nA}}^{x_{nB}}\phi_{n\ell}(x)\left[\frac{dk_{n\ell}(x)}{dx}\right]\left[\frac{d\phi_{n\ell}(x)}{dx}\right]^{-1}dx\equiv\int_{x_{0A}}^{x_{0B}}\phi_{0\ell}(x)\left[\frac{dk_{0\ell}(x)}{dx}\right]\left[\frac{d\phi_{0\ell}(x)}{dx}\right]^{-1}dx\,. \qquad (5)$$

where Q_c is the quantum correction term. The exact quantization rule can easily be generalized to three dimension, in spherical coordinates it assumes the form:

$$\int_{r_{nA}}^{r_{nB}}k_{n\ell}(r)dr=N\pi+\int_{r_{nA}}^{r_{nB}}\phi_{0\ell}(r)\left[\frac{dk_{0\ell}(r)}{dr}\right]\left[\frac{d\phi_{0\ell}(r)}{dr}\right]^{-1}dr\,. \qquad (6)$$

In compact form, Eq. (6) can be expressed as:

$$I=N\pi+Q_c\,. \qquad (7)$$

where

$$I=\int_{r_{nA}}^{r_{nB}}k_{n\ell}(r)dr\,. \qquad (8)$$

$$Q_c=\int_{r_{nA}}^{r_{nB}}\phi_{0\ell}(r)\left[\frac{dk_{0\ell}(r)}{dr}\right]\left[\frac{d\phi_{0\ell}(r)}{dr}\right]^{-1}dr\,. \qquad (9)$$

The Schrödinger equation in three dimensions for a spherically symmetric potential [18] is given as:

$$\frac{d^2R_{n\ell}(r)}{dr^2}+\frac{2\mu}{\hbar^2}\left\{E_{n\ell}-V_{eff}(r)\right\}R_{n\ell}(r)=0\,. \qquad (10)$$

$R_{n\ell}$ (r) being the radial wave function.

2.2 Energy Eigenvalues of the Eckart Potential

The effective Eckart potential [17] is given by:

$$V_{eff}(r)=V(r)+V_\ell(r)\,. \qquad (11)$$

where V (r) and V_ℓ (r) are respectively the Eckart potential and centrifugal term potential respectively, they are given by:

$$V(r)=-\alpha\frac{e^{-r/a}}{1-e^{-r/a}}+\beta\frac{e^{-r/a}}{\left(1-e^{-r/a}\right)^2}\,. \qquad (12)$$

with a as the range of the potential, α and β(α > 0, β > 0, α < β) are the depths of the potential.

$$V_\ell(r)=\frac{L\hbar^2}{2\mu r^2}\cdot \tag{13}$$

where L = ℓ (ℓ + 1). Substituting Eq. (12) and Eq. (13) in Eq. (11), ones obtains:

$$V_{eff}(r)=-\alpha\frac{e^{-r/a}}{1-e^{-r/a}}+\beta\frac{e^{-r/a}}{\left(1-e^{-r/a}\right)^2}+\frac{L\hbar^2}{2\mu r^2}\cdot \tag{14}$$

Eq. (10), with the effective potential given by Eq. (14) can be solved exactly only for the case of s-wave (ℓ = 0), to obtain analytical solution for ℓ ≠ 0, we must use a suitable approximation on the centrifugal term. one could employ the Greene and Aldrich approximation [ref] on the centrifugal term, this approximation is however unsuitable for short range region of the screening parameter, thus, we invoke the approximation due to Taskin and Kocal [17], known to give accurate results for both short and long range parameter, therefore, we assume:

$$\frac{1}{r^2}\approx\frac{1}{a^2}\left\{\eta\frac{e^{-r/a}}{1-e^{-r/a}}+\lambda\frac{e^{-2r/a}}{\left(1-e^{-r/a}\right)^2}\right\}\cdot \tag{15}$$

where η and λ are adjustable parameters. By replacing Eq. (15) in Eq. (14) and simplifying, get:

$$V_{eff}(r)=\left(\frac{L\hbar^2\eta}{2\mu a^2}-\alpha\right)\frac{1}{e^{r/a}-1}+\beta\frac{e^{r/a}}{\left(e^{r/a}-1\right)^2}+\frac{L\hbar^2\lambda}{2\mu a^2}\frac{1}{\left(e^{r/a}-1\right)^2}\cdot \tag{16}$$

Letting

$$y=\frac{1}{e^{r/a}-1}\cdot \tag{17}$$

Eq. (17) transforms to:

$$V_{eff}(y)=\left(\frac{L\hbar^2\eta}{2\mu a^2}-\alpha\right)y+\beta\left(y^2+y\right)+\frac{L\hbar^2\lambda}{2\mu a^2}y^2\cdot \tag{18}$$

Eq. (18) leads to:

$$V_{eff}(y)=\frac{\hbar^2}{2\mu a^2}\left(L\lambda+\frac{2\mu a^2\beta}{\hbar^2}\right)y^2+\frac{\hbar^2}{2\mu a^2}\left\{\frac{2\mu a^2\beta}{\hbar^2}-\left(\frac{2\mu a^2\alpha}{\hbar^2}-L\eta\right)\right\}y\cdot \tag{19}$$

By defining the following constants:

$$\varepsilon=\frac{2\mu a^2\beta}{\hbar^2}\cdot \tag{20}$$

$$\kappa=\frac{2\mu a^2\alpha}{\hbar^2}-L\eta\cdot \tag{21}$$

Eq. (19) assumes the following simplest form

$$V_{eff}(y)=Ay^2+By\cdot \tag{22}$$

where

$$A=\frac{\hbar^2}{2\mu a^2}\left(L\lambda+\varepsilon\right)\cdot \tag{23}$$

$$B=\frac{\hbar^2}{2\mu a^2}\left(\varepsilon-\kappa\right)\cdot \tag{24}$$

The turning points, y_{nA} and y_{nB} are determined by solving the equation $V_{eff}(y) = E_{n\ell}$, this results in:

$$Ay^2+By-E_{n\ell}=0\cdot \tag{25}$$

Eq. (25) has roots given by:

$$y_{nA}=-\frac{B+\sqrt{B^2+4AE_{n\ell}}}{2A}\cdot \tag{26}$$

$$y_{nB}=-\frac{B-\sqrt{B^2+4AE_{n\ell}}}{2A}\cdot \tag{27}$$

From Eq. (26) and Eq. (27) it is obvious that the sum and product of y_{nA} and y_{nB} are given by:

$$y_{nA}+y_{nB}=-\frac{B}{A}\cdot \tag{28}$$

$$y_{nA}\,y_{nB}=-\frac{E_{n\ell}}{A}\cdot \tag{29}$$

For the ground (n = 0) state, Eq. (28) and Eq. (29) gives respectively:

$$y_{0A}+y_{0B}=-\frac{B}{A}\cdot \tag{30}$$

$$y_{0A}\,y_{0B}=-\frac{E_{0\ell}}{A}\cdot \tag{31}$$

The expression for the momentum is given by, following Eq. (2), this gives:

$$k_{n\ell}(y)=\sqrt{\frac{2\mu}{\hbar^2}\left(E_{n\ell}-Ay^2-By\right)}\cdot \tag{32}$$

With slight modification, Eq. (32) can be written as:

$$k_{n\ell}(y)=\sqrt{\frac{2\mu A}{\hbar^2}\left(\frac{E_{n\ell}}{A}-y^2-\frac{B}{A}y\right)}\cdot \tag{33}$$

An important relation which will be required later is the expression relating $k_{n\ell}$ (y) and the turning points y_{nA} and y_{nB}, inserting Eq. (28) and Eq. (29) in Eq. (33), we find:

$$k_{n\ell}(y)=\sqrt{\frac{2\mu A}{\hbar^2}\left(y-y_{nA}\right)\left(y_{nB}-y\right)}\cdot \tag{34}$$

The derivative of Eq. (34) with respect to y is given by:

$$k'_{n\ell}(y)=-\frac{\sqrt{2\mu A}}{\hbar}\frac{\left\{y-\frac{1}{2}\left(y_{nA}+y_{nB}\right)\right\}}{\sqrt{\left(y-y_{nA}\right)\left(y_{nB}-y\right)}}\cdot \tag{35}$$

Eq. (35) gives for the ground state(n = 0), and employing Eq. (30):

$$k'_{0\ell}(y)=-\frac{\sqrt{2\mu A}}{\hbar}\frac{\left(y+B/2A\right)}{\sqrt{\left(y-y_{0A}\right)\left(y_{0B}-y\right)}}\cdot \tag{36}$$

Using Eq. (2) and Eq. (3), the Riccati equation in three dimensions in spherical coordinates is:

$$\phi'_{n\ell}(r)+\frac{2\mu}{\hbar^2}\left\{E_{n\ell}-V_{eff}(r)\right\}+\phi^2_{n\ell}(r)=0\cdot \tag{37}$$

To obtain the corresponding equation in terms of variable y, we substitute Eq. (17) in Eq. (37) and use Eq. (22) to eliminate $V_{n\ell}$ (y), this leads to:

$$-\frac{1}{a}y(1+y)\phi'_{n\ell}(y)+\frac{2\mu}{\hbar^2}\left(E_{n\ell}-Ay^2-By\right)+\phi^2_{n\ell}(y)=0\cdot \tag{38}$$

Eq. (38) has for the ground state;

$$-\frac{1}{a}y(1+y)\phi'_{0\ell}(y)+\frac{2\mu}{\hbar^2}\left\{E_{0\ell}-Ay^2-By\right\}+\phi^2_{0\ell}(y)=0\quad\cdot \tag{39}$$

Since $\phi_{0\ell}$ (y) has one zero and no pole, it has to assume a linear form in y, for a trial solution, we take:

$$\phi_{0\ell}(y)=-c_1y+c_2\cdot \tag{40}$$

c_1 and c_2 being constants, substituting Eq. (40) in Eq. (39), get:

$$\frac{c_1}{a}\left(y+y^2\right)+\frac{2\mu}{\hbar^2}\left\{E_{0\ell}-Ay^2-By\right\}+c_1^2y^2-2c_1c_2y+c_2^2=0\cdot \tag{41}$$

Eq. (41) simplifies to;

$$\left(\frac{c_1}{a}-\frac{2\mu A}{\hbar^2}+c_1^2\right)y^2+\left(\frac{c_1}{a}-\frac{2\mu B}{\hbar^2}-2c_1c_2\right)y+\frac{2\mu E_{0\ell}}{\hbar^2}+c_2^2=0\cdot \tag{42}$$

By equating corresponding coefficients of y^2, y and y^0 respectively on both sides of Eq. (42), we arrive at the following relations:

$$c_1^2+\frac{c_1}{a}=\frac{2\mu A}{\hbar^2}\cdot \tag{43}$$

$$\frac{c_1}{a}-2c_1c_2=\frac{2\mu B}{\hbar^2}. \tag{44}$$

$$-c_2^2=\frac{2\mu E_{0\ell}}{\hbar^2}. \tag{45}$$

Solving for c_1 in Eq. (43), get:

$$c_1=-\frac{1}{2a}-\sqrt{\frac{1}{4a^2}+\frac{2\mu A}{\hbar^2}}. \tag{46}$$

By substituting Eq. (23) in Eq. (46), c_1 can be expressed in more compact form as:

$$c_1=-\frac{\sigma}{a}. \tag{47}$$

where

$$\sigma=\tfrac{1}{2}\left\{1+\sqrt{1+4L\lambda+4\varepsilon}\right\}. \tag{48}$$

Evidently, Eq. (48) gives;

$$L\lambda+\varepsilon=\sigma^2-\sigma. \tag{49}$$

Eq. (44) gives:

$$c_2=\frac{1}{2a}-\frac{\mu B}{c_1\hbar^2}. \tag{50}$$

Having obtained c_1 and c_2, we are now in position to compute the various integrals which appear in Eq. (7), starting with the right hand side of this equation, Eq. (9) can be used to obtain the quantum correction, using the transformation given by Eq. (17), we have;

$$Q_c=-a\int_{y_{0A}}^{y_{nB}}\frac{\phi_{0\ell}(y)}{\phi'_{0\ell}(y)}k'_{0\ell}(y)\frac{dy}{y(1+y)}. \tag{51}$$

Using Eq. (40) and Eq. (36) in Eq. (51), we have:

$$Q_c=\frac{a\sqrt{2\mu A}}{\hbar}\int_{y_{0A}}^{y_{nB}}\frac{(y-c_2/c_1)(y+B/2A)}{y(1+y)}\frac{dy}{\sqrt{(y-y_{0A})(y_{0B}-y)}}. \tag{52}$$

By expanding out the numerator in Eq. (52) and splitting into partial fractions, we obtained:

$$Q_c=\frac{a\sqrt{2\mu A}}{\hbar}\int_{y_{0A}}^{y_{nB}}\left\{1-\frac{c_2B}{2c_1Ay}+\frac{(c_1+c_2)(B-2A)}{2c_1A(1+y)}\right\}\frac{dy}{\sqrt{(y-y_{0A})(y_{0B}-y)}}. \tag{53}$$

the definite integral in Eq. (53) can be evaluated by means of the following standard integral [5] given by:

$$\int_{y_{nA}}^{y_{nB}}\frac{dy}{(p+qy)\sqrt{(y-y_{nA})(y_{nB}-y)}}=\frac{\pi}{\sqrt{(p+qy_{nB})(p+qy_{nA})}}. \tag{54}$$

applying Eq. (54) in Eq. (53) yields the following results:

$$Q_c=\frac{\pi a\sqrt{2\mu A}}{\hbar}\left\{1-\frac{c_2B}{2c_1A}I_1+\frac{(c_1+c_2)(B-2A)}{2c_1A}I_2\right\}. \tag{55}$$

where

$$I_1^{-2}=y_{0A}y_{0B}\equiv-\frac{E_{0\ell}}{A}. \tag{56}$$

Thus, by putting Eq. (45) in Eq. (56), we find:

$$I_1=\frac{\sqrt{2\mu A}}{c_2\hbar} \tag{57}$$

similarly; we find:

$$I_2^{-2}=1+y_{0A}+y_{0B}+y_{0A}y_{0B}. \tag{58}$$

On substituting Eq. (30) and Eq. (31) in Eq. (58), this gives:

$$I_2^{-2}=1-\frac{B}{A}-\frac{E_{0\ell}}{A}. \tag{59}$$

To further simplify Eq. (59), divide Eq. (44) and Eq. (45) each by Eq. (43), and the results substituted in Eq. (59), get:

$$I_2^{-2} = 1 - \frac{\frac{c_1}{a} - 2c_1 c_2}{c_1^2 + \frac{c_1}{a}} + \frac{c_2^2}{c_1^2 + \frac{c_1}{a}} \equiv \frac{(c_1 + c_2)^2}{c_1^2 + \frac{c_1}{a}} \cdot \quad (60)$$

Using Eq. (43) to eliminate the denominator in Eq. (60), we obtained:

$$I_2 = \frac{\sqrt{2\mu A}}{\hbar(c_1 + c_2)} \cdot \quad (61)$$

Substituting Eq. (57) and Eq. (61) in Eq. (55), we have for the quantum correction:

$$Q_c = \frac{\pi a \sqrt{2\mu A}}{\hbar}\left(1 - \frac{\sqrt{2\mu A}}{c_1 \hbar}\right) \cdot \quad (62)$$

The other integral on the right hand side of Eq. (7) is given in terms of variable y as:

$$I = -a \int_{y_{nA}}^{y_{nB}} k_{n\ell}(y) \frac{dy}{y(1+y)} \cdot \quad (63)$$

Putting Eq. (34) in Eq. (63), this gives:

$$I = -\frac{a\sqrt{2\mu A}}{\hbar} \int_{y_{nA}}^{y_{nB}} \frac{\sqrt{(y - y_{nA})(y_{nB} - y)}}{y(1+y)} dy \quad (64)$$

In order to evaluate the integral in Eq. (64) we use the following standard integral [8], viz:

$$\int_{y_{nA}}^{y_{nB}} \frac{\sqrt{(y - y_{nA})(y_{nB} - y)}}{y(1+qy)} dy = \pi\left\{\frac{\sqrt{(q y_{nA} + 1)(q y_{nB} + 1)}}{q} - \frac{1}{q} - \sqrt{y_{nA} y_{nB}}\right\}. \quad (65)$$

By using the definite integral in Eq. (65) we obtained,

$$I = \frac{a\pi\sqrt{2\mu A}}{\hbar}\left\{1 + \sqrt{(-E_{n\ell}/A)} - \sqrt{1 - B/A + (-E_{n\ell}/A)}\right\} \cdot \quad (66)$$

Substitute Eq. (66) and Eq. (62) in Eq. (7) to get:

$$\sqrt{(-E_{n\ell}/A)} - \sqrt{1 - B/A + (-E_{n\ell}/A)} = \chi \cdot \quad (67)$$

where

$$\chi = \frac{N\hbar}{a\sqrt{2\mu A}} - \frac{\sqrt{2\mu A}}{c_1 \hbar} \cdot \quad (68)$$

Substituting Eq. (23) in Eq. (68) and then inserting Eq. (47) in the resulting expression, Eq. (68) reduces to:

$$\chi = \frac{n + \sigma}{\sqrt{\sigma^2 - \sigma}} \cdot \quad (69)$$

Where we have replaced N by n + 1 in Eq. (68). Eq. (67) and Eq. (69) gives the energy eigenvalues, $E_{n\ell}$ as:

$$E_{n\ell} = -\frac{\hbar^2}{2\mu a^2}\left\{\frac{\kappa + L\lambda}{2(n+\sigma)} - \frac{n+\sigma}{2}\right\}^2 \cdot \quad (70)$$

2.3 Energy Eigenfunctions

Having obtained the expression for the energy eigenvalues of the Eckart potential as given by Eq. (70), for completeness, we now derive the corresponding expression for the energy eigenfunctions, this is achievable by solving the nonlinear Riccati equation given by Eq. (38), alternatively, the Schrödinger equation given by Eq. (10) can be solved directly. Inserting the transformation Eq. (17) in Eq. (10) and the effective potential, Eq. (22), and simplifying, we obtained:

$$y^2(1+y)^2 R''_{n\ell}(y) + y(1+y)(1+2y)R'_{n\ell}(y) + \frac{2\mu a^2}{\hbar^2}\left(E_{n\ell} - Ay^2 - By\right)R_{n\ell}(y) = 0 \cdot \quad (71)$$

Eq. (71) assumes the form:

$$y^2(1+y)^2 R''_{n\ell}(y) + y(1+y)(1+2y)R'_{n\ell}(y) - \left\{(L\eta + \varepsilon)y^2 + (\varepsilon - \kappa)y + \omega_{n\ell}\right\}R_{n\ell}(y) = 0 \cdot \quad (72)$$

when Eq. (23) and Eq. (24) are used in Eq. (71). For brevity we have defined:

$$\omega_{n\ell} = -\frac{2\mu a^2 E_{n\ell}}{\hbar^2} \cdot \quad (73)$$

Letting

$$y=\frac{u}{1-u}\cdot \quad (74)$$

Eq. (72) takes the following form:

$$u^2 R''_{n\ell}(u)+u R'_{n\ell}(u)-\left\{(L\eta+\varepsilon)\frac{u^2}{(1-u)^2}+(\varepsilon-\kappa)\frac{u}{1-u}+\omega_{n\ell}\right\}R_{n\ell}(u)=0\cdot \quad (75)$$

We can obtain an ansatz solution for Eq. (75), following ref. [17, 19], assume a solution of the form:

$$R_{n\ell}(u)=N_{n\ell}\,u^p\,(1-u)^{q+1}\,\psi_{n\ell}(u)\cdot \quad (76)$$

where $N_{n\ell}$ is the normalization constant, p and q are constants to be determined such that Eq. (76) is satisfied and $\psi_{n\ell}$ (u) is an unknown function. By successively differentiating Eq. (76) we have:

$$R'_{n\ell}(u)=\left\{\frac{\psi'_{n\ell}(u)}{\psi_{n\ell}(u)}+\frac{p}{u}-\frac{q+1}{1-u}\right\}R_{n\ell}(u)\cdot \quad (77)$$

$$R'_{n\ell}(u)=\left\{\frac{\psi''_{n\ell}(u)}{\psi_{n\ell}(u)}+\left(\frac{2p}{u}-\frac{2q+2}{1-u}\right)\frac{\psi'_{n\ell}(u)}{\psi_{n\ell}(u)}-\frac{2p\,(q+1)}{u(1-u)}+\frac{q(q+1)}{(1-u)^2}+\frac{p^2-p}{u^2}\right\}R_{n\ell}(u)\cdot \quad (78)$$

Substituting Eq. (78) and Eq. (77) in Eq. (75) and simplifying, get:

$$\left\{\begin{array}{l} u(1-u)\psi''_{n\ell}(u)+[2p+1-(2p+2q+3)]\psi'_{n\ell}(u) \\ -\left[(q+1)(2p+1)+\varepsilon-\kappa-(q^2+q-L\eta-\varepsilon)\frac{u}{1-u}-\frac{(p^2-\omega_{n\ell})(1-u)}{u}\right]\psi_{n\ell}(u)\end{array}\right\}R_{n\ell}(u)=0\cdot \quad (79)$$

Subject to the following constraints, Eq. (79) is Gaussian hypergeometric differential equation, iff:

$$p^2=\omega_{n\ell}\cdot \quad (80)$$

and

$$q^2+q=L\eta+\varepsilon\cdot \quad (81)$$

Eq. (80) and Eq. (81) gives respectively:

$$p=\omega_{n\ell}^{\frac{1}{2}}\cdot \quad (82)$$

$$q=\sigma-1\cdot \quad (83)$$

with Eq. (80) and Eq. (81), Eq. (79) is reduced to:

$$u(1-u)\psi''_{n\ell}(u)+\{2p+1-(2p+2q+3)\}\psi'_{n\ell}(u)-\{(q+1)(2p+1)+\varepsilon-\kappa\}\psi_{n\ell}(u)=0\cdot \quad (84)$$

whose solution is the hypergeometric function given by:

$$\psi_{n\ell}(u)={}_2F_1(-n,n+2p+2q+2;2p+1;u)\cdot \quad (85)$$

It follows that the wave function given in Eq. (76) can be expressed in terms of variable y as:

$$R_{n\ell}(y)=N_{n\ell}\,y^p\,(1+y)^{-p+q+1}\,{}_2F_1(-n,n+2p+2q+2;2p+1;y/(1+y))\cdot \quad (86)$$

Having obtained the constants p and q as given by Eq. (82) and Eq. (83) respectively, next is to determine the normalization constant. The normalization of wave functions [20] requires that:

$$\int_0^\infty \left|R_{n\ell}(r)\right|^2 dr=1\cdot \quad (87)$$

If we insert Eq. (17), then Eq. (74) in Eq. (87), get:

$$a\int_0^1 u^{-1}\left|R_{n\ell}(u)\right|^2 du=1\cdot \quad (88)$$

After substituting Eq. (76), Eq. (85) in Eq. (88), we have the following definite integral:

$$a\,N_{n\ell}^2\int_0^1 u^{2p-1}(1-u)^{2q+2}\left|{}_2F_1(-n,n+2p+2q+2;2p+1;u)\right|^2 du=1\cdot \quad (89)$$

Using the result of Dong and Qiang [20] to evaluate the integral in Eq. (89), we find:

$$N_{n\ell}=\frac{1}{\sqrt{M}}\cdot \quad (90)$$

where

$$M = \frac{n!a(2n+2q+1)\Gamma(n+2p+3)\Gamma(2p)\Gamma(2p+1)}{(2n+2p+2q+3)\Gamma(n+2p+1)\Gamma(n+2p+2q+3)}. \quad (91)$$

with Γ(x) being the gamma function of the argument x.

3.0 Results and Discussion

If we let κ = ϕ, η = ξ and L = γ in Eq. (20), Eq. (21) and Eq. (48), the energy eigenvalues given by Eq. (70) coincides with the energy eigenvalues of the Eckart potential in ref. [17] which was obtained by solving a hypergeometric differential equation via the ansatz solution method. As a special case of the Eckart potential, for some values of the parameters a, α, β, ε, κ, σ, η and λ, the effective Eckart potential in Eq. (16) reduces to Hulthén potential, thus, by comparing the effective potential given by Eq. (8) in ref. [19] with Eq. (16) of this article, it is obvious that $a = 1/\delta$, $\alpha = Ze^2\delta$, β = 0, $L = \ell(\ell+1)$; η = ω, and λ = 1, substituting these values in Eq. (20), Eq. (21) and Eq. (48), we obtained respectively, ε = 0, $\kappa = 2\mu Ze^2/\delta\,\hbar^2$ and $\sigma = \ell+1$, which when used in Eq. (70), yields:

$$E_{n\ell} = -\frac{\hbar^2\delta^2}{2\mu}\left\{\frac{\frac{2\mu Ze^2}{\hbar^2\delta} + \ell(\ell+1)(1-\omega)}{2(n+\ell+1)} - \frac{n+\ell+1}{2}\right\}^2. \quad (92)$$

Eq. (92) is the energy eigenvalues of the Hulthén potential as deduced by Wang *et al*. [19]. We have used the values α = 1/a. λ = 0.98 and η = 1.10 [17] to plot the unnormalized wave functions of Eq. (76) for states n = 2, 3 and 5 in atomic units (ħ = μ = 1). The plots are shown in Figures 1, 2 and 3. In conclusion, we have solved the radial Schrödinger equation with the Eckart potential and obtained closed form expressions for the energy eigenvalues and normalized radial wave functions of the Eckart potential by exact quantization rules, the results of this research work could be useful in areas of solid state physics, atomic physics, nuclear physics and molecular physics.

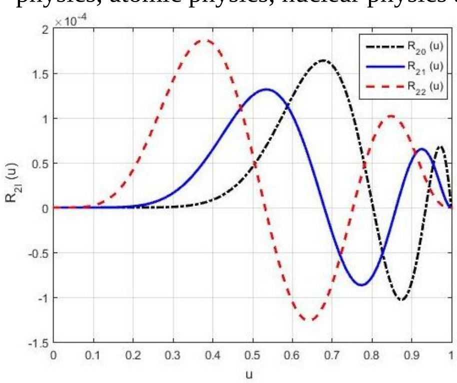

Figure 1 Plot of Unnormalized Radial Wave Functions R_{20}, R_{21} and R_{22} vs. u

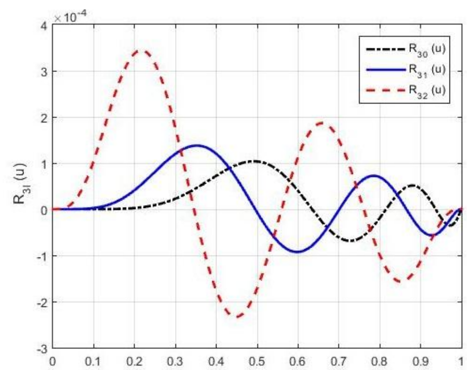

Figure 2 Plot of Unnormalized Radial Wave Functions R_{30}, R_{31} and R_{32} vs. u

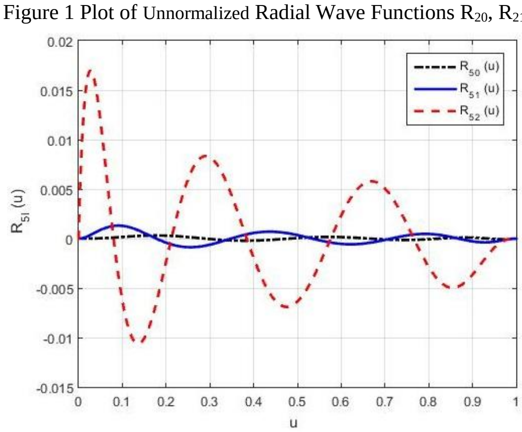

Figure 3 Plot of Unnormalized Radial Wave Functions R_{50}, R_{51} and R_{52} vs. u

4.0 Conclusion

We have solved the radial Schrödinger equation and obtained closed form expressions for the bound state enrtgy eigenvalues and eigenfunctions of the Eckart potential by exact quantization rule, a Pekeris-type approximation scheme was employed to deal with the spin-orbit centrifugal term of the effective Eckart potential. Closed form expressions for the bound state energy eigenvalues and eigenfunctions of the Hulthén potential was also deduced as a special case of the Eckart potential.

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