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ON THE SOLUTION AND ANALYSIS OF HARMONIC DIRICHLET AND NEUMANN PROBLEMS OF IDEAL FLUID FLOWS USING CONFORMAL MAPS

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Abstract

A purely conformal mapping method for solving harmonic Dirichlet and Neumann problems of ideal fluid flows in domains whose boundaries have inconvenient geometries is presented. The method uses an appropriate mapping function to transform the given problem in a domain of the w plane onto a standard domain in the z plane where the complex potential for the uniform to the right is well known. The simplified problem was then solved using the composition of two analytic functions to obtain the required complex potential for the flow in the w plane. The stream function and solution of the flow problem was then isolated from the complex potential and the streamlines of each flow generated to show the unique features of each flow and the flow pattern analyzed in terms of fluid speed by the spacing of the streamlines. The fluid velocity was also determined from the complex potential and the stagnation points on the boundary in each flow identified. This method gave exact general analytical solutions to the problems considered and could therefore be a useful alternative choice for solving Laplace's equation for two dimensional fluid flows.

***Keywords*:** Conformal Map, Schwarz-Christoffel Map, Joukowski map, Analytic Function, Branch of a Multiple Valued Function, Non-viscous, Incompressible, Coefficient of Pressure, Joukowski Airfoils, NACA Airfoils.

1. Introduction

Most problems of fluid flow when modelled mathematically under the assumptions that the flows are non-viscous and incompressible lead to the problem of solving an elliptic second order linear partial differential equation

$$\nabla^2\psi = 0 \qquad (1)$$

called Laplace's equation subject to some specified boundary conditions which depend on the problem in question. If the boundary condition is such that ψ takes prescribed values along the boundary, then the problem is called a Dirichlet problem while a Neumann problem is one in which the normal derivative $\frac{\partial\psi}{\partial n}$ takes prescribed values on the boundary. In this paper we shall consider both types of boundary value problems. It is well known in the theory of analytic functions of a complex variables that if a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain Ω then its component functions $u(x, y)$ and $v(x, y)$ are harmonic there. The solution to problem (1) therefore reduces to finding a function which is analytic in Ω and whose real and imaginary parts satisfy the boundary conditions. The complex variable method of conformal mapping is a useful intermediate step in the solution and analysis of ideal flows as is evident in the works in [1-12] as well as other none flow problems in electrostatics [1,2,10,12,13], electromagnetism [10,12], thermal physics [1,2,10,15] and other areas of computational and applied mathematics[15-19].The technique involves the transformation of the problem from a domain with an inconvenient geometry in one complex plane into a domain with a simpler geometry in another complex plane by means of an appropriate mapping function which preserves the magnitude of the angles between curves as well as their orientation. Amongst a variety of conformal transformations, the ones commonly used in analyzing ideal fluid flows are the Joukowski map, the Karman-Trefftz map (a generalization of the Joukowski map), and the

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Schwarz-Christoffel map. In this paper, we shall focus on the Schwarz-Christoffel and Joukowskimaps. The Schwarz-Christoffel transformation which is given by [1] as

$$w = f(z) = A\int_{z_0}^{z} \prod_{j=1}^{n-1} \left(s - x_j\right)^{-k_j} ds + B \quad (2)$$

or

$$\frac{dw}{dz} = f'(z) = A\prod_{j=1}^{n-1} (z - x_i)^{-k_j} \quad (3)$$

is one that conformally maps the upper half Im z > 0 of the z plane and the entire x axis except for a finite number of points $x_1, x_2, \dots, x_{n-1}, \infty$ in a one-to-one correspondence onto the interior of a given simple closed polygon and its boundary, respectively, such that $w_j = f(x_j)(j = 1,2,\dots,n-1)$ and $w_n = f(\infty)$ are the vertices of the polygon. The points $z = x_j$ $(j = 1,2,\dots,n-1)$ are arranged such that the order relation $x_1 < x_2 < \cdots < x_{n-1}$ is satisfied. The complex constants A and B in formula (2) determine the size, orientation and position of the polygon, the k_j's are real constants between -1 and 1 determined from the relation $-\pi < k_j\pi < \pi$, where $k_j\pi$ $(j = 1,2,\dots,n-1)$ are the exterior angles at the vertices w_j $(j = 1,2,\dots,n-1)$ of the polygon, while the limits of integration z_0 and z are respectively fixed and variable points in the region Im z ≥ 0 of analyticity of the Schwarz-Christoffel function.

The other conformal map to be considered in this paper is the Joukowski transformation which is defined by [6] as

$$w = z + \frac{c^2}{c} \quad (4)$$

and has critical points at $z = \pm c$ which map into the critical points $w = \pm 2c$, respectively. This transformation is extensively used in aerodynamics (see for instance, [9], [11], and [12]) to simplify the oblong shape of an airfoil and the flow around it onto a pseudo circle and the flow exterior to it, respectively. It is also used in aerodynamics to model the lift around NACA family of four digit series airfoils based on a Joukowski transformation program developed by the National Aeronautics and Space Administration(NASA) and used by [6] in their research paper. In addition, it is used in aerodynamics to solve for the two dimensional potential flow around a class of airfoils called Joukowski airfoils (the ones also being researched into). Furthermore, the analytic pressure distribution on their upper and lower surfaces is quite valuable since its exact solution is often used as test case for measuring the performance and accuracy of other methods of computing pressure distribution on arbitrary airfoil. We note here that Joukowski airfoils are foil shapes generated by conformal transformation of circles whose centres are offset from the origin of the z plane and the circle is made to pass through one of the critical points of the transformation $z = \pm c$ and envelops the other (see figure 3(a)). The transformation (4) also maps the flow around the circle onto the flow exterior to the corresponding Joukowski airfoil. A detailed presentation of the manner in which the transformation (4) map circles into various profiles can be found in [1].In this research paper, we shall apply theSchwarz-Christoffel transformation in the solution and analysis of aharmonic Dirichlet problem of ideal fluid flow in a semi-infinite strip while the Joukowski transformation will be used to solve the harmonic Dirichlet and Neuman problems for the ideal flows in the domain Im z > 0 of the z plane which is above the semi-circular are$w = e^{i\theta}(0 \leq \theta \leq \pi)$ and the lines $u < -1$ and $u > 1$ and around aJoukowski airfoil, respectively.

2. METHODOLOGY

Consider an ideal fluid flow in a domain Ω of the w plane whose boundary $\partial\Omega$ has an inconvenient geometry. The flow field is the solution of the mathematical problem (1) subject to some boundary conditions. In order to solve this problem the appropriate mapping function $z = G(w)$ is used to simplify the given problem by transforming it onto a standard domain such as the upper half Im z > 0 of the z plane or the domain exterior to the unit$z = e^{i\theta}(0 \leq \theta < 2\pi)$ circle for which an analytic form of the solution for potential flow is well known. If $F(z)$ is the complex potential for the flow in the z plane, then the function $F\big(G(w)\big) = \mathrm{H}(w)$ is analytic in the problem domain of the w plane and hence represent the complex potential for the flow there. If$\mathrm{F}(z) = \phi(x,y) + i\psi(x,y)$, $z = G(w) = g(u,v) + ih(u,v)$, and $\mathrm{H}(w) = \Phi(u,v) + i\Psi(u,v)$, then

$$\Phi(u,v) = \phi[g(u,v), h(u,v)] \quad (5)$$

and

$$\Psi(u,v) = \psi[g(u,v), h(u,v)] \quad (6)$$

are respectively the velocity potential and the stream functions of the flow. The stream function $\Psi(u,v)$ is the required solution of the flow problem although its harmonic conjugate $\Phi(u,v)$ is also a solution but not the one desired or really sought after. On setting $\phi[g(u,v), h(u,v)] = c_1$ and $\psi[g(u,v), h(u,v)] = c_2$, where $c_1$andc_2 are real constants, the equipotential lines and streamlines of the flow are obtained, respectively. Alternatively, the equipotential lines and streamlines of the flow in the problem domain can be generated by finding the images of the equipotential lines and streamlines of the uniform flow in the simplified domain of the z plane using the inverse function $w = f(z) = G^{-1}(z)$. If the inverse function turns out to be multiple valued, then its appropriate branch should be found using the complex variable method of branch cuts and then used. The fluid velocity $v(w)$ is simply the conjugate of the derivative of the complex potential and is

$$v(w)=\overline{\frac{d\mathrm{H}}{dw}}=\overline{\frac{dF}{dz}}.\overline{\frac{dz}{dw}}$$

or

$$v(w)=\overline{\frac{d\mathrm{H}}{dw}}=\frac{\overline{\left(\frac{dF}{dz}\right)}}{\overline{\left(\frac{dw}{dz}\right)}} \tag{7}$$

where $\frac{dw}{dz}$ is the derivative of the mapping function.The fluid speed is the modulus of the fluid velocity and from equation (7), we have that

$$|v(w)|=\frac{\left|\frac{dF}{dz}\right|}{\left|\frac{dw}{dz}\right|} \tag{8}$$

Equation (8) is important and shows that the fluid speed in the problem domain of the w plane can be obtained from the corresponding one in the z plane and the derivative of the mapping function.

3. RESULTS

In this section we present the solution to some harmonic Dirichlet problems and a Neumann problem in ideal fluid flows based on the purely conformal mapping techniques outlined in the methodology.

Problem 1: (Flow in a Semi-Infinite Strip of Width a)

We first consider the harmonic Dirichlet problem in equation (1) for the ideal fluid flow in a semi-infinite strip shown in Figure 1(a) and described by the equation

$$-\frac{a}{2}\le u\le\frac{a}{2},\qquad v\ge 0$$

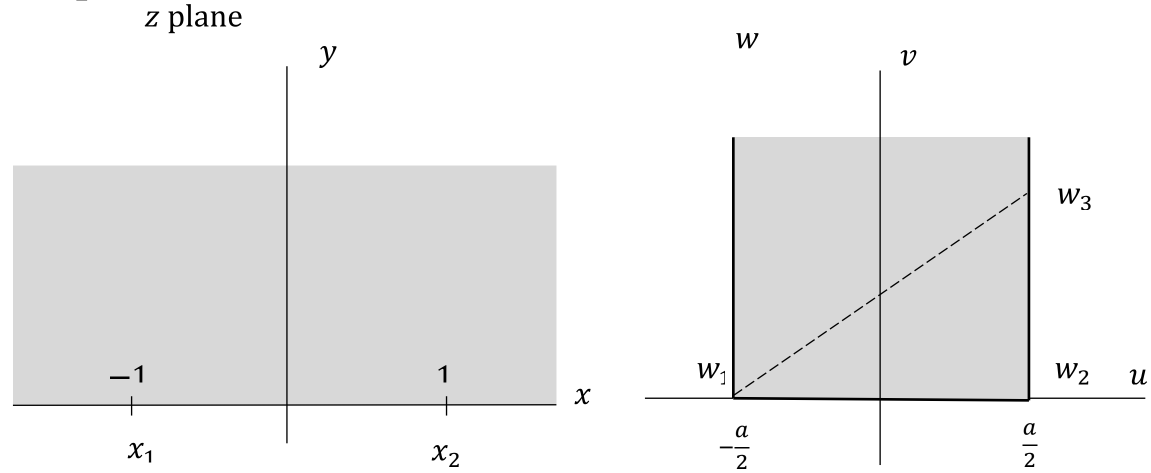

Figure 1(a): One-to-One Mapping of the Half Plane Im z ≥ 0 in the *z* plane Onto the Semi-Infinite Strip in the *w* plane

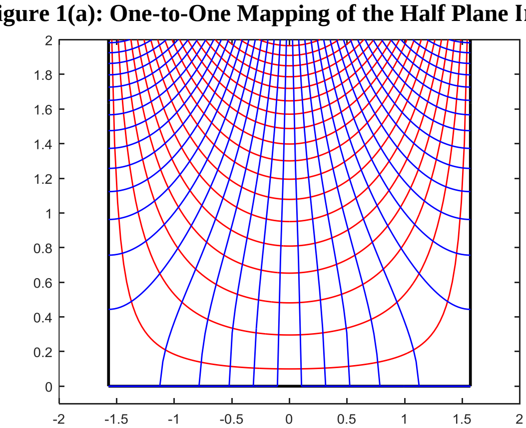

Figure 1(b): Equipotential (Blue) and Streamlines (Red) of flow in the Interior of a Semi-Infinite Strip the *w* plane.

Here, the Schwarz-Christoffel transformation $w = f(z)$ that maps the upper half $\text{Im}\, z > 0$ of the z plane in a one-to-one manner onto the interior of the semi-infinite strip is found to be

$$w = \frac{a}{\pi}\sin^{-1} z \qquad (9)$$

by considering the strip as a limiting form of a triangle with vertices at $w_1 = -a/2,\ \ w_2 = a/2$, and w_3 as the imaginary part of w_3 tends to infinity or simply using the table of transforms given by [2]. The part $x \le -1$ of the x axis is mapped by the transformation (9) onto the vertical line $x = -a/2$, the part $-1 < x < 1$ maps onto the segment $-a/2 < u < a/2$ of the strip, the part $x \ge 1$ map onto the vertical line $x = a/2$, while the points $z = -1\ and\ z = 1$ map into the points $-\frac{a}{2}$ and $\frac{a}{2}$, respectively. The inverse of the transformation (9) is

$$z = \sin\left(\frac{w\pi}{a}\right) \qquad (10)$$

and maps the interior of the semi-infinite strip and its boundaries in a one-to-one manner onto the upper half $\text{Im}\, z > 0$ of the z plane and the parts of the x axis as already stated. Now, the complex potential for the uniform flow to the right in the upper half $\text{Im z} > 0$ of the z plane is analytic and well known and given by [1] and [6] as

$$\text{F(z)} = v_0 z \qquad (11)$$

where v_0 is a positive real constant corresponding to the speed of the flow. Since the inverse map is analytic inside the strip, the composite function

$$\text{H}(w) = v_0 \sin\left(\frac{w\pi}{a}\right) \qquad (12)$$

is also analytic in the interior of the semi-infinte strip and hence represents the complex potential for the flow there. If $\text{H}(w) = \Phi(u,v) + i\Psi(u,v)$ and $w = u + iv$, then the velocity potential and stream function of the flow are

$$\Phi(u,v) = v_0 \sin\frac{\pi u}{a}\cosh\frac{\pi v}{a} \qquad (13)$$

and

$$\Psi(u,v) = v_0 \cos\frac{\pi u}{a}\sinh\frac{\pi v}{a} \qquad (14)$$

respectively. Hence the equipotential lines and streamlines of the flow are

$$v_0 \sin\frac{\pi u}{a}\cosh\frac{\pi v}{a} = c_1 \qquad (15)$$

and

$$v_0 \cos\frac{\pi u}{a}\sinh\frac{\pi v}{a} = c_2 \qquad (16)$$

respectively, where $c_1\ and\ c_2$ are real constants. The velocity of the flow is obtained from the complex potential of the flow in equation (12) as

$$v(w) = \overline{\text{H}'(w)} = \frac{\pi v_0}{a}\cos\frac{\pi}{a}\bar{w} = \cos\frac{\pi u}{a}\cosh\frac{\pi v}{a} + i\sin\frac{\pi u}{a}\sinh\frac{\pi v}{a} \qquad (17)$$

while the fluid speed is

$$|v(w)| = \sqrt{\cos^2\frac{\pi u}{a} + \sinh^2\frac{\pi v}{a}} \qquad (18)$$

Problem 2: (Flow in the Upper Half Im w > 0 of the w plane Bounded by the Semi-Circular Arc $w = e^{i\theta}(0 \le \theta \le \pi)$ and the Lines $u < -1$ and $u > 1$)

We next consider the harmonic Dirichlet problem in equation (1) for the ideal fluid flow in the domain $\text{Im w} > 0$ bounded by the semi-circular arc $w = e^{i\theta}(0 \le \theta \le \pi)$ and the parts $u < -1$ and $u > 1$ of the u axis as shown in Figure 2(a).

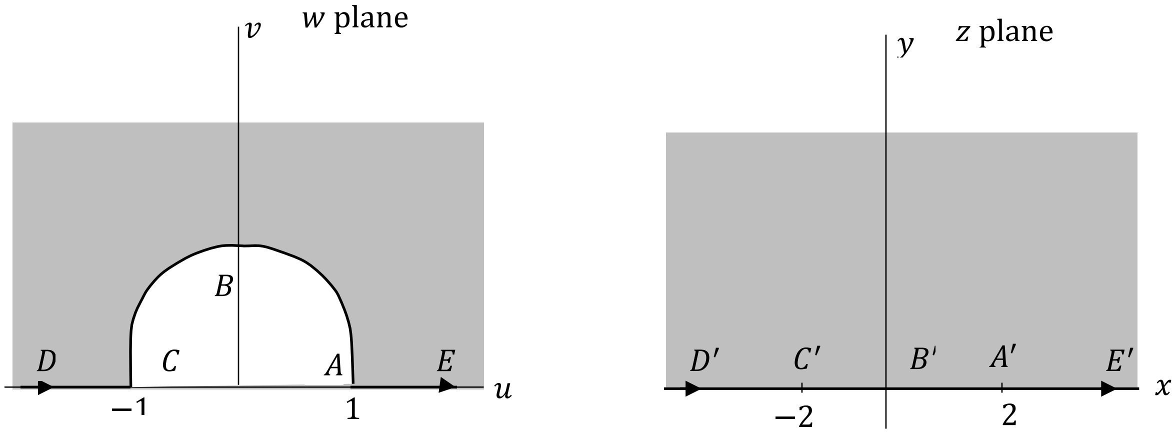

Figure 2(a): One-to-One Mapping of the Shaded Region in the w plane Onto the Upper Half Im z > 0.

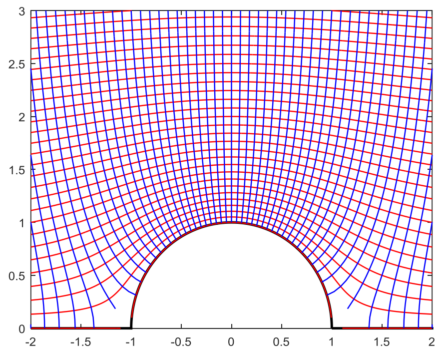

Figure 2(b): Equipotential Lines (Blue) and Streamlines (Red) of flow in the Domain of the First Diagram of figure 2(a).

This problem can be interpreted as a two dimensional flow over a solid cylinder that is cut into two equal parts through its axis and one part placed at the bottom of a deep flowing stream. Here the mapping function is the Joukowski transformation and it maps the half circle $w = e^{i\theta}$ $(0 \le \theta \le \pi)$ in a one-to-one manner onto the line segment $-2 \le x \le 2$, the parts $u < -1$ and $u > 1$ of the u axis in a one-to-one manner onto the portions $x < -2$ and $x > 2$ of the x axis, respectively. It also maps the entire domain above the semi-circular arc in a one-to-one manner onto the entire upper half of the z plane ([1] and [2]). The complex potential for the flow in the problem domain of the w plane is therefore

$$H(w) = v_0\left(w + \frac{1}{w}\right) \quad (19)$$

If $H(w) = \Phi(u, v) + i\Psi(u, v)$ and $w = u + iv$, then separation of the real and imaginary parts yield the velocity potential and stream function of the flow as

$$\Phi(u, v) = v_0\left[\frac{u(u^2+v^2+1)}{u^2+v^2}\right] = v_0\left[\frac{(\rho^2+1)\cos\sigma}{\rho}\right] \quad (20)$$

and

$$\Psi(u, v) = v_0\left[\frac{v(u^2+v^2-1)}{u^2+v^2}\right] = v_0\left[\frac{(\rho^2-1)\sin\sigma}{\rho}\right] \quad (21)$$

respectively, where$w = \rho e^{i\sigma}$.The conjugate of the fluid velocity is

$$\overline{v(w)} = F'(w) = v_0\left(1 - \frac{1}{w^2}\right) = v_0\left(\frac{\rho^2 e^{2\sigma i} - 1}{\rho^2 e^{2\sigma i}}\right) \quad (22)$$

while the fluid speed is

$$|v(w)| = \sqrt{\rho^2 - 2\rho\cos 2\sigma - \rho^{-2}} \quad (23)$$

Problem 3:(Flow around a Circular Cylinder and Mapped Joukowski Airfoil)

Finally, we shall consider equation (1) for the determination of the velocity field and pressure coefficient distribution around Joukowskiairfoils. In this problem, the mapping function is the Joukowski transformation (4) and we consider the case for which $c = 1$.

Now, suppose that $\mathrm{H}(w)$ is the complex potential for the flow around a Joukowski airfoil in the w plane. The composite function

$$\mathrm{H}[(f(z))] = \mathrm{F}(z) \quad (24)$$

where$w = f(z) = z + \frac{1}{z}$ is the Joukowski map,is analytic in the domain exterior to the corresponding circle in the z plane and hence represents the complex potential for the flow there. The complex potential$\mathrm{F}(z)$for the flow around a circular cylinder is well known and given by [6]as

$$\mathrm{F}(z) = v_\infty e^{-i\alpha} z + \frac{v_\infty R^2 e^{i\alpha}}{z - z_0} - \frac{\mathrm{i}\Gamma}{2\pi}\ln(z - z_0) \quad (25)$$

where z_0 is the circle centre, R is the circle radius, α is the flow angle of attack, and Γ is the circulation given as$\Gamma = -4\pi v_\infty R\sin(\alpha + \beta)$ and β is the acute angle between the straight line through the centre of the circle and parallel to the x axis and the radius of the circle from the circle centre to the critical point $z = c$. The circulation in equation (25) is set at the value for which the Kutta condition is satisfied so that when the circle is transformed the critical point $z = 1$ moves to the trailing edge $w = 2$ of the joukowski airfoil while the other x intercept of the circle becomes the leading edge of the airfoil. Differentiating equation (25) with respect to z, we obtain

$$\frac{dF}{dz} = v_\infty e^{-i\alpha} - \frac{v_\infty R^2 e^{i\alpha}}{(z - z_0)^2} - \frac{\mathrm{i}\Gamma}{2\pi(z - z_0)} \quad (26)$$

The fluid speed around a Joukowski airfoil is therefore

$$|v(w)| = \frac{\left|\frac{dF}{dz}\right|}{\left|\frac{dw}{dz}\right|} \qquad (27)$$

The pressure coefficient distributionc_p on an airfoil in an inviscid and incompressible flow is given by [6] as

$$c_p = 1 - \left(\frac{v}{v_\infty}\right)^2 \qquad (28)$$

Substituting equations (26) into equation (27) the pressure coefficient distribution on a Joukowski airfoil can now be computed. Figure 3(b) show the graph of the pressure coefficient distribution on the cambered Joukowski airfoil in figure 3(a) at 10^0 flow angle of attack.

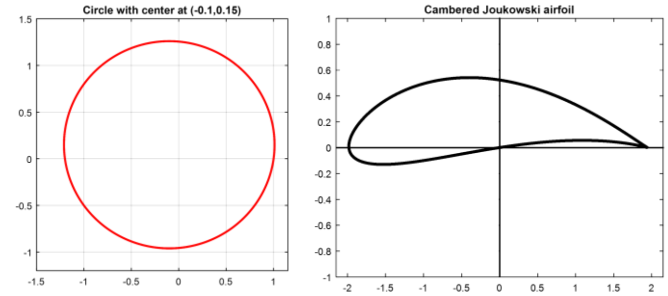

Figure 3(a):Circle with Centre at $z = -0.1 + 0.15i$ and Corresponding Mapped Cambered Joukowski Airfoil for the Case $c = 1$ in the Joukowski Transformation

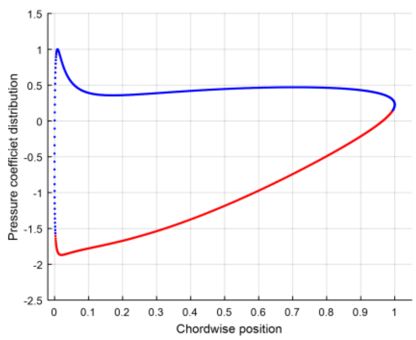

Figure 3(b): Pressure Coefficient Distribution Against Chordwise Position on the Cambered Joukowski Airfoil in figure 3(a) at 10^0 Flow Angle of Attack (Blue Line-c_p on Lower Surface, Red Line-c_p on Upper Surface)

Discussion

Flow in a Semi-Infinite Strip of Width a Units

In analyzing this flow, we first note that the stream function in equation (14) is harmonic throughout the interior of the semi-infinite strip and vanished everywhere on its boundary. Figure 1(b) shows the streamlines and equipotential lines of flow generated in the strip of width π units with $a = \pi$ in the problem. As expected the equipotential and streamlines of the flow are orthogonal at their points of interception since they are level curves of the real and imaginary parts of an analytic function. Closer streamlines in the flow indicate regions of higher fluid speed. The fluid speed therefore decreases as the flow approaches the finite end of the semi-infinite strip as indicated by the pattern of the streamlines. This is expected and in agreement with reality. Notice that the points $w = (\pm\frac{a}{2}, 0)$ corresponding to the vertices of the semi-infinite strip are stagnation points where the fluid speed in equation (18) vanishes identically. The streamlines indicate the actual path taken by the fluid particles in a steady flow while the equipotential lines are curves along which the fluid velocity is constant.

Flow in the Upper Half Im w > 0 of the w plane Bounded by the Semi-Circular Arc $w = e^{i\theta}(0 \le \theta \le \pi)$ and the Lines $u < -1$ and $u > 1$

In this problem too, the stream function of the flow in equation (21) is harmonic in the indicated domain and vanishes everywhere on the boundary. Figure 2(b) show streamlines and equipotential lines of the flow field generated using a MATLAB code. The streamline pattern show that the fluid speed is highest around the surrounding region of the semi-circular arc or barrier. Observe that the flow speed decreases further away from this region and become gradually normal as indicated by the fairly straight and parallel streamlines. Notice from equation (22) that the points $w = (\pm 1,0)$are stagnation points on the boundary of the domain of flow.

Flow around a Circular Cylinder and Mapped Joukowski Airfoil
The velocity field and hence pressure distribution on the upper and lower surfaces of an airfoil is primarily responsible for the lift on it. Figure 3(b) show MATLAB plot of the pressure coefficient distribution against chord wise position on the cambered Joukowski airfoil in figure 3(a). Observe that at each chordwise position the pressure distribution on the lower airfoil surface is higher than that on the upper airfoil surface. This airfoil therefore experiences lift on it due to the net upward vertical pressure. This is in agreement with experimental data on real airfoils in wind tunnels.

Conclusion
In applied mathematics two boundary value problems of great importance are the Dirchlet and Neumann problems. Consequently, we have in this research paper presented a simple but efficient method for solving such problems arising from ideal fluid flows in domains whose boundaries consists of inconvenient geometries. The method which is purely conformal based was applied to ideal flows in asemi-infinite strip, flow in the domain$\text{Im}z > 0$above the semi-circular arc $w = e^{i\theta}(0 \le \theta \le \pi)$and parts of the u axis $-1 < u$ and $u > 1$, and flow around a circular cylinder and corresponding mapped joukowski airfoil. The method gave exact general analytic solutions which were visualized through its equipotential lines and streamlines pattern. However, the method is not without limitations. One obvious limitation has to do with the ability of identifying the mapping function to use for a particular problem. Unfortunately, there is no systematic way of knowing this function but its identification depends largely on someone's experience and familiarity with the manner in which curves and regions are mapped by most analytic functions. Another limitation has to do with the fact that the method is conformal based and hence limited to problems which can be reduced to ones in two dimensions and have a high degree of symmetry. This technique is often difficult to apply when the symmetry is broken.
In closing, we note that the problems considered in this paper were either of the Dirichlet or Neumann type and hence suggest that further research in this field should focus on extending the work to include domains with mixed boundary conditions of both the Dirichlet and Neumann types.

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