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# STRUCTURE OF ELEMENTS IN FINITE PARTIAL TRANSFORMATION SEMIGROUPS

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***Abstract***

***In a finite semigroups S the index and period of an element $\alpha \in S$ are respectively the smallest values of integers m ≥ 1 and r ≥ 1 such that $\alpha^{m+r} = \alpha^{m}$. In the literature, it was shown that every element α of a finite full transformation semigroup $T_n$can be uniquely factorised into a product of a permutation and a element of period 1. In this paper, we extend this concept to element of the larger semigroup $P_n$, of all partial transformation of a finite set of n elements. We show that each $\alpha \in P_n$is factorisable into a product of a permutation and an element of period 1. In line with the literature, factorisation is used to count the number of all elements in $P_n$having period 1.***

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## 1 Introduction

For a fix positive integer *n*, write$X_n = \{1,2,\dots,n\}$, and denote by $S_n$,$T_n$ and $P_n$the symmetric-group, full and partial transformations semigroups on $X_n$respectively. The semigroup $T_n$have been much studied over the last sixty years, see example [1,2 and 3]. Many general concepts in a semigroup have been characterised and examine of Green's relations, ideals, product of idempotent, generating sets, rank and congruence's, have all been examined in the full transformation semigroup $T_n$, see for example [1, 2, 4 and 5]. Ayik et al [6] studiedthe notion of index and period of elements in $T_n$. Let$S$be a semigroup and $a \in S$. If there exist positive integers $m$ and $r$ such that$a^{m+r} = a^{m}$with $a, a^2, \dots, a^{m+r-1}$all pairwise distinct, then $a$ is called an $(m\,,r) - potent$and the integers $m, r$are called the index and period of $a$ respectively, denoted by *index*($a$) and *period*($a$). An element of index $m$and period 1 is called an *m−potent*, and that of both index and period equal to 1 is called an idempotent.

In [6], it was proved that an $(m\,,r) - potent$element $\alpha$ of $T_n$can be uniquely factorized as $\alpha = \sigma\beta$where$\sigma$is a permutation in $S_n$of order $r$ and $\beta$is an $m - potent$element in $T_n$they used this factorisation to count the number of $m - potents$in $T_n$. Here, we extent this idea to cover elements of the partial transformation semigroup $P_n$

## 2 Preliminaries

Let $X_n^0 = X_n \cup \{0\}$ and denote the semigroup of all full transformations of $X_n^0 by T_{X_n^0}$, where $0\alpha = 0$. By a result of [7], quoted in [8] there is an isomorphism between $P_n$and a subsemigroup$X_n^0\ of\ T_{X_n^0}$. This isomorphism proves to be a powerful tool in translating results on $T_n$to very similar results concerning $P_n$, [9]. For each $\alpha \in P_n$, the map $\alpha^* \in T_{X_n^0}$, define by:

$$x\alpha^* = \begin{cases} x\alpha & \text{if } x \in dom(\alpha) \\ 0 & \text{if } x \notin dom(\alpha). \end{cases}$$

For convenience the following result is recorded from [7], also to be found in [8].

**Theorem 2.1** *For each $\alpha \in P_n$, the mappings $\alpha \mapsto \alpha^*$and $\alpha^* \mapsto \alpha^* |X_n$ (the restriction of $\alpha^*$to$X_n$) are mutually inverse isomorphisms of $P_n$onto the subsemigroup$P_n^*\ of\ T_{X_n^0}$ and vice-verse.*

Here, the following important remark is made and it will be effectively used throughout the next sections.

**Remark**: For $r \geq 1$,$m \geq 1$, an $(m, r) - potents\ in\ P_n^*$ corresponds in these isomorphism's to $(m, r)\ -\ potents$in $P_n$.

For each $\alpha \in P_n$, let $Fix(\alpha)$ denote the set of all fixed points of $\alpha$ in $dom(\alpha)$, that is,

$Fix(\alpha) = \{x \in dom(\alpha) | x\alpha = x\}$.

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The set of all shifting points of $\alpha$ in $dom(\alpha)$ is denoted by $Shift(\alpha)$, that is
$Shift(\alpha) = \{x \in dom(\alpha) | x\alpha \neq x\} = dom(\alpha) \setminus Fix(\alpha)$.
Since the semigroup $P_n$ is finite, for each $\alpha \in P_n$, there must be positive integers $m, r$ such that
$\alpha^{m+r} = \alpha^m$.
Thus, smallest such $m$ and $r$ are called index and period of $\alpha$ respectively. Let $T_n$ be *full transformation semigroup* and $P_n$ be *partial transformation semigroup*.
Let $\alpha \in P_n$. For each $x \in dom(\alpha) \setminus im(\alpha)$, the sequence
$x, x\alpha, x\alpha^2, ...$
either arrives into a cycle or at a terminal point.
In the former, we define length of the sequence $l^\alpha(x)$ to be the number of distinct terms in the sequence that are not in the cycle. In the latter, we define $l^\alpha(x)$ to be the number of distinct terms in the sequence.
**Example 2.2** Let $\alpha^* \in P^*_{15}$ be the map

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 0 & 2 & 3 & 4 & 5 & 6 & 4 & 0 & 9 & 10 & 0 & 12 & 0 & 14 & 13 & 15 \end{pmatrix}.$$

We can clearly see that for the above map $\alpha^*$, we have
$x \in \{1, 7, 8, 11\} = X_n^0 \setminus im(\alpha^*)$
, and so $l^{\alpha^*}(1) = 3$, $l^{\alpha^*}(7) = 1$, $l^{\alpha^*}(8) = 3$ and $l^{\alpha^*}(11) = 2$. There for we have the maximum of $l^{\alpha^*}(x)$ to be 3. So the index of $\alpha^*$ is 3 and the period is 6 .
Is isomorphic to:
**Example 2.3** Let $\alpha \in P_{15}$ be the map

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 11 & 13 & 14 & 15 \\ 2 & 3 & 4 & 5 & 6 & 4 & 9 & 10 & 12 & 14 & 13 & 15 \end{pmatrix}.$$

We can clearly see that for the above map $\alpha$, we have
$x \in \{1,8,11\} = dom(\alpha) \setminus im(\alpha)$,
and so $l^\alpha(1) = 3$, $l^\alpha(7) = 1$ and $l^\alpha(11) = 2$. Therefor the maximum of $l^\alpha(x)$ is 3. We say that the index of this $\alpha$ is 3 and the period is 6. Because the period of the maps $\alpha^*$ and $\alpha$ is the least common multiple of the lengths of the cycles. Here we can clearly see that maximum of
$l^{\alpha^*}(x) = l^\alpha(x) = 3$.
The next Lemma shows that the index of each $\alpha$ *in* $P_n$ can be conveniently defined in term of this length function.

**Lemma 2.4** *Let* $\alpha \in P_n$. *Then* $index(\alpha) = max\{l\alpha(x) | x \in dom(\alpha) \setminus im(\alpha)\}$ .
*Proof.* It is clear that every path into a cycle of $\alpha \in P_n$, remains to be the same path into a cycle of $\alpha^* \in P_n^*$. Also a path into a terminal point of $\alpha \in P_n$, remains the same path into a fixed point of $\alpha^* \in P_n^*$. Therefore, by the definition of length of a path $l^\alpha(x) = l^{\alpha^*}(x)$ *for all* $x \in dom(\alpha) \setminus im(\alpha)$. Now, since $l^{\alpha^*}(x) = 1$ *for all* $y \notin dom(\alpha) \cup im(\alpha)$ and in full transformation $T_{X_n^0}$ the index of each map $\alpha^*$ is obtained to be the maximum of $\{l^{\alpha^*}(x) | x \in X_n^0 \setminus im(\alpha)\}$, the result follows.
From the proof of Lemma 2.4, the below corollary 2.5 follows.
**Corollary 2.5** *For each* $\alpha \in P_n$, $index(\alpha) = index(\alpha^*)$ *and* $period(\alpha) = period(\alpha^*)$.

## 3 Decomposition of each $\alpha \in P_n$

In this section we decomposed each $\alpha$ in $P_n$, of index $m$ and period $r$, as a product of permutation $\sigma$ of order $r$ and an $m - potent$ $\beta$ of index $m$.
**Theorem 3.1** *Let $\alpha$ be an element of* $P_n$ *of index* $m$ *and period* $r$. *Then there exist a permutation $\sigma$ of order $r$ in* $S_n$ *(the symmetric group of degree n) and an* $m - potent$ $\beta$ *in* $P_n$ *such that* $\alpha = \sigma\beta$ *and* $\text{Shift}(\sigma) \cap \text{Shift}(\beta) = \emptyset$.

*Proof.* Let $\alpha \in P_n$ be of index $m$ and period $r$. Then by Corollary 2.5 the map $\alpha^* \in P_n^*$ has index $m$ and period $r$ as well. Using the decomposition algorithms described in Theorem 2 of [6], $\alpha^*$ is expressible as a product, $\alpha^* = \sigma^*\beta^*$, where $\sigma^*$ is a permutation of order $r$ *in* $P_n^*$ and $\beta^*$ is an $(m, r) - potents$ *in* $P_n^*$ such that $\text{Shift}(\alpha^*) \cap \text{Shift}(\beta^*) = \emptyset$.
Now $\alpha = \alpha^*|_{X_n} = (\alpha^*\beta^*)|_{X_n} = \alpha^*|_{X_n}\beta^*|_{X_n} = \sigma\beta$
, where $\sigma$ is a permutation of order $r$ since zero (0) is fixed by $\sigma^*$, and $\beta$ is an $m$–*potent* in $P_n$ by Lemma 2.4. It is clear that $\text{Shift}(\sigma) = \text{Shift}(\alpha^*)$ and $\text{Shift}(\beta) \subseteq \text{Shift}(\beta^*)$. Thus $\text{Shift}(\sigma) \cap \text{Shift}(\beta) = \emptyset$ as required.
**Theorem 3.2** *Let $\sigma$ be a permutation of order $r$ in* $S_n$ *and let $\beta$ be an* $m - potent$ *in* $P_n$ *such that* $\text{Shift}(\sigma) \cap \text{Shift}(\beta) = \emptyset$. *Let* $\alpha = \sigma\beta$. *Then, $\alpha$ has index $m$ and period $r$.*

*Proof.* This is a consequence of the corresponding result in full transformation in [6] along with Vagner representation.

**Theorem 3.3** *Let $\alpha$ be an element of $P_n$. Then $\alpha$ is an $(m,r)-potent$ if and only if there exist a unique permutation $\sigma$ of order $r$ and a unique $m-potent$ element $\beta$ such that $\alpha = \sigma\beta$ and* $\text{Shift}(\sigma) \cap \text{Shift}(\beta) = \emptyset$.

*Proof.* From Theorems 3.1 and 3.2, it is clear that the decomposition of $\alpha$ given by theorem 3.1 is unique. It follows from theorem 3.2 that $\alpha$ has index $m$ and period $r$. It is also clear from theorem 3.1 that Shift($\sigma$)∩Shift($\beta$) = ∅.

## 4 Formula for the number of *m−potent* in $P_n$

in this section we identify the number of $m-potent$ elements in the partial transformation semigroup $P_n$. First, we start by partitioning the integers of $m-potent$ paths in $P_n$.

Let $P_{m+1}(n+1)$ be the set of all partitions of the integer $n+1$ into $m+1$ non-zero parts.

**Theorem 4.1** *The number of $m-potent$ elements in $P_n^*$ is*

$$\sum_{\{k_0,k_1,\ldots,k_m\}\in P_{m+1}(n+1)} \binom{n+1}{k_0,k_1,\ldots,k_m} k_0^{k_1} k_1^{k_2} \cdots k_{m-1}^{k_m}.$$

*Proof.* First it can be noticed that, by the power of Vagner representation, it suffices to count the number of *m−potent* elements in $P_n^*$. From Theorem 1 in [6] every $m-potent\ \alpha^*\ in\ P_n^*$ admits a partition of $X_n \cup \{0\}$ into $m+1$ parts $A_0, A_1, \ldots A_m$, in which elements of $A_{j+1}$ are mapped in to elements of ($A_j$ for $j = 0,1,2,\ldots,n-1$).This can be done in $|A_j|^{|A_{j+1}|}$ ways. Now, given a partition $k_0, k_1, \ldots k_m$, of $n+1$ into $m+1$ non-zero parts, we can assign sets $A_0, A_1, \ldots A_m$, with $|A_j| = k_j$ for each $j$ in

$$\binom{n+1}{k_0,k_1,\ldots,k_m} = \frac{(n+1)!}{k_0!k_1!\cdots k_m!}$$

ways. Thus, the number of $m-potent\ \alpha^*\ \ in\ P_n^*$ admitting the partition $A_0, A_1, \ldots A_m$, is

$$\binom{n+1}{k_0,k_1,\ldots,k_m} k_0^{k_1} k_1^{k_2} \cdots k_{m-1}^{k_m}$$

. Therefore, the total number of $m-potent$ elements in $\mathcal{P}_n^*$ is

$$\sum_{\{k_0,k_1,\ldots,k_m\}\in P_{m+1}(n+1)} \binom{n+1}{k_0,k_1,\ldots,k_{m+1}} k_0^{k_1} k_1^{k_2} \cdots k_{m-1}^{k_m}.$$

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