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VIBRATION ANALYSIS OF BEAMS WITH UNIFORM PARTIALLY DISTRIBUTED MASSES UNDER GENERAL BOUNDARY CONDITIONS.

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Abstract

An elegant mathematical solution procedure is developed and used to treat the problem of the dynamic interactions of a beam-load system. Two structural models are considered in this study, approximate analytical solutions of the governing differential equation describing the motions of the vibrating system are presented. Effects of some vital structural parameters on response characteristics and stability of thin beams traversed by travelling loads are established. Influence of the mass ratio and load distribution on the dynamic behaviour of the structural member carrying distributed moving masses is carefully scrutinized. Conditions under which the amplitude of vibration of this structure-mass system may grow without bound for both the moving force and moving mass models are clearly indicated. Results further revealed that, for the same natural frequency, the critical speed for the beam-force system is higher than that of the beam-mass system. Hence, the risk of resonance effects is higher in a beam-mass system.

***Keywords*:** Vibration analysis, Distributed loads, boundary conditions, resonance, bearing member, vibrating system, Dynamic Characteristics.

1.0 INTRODUCTION

The basic understanding of the complexity of the interactions between structural members and the masses traversing them at different velocities is very vital as it helps in controlling structural vibrations and safe operations of such a system. Thus, the practical problem of vibration of structures due to the passage of moving loads is of technological importance and for this reason, there has been increasing need for continuous study of the behaviour of (elastic/inelastic) solid bodies traversed by travelling loads. Such studies are for instance very useful in the design of aircraft which are under the influence of various types of moving pressure loads during flight, design of bridges and runways and it also provides a safer, reliable and more economical design of structural members on which loads of various categories move [1]. Hence, many researchers in engineering, applied mathematics and other related fields have taken considerable interest in the problem of assessing the dynamic response of elastic structures under the action of moving loads [2-16]. The problem of elastic structures under the actions of moving loads has been extensively studied by many authors during the last few years. However, from historical point of view, most of the theories proposed by these authors and the applications of their solution techniques are limited to cases: one; where the complex interactions between structural members and loads traversing them at various velocities are modelled without incorporating foundation stiffness into the governing equation; two, where the inertia effects of the moving load is neglected in the governing equation of motion. Guiseppe and Alessandro [17], once described this type of beam model as the crudest approximation known to literature; Three, where the problem of complex interactions of beam-like structures and fast travelling loads has been significantly simplified by assuming the moving load to be a lumped mass see for instance [18-22] and the references therein. Nonetheless, in practice, moving loads are in the

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form of moving distributed mass rather than moving lumped mass; four, where only simply supported boundary conditions are employed to illustrate the characteristics of the vehicle-load interactions. Other boundary conditions of interest are rarely considered. The reason for all these limitations is not far-fetched when all these all-important factors are involved in governing differential equation of motion, most known analytical procedures or computational techniques break down. Even when they appear to work, a great amount of computational labour is required to solve the resulting complex and much complicated governing differential equation of motion.

Many authors who have made bold effort to address all or some of these aforementioned limitations in their studies, however, are constrained by some great mathematical difficulties in obtaining analytical solutions to the load-structure problems and resorted to numerical simulations [23-29]. It is very important to note that, when considering the dynamic characteristics of a vehicle-load interaction during the passage of the heavy subsystems, analytical method of solution is desirable as it gives more insights and useful information about the dynamical system [30-35]. In our recent study [36], an effort was made to investigate the behavioural study of a finite beam resting on elastic foundation and subjected to travelling distributed masses. All the aforementioned limitations were addressed in this work. Nevertheless, boundary conditions other than the simple ones are not considered. The objective of this study, therefore, is to extend the work in [36] to cover more pertinent boundary conditions of interest.

The specific aim of this study among others is, to obtain the approximate analytical solutions of the governing equations of motions of elastic thin beam on elastic subgrade and under the actions travelling distributed loads for the problems of moving distributed forces and moving distributed masses, establish the dynamic effects of elastic bearing member and other beam parameters on the response characteristics of the beam under moving loads, indicate the conditions under which the vibration amplitude of this dynamical systems grow without bound for both the moving force model and the moving mass model, indicate the influence of the mass ratio on the structure-load interactions of the vibrating system and to determine the effects of load distribution on the dynamic behaviour of the uniform thin beam under forced vibration.

2.0 FORMULATION OF THE EQUATION OF MOTION

Consider a homogeneous beam supported by elastic subgrade and under the actions of travelling masses M. At time t=0, the mass is assumed to strike the beam at the point x = 0 and then continue to travel along with a constant velocity type of motion. The beam mechanical properties are assumed to be constant along the span L of the beam. The deflection $W(x,t)$ describing the motion of the vibrating beam is given by the partial differential equation [36]

$$EIW''''(x,t)-\left(\bar{F}+\bar{S}\right)W''(x,t)+\bar{m}\ddot{W}(x,t)+\bar{K}W(x,t)=P(x,t) \tag{1}$$

where the prime and the over-dot represent the partial derivatives with respect to the spatial coordinate x and the time t respectively, EI is the flexural rigidity, $\bar{F}$ is the axial force, $\bar{K}$ is the foundation constant, $\bar{S}$ is the shear rigidity and $P(x,t)$ is the travelling distributed load.

The structure is assumed to be under general boundary and the initial conditions are taken to be

$$W(x,0)=0=\frac{\partial W(x,0)}{\partial t} \tag{2}$$

Considering the load-track inertia $P(x,t)$ can be given as

$$P(x,t)=P_f(x,t)\left[1-\frac{\Theta[Z(x,t)]}{g}\right] \tag{3}$$

where the travelling force $P_f(x,t)$ traversing this structural member is given as

$$P_f(x,t)=\frac{P_0}{\xi}\left[H\left(x-\gamma+\frac{\xi}{2}\right)-H\left(x-\gamma-\frac{\xi}{2}\right)\right] \tag{4}$$

where P_0 is the constant travelling load and H is the Heaviside step function defined by the property,

$$H(x)=\begin{Bmatrix}0, & x<0\\ 1, & x>0\end{Bmatrix} \tag{5}$$

and

$$\gamma=vt+\frac{\xi}{2} \tag{6}$$

where v is the travelling velocity. ξ approaches zero leads to

$$\delta(x-v)=\frac{1}{\xi}\left[H\left(x-\gamma+\frac{\xi}{2}\right)-H\left(x-\gamma-\frac{\xi}{2}\right)\right] \tag{7}$$

where $\delta(\cdot)$ is the Dirac delta function.

Operator Θ in (3) is given as

$$\Theta[\cdot]=\left(\frac{\partial^2}{\partial t^2}+2\dot{\gamma}\frac{\partial^2}{\partial x\partial t}+\dot{\gamma}^2\frac{\partial^2}{\partial x^2}+\ddot{\gamma}\frac{\partial}{\partial x}\right)[\cdot] \tag{8}$$

equation (7) would naturally reduce the formulation to that of concentrated moving mass problem.
Substituting (3) into (1) and taking into account (4) and (8) one obtains

$$EI\frac{\partial^4 W(x,t)}{\partial x^4}-\bar{F}\frac{\partial^2 W(x,t)}{\partial x^2}+\bar{m}\frac{\partial^2 W(x,t)}{\partial t^2}+\bar{K}W(x,t)-\bar{S}\frac{\partial^2 W(x,t)}{\partial x^2}+\frac{M}{\xi}\left\{H\left(x-\gamma+\frac{\xi}{2}\right)\right. \tag{9}$$

$$\left.-H\left(x-\gamma-\frac{\xi}{2}\right)\right\}\left[\frac{\partial^2 W(x,t)}{\partial t^2}+2v\frac{\partial^2 W(x,t)}{\partial x\partial t}+v^2\frac{\partial^2 W(x,t)}{\partial x^2}\right]=\frac{P_0}{\xi}\left[H\left(x-\gamma+\frac{\xi}{2}\right)-H\left(x-\gamma-\frac{\xi}{2}\right)\right]$$

Where (9) is the equation describing the motions of the beam under the actions of the travelling distributed masses. Evidently, obtaining an exact solution of the governing partial differential equation (9) is not possible. Therefore, in what follows, an approximate analytical solution to the governing equation (9) above is sought.

3.0 SOLUTION PROCEDURE

To calculate the beam deflection $W(x,t)$, use is made of the mode superimposition method. By this method, the beam deflection $W(x,t)$ can be written as

$$W_i(x,t)=\sum_{i=1}^{\infty}R_i(t)S_i(x) \tag{10}$$

where $R_i(t)$ are coordinates in modal space and $S_i(x)$ are the normal modes of free vibration written as

$$S_i(x)=\sin\frac{\lambda_i x}{L}+A_i\cos\frac{\lambda_i x}{L}+B_i\sinh\frac{\lambda_i x}{L}+C_i\cosh\frac{\lambda_i x}{L} \tag{11}$$

Where, λ_i is the mode frequency and the constants A_i, B_i and C_i define the shape and amplitudes of the beam vibration. Their values depend on the boundary conditions associated with the structure. The following property

$$\frac{1}{\xi}\int_0^L f(x)\left[H\left(x-\gamma+\frac{\xi}{2}\right)-H\left(x-\gamma-\frac{\xi}{2}\right)\right]dx=f(\xi)+\left(\frac{\xi}{2}\right)^2\frac{f''(\xi)}{3!}+\left(\frac{\xi}{2}\right)^4\frac{f^{iv}(\xi)}{5!}+\ldots \tag{12}$$

Shall be employed and also the function $P(x,t)$ will be expressed as

$$P(x,t)=\sum_{i=1}^{\infty}S_i(x)\Omega_i(t) \tag{13}$$

where $\Omega_i(t)$ are unknown functions of time.

Thus, substituting the expressions (10) and (12) and (13) into (9) and after some rearrangements yields

$$\sum_{i=1}^{\infty}\bar{m}_i(x)\ddot{R}_i(t)+\left(EIS_i^{iv}(x)-\bar{F}S_i''(x)+\bar{K}S_i(x)-\bar{S}S_i''(x)\right)R_i(t)$$

$$=\sum_{i=1}^{\infty}\left\{\frac{P_0}{S_0(i,j)}\left(S_j(\xi)+\frac{\xi^2}{24}S_j''(\gamma)\right)-\frac{M}{S_0(i,j)}\sum_{\substack{j=1\\ j\neq i}}^{\infty}\left(S_i(\gamma)S_j(\gamma)+\frac{\xi^2}{24}\left[S_i(\gamma)S_j''(\gamma)+S_j''(\gamma)S_j(\gamma)+2S_i'(\gamma)S_j'(\gamma)\right]\right)\ddot{R}_j(t)\right. \tag{14}$$

$$+2v\left(S_i'(\gamma)S_j(\gamma)+\frac{\xi^2}{24}\left[S_i'(\gamma)S_j''(\gamma)+2S_j''(\gamma)S_j'(\gamma)+S_i'''(\gamma)S_j(\gamma)\right]\right)\dot{R}_j(t)$$

$$\left.+v^2\left(S_i''(\gamma)S_j(\gamma)+\frac{\xi^2}{24}\left[S_i''(\gamma)S_j''(\gamma)+2S_j'''(\gamma)S_j'(\gamma)+S^{iv}{}_i(\gamma)S_j(\gamma)\right]\right)R_j(t)\quad\right]\right\}\times S_i(x)$$

To obtain an expression for $R_i(t)$, the expression on the left-hand side of equation (14) is required to be orthogonal to the function $S_j(x)$. Thus, multiplying equation (14) by $S_j(x)$ and integrating from end x = 0 to end x=L leads to

$$\sum_{i=1}^{\infty}\left\{ \ddot{R}_i(t)+\gamma_{mf}^2 R_i(t)+\frac{M}{\mu L}H_1^*(i,j)\sum_{j=1}^{\infty}\left\{ \left(S_i(\gamma)S_j(\gamma)+\frac{\xi^2}{24}\left[S_i(\gamma)S_j''(\gamma)+S_j''(\gamma)S_j(\gamma)+2S_i'(\gamma)S_j'(\gamma)\right]\right)\ddot{R}_j(t)\right.\right.$$
$$+2v\left(S_i'(\gamma)S_j(\gamma)+\frac{\xi^2}{24}\left[S_i'(\gamma)S_j''(\gamma)+2S_j''(\gamma)S_j'(\gamma)+S_i''(\gamma)S_j(\gamma)\right]\right)\dot{R}_j(t)$$
$$\left.\left.+v^2\left(S_i''(\gamma)S_j(\gamma)+\frac{\xi^2}{24}\left[S_i''(\gamma)S_j''(\gamma)+2S_j''(\gamma)S_j'(\gamma)+S^{iv}{}_i(\gamma)S_j(\gamma)\right]\right)R_j(t)\right\}\right\}=\frac{P_0}{\bar{m}H_1(i,j)}\left(S_j(\gamma)+\frac{\xi^2}{24}S_j''(\gamma)\right) \tag{15}$$

Where

$$H_1(i,j)=\int_0^L S_i(x)S_j(x)dx \qquad H_2(i,j)=\int_0^L S_i^{iv}(x)S_j(x)dx \qquad H_3(i,j)=\int_0^L S_i''(x)S_j(x)dx \tag{16}$$

$$H_4(i,j)=\int_0^L S_i(x)S_j(x)dx \qquad H_5(i,j)=\int_0^L S_i''(x)S_j(x)dx$$

and

$$\gamma_{mf}^2=\frac{EIH_2(i,j)-\bar{F}H_3(i,j)+\bar{K}H_4(i,j)-\bar{S}H_5(i,j)}{\mu H_1(i,j)} \quad \text{and} \quad H_1^*(i,j)=\frac{L}{H_1(i,j)} \tag{17}$$

When the ith particle of the system is considered, we have

$$\ddot{R}_i(t)+\gamma_{mf}^2 R_i(t)+\Gamma_a\sum_{\substack{j=1\\ j\neq i}}^{\infty}\left\{ \left(S_i(\gamma)S_j(\gamma)+\frac{\xi^2}{24}\left[S_i(\gamma)S_j''(\gamma)+S_j''(\gamma)S_j(\gamma)+2S_i'(\gamma)S_j'(\gamma)\right]\right)\ddot{R}_j(t)\right.$$
$$+2v\left(S_i'(\gamma)S_j(\gamma)+\frac{\xi^2}{24}\left[S_i'(\gamma)S_j''(\gamma)+2S_j''(\gamma)S_j'(\gamma)+S_i''(\gamma)S_j(\gamma)\right]\right)\dot{R}_j(t)$$
$$\left.+v^2\left(S_i''(\gamma)S_j(\gamma)+\frac{\xi^2}{24}\left[S_i''(\gamma)S_j''(\gamma)+2S_j'''(\gamma)S_j'(\gamma)+S^{iv}{}_i(\gamma)S_j(\gamma)\right]\right)R_j(t)\right\}\Bigg\}=\frac{P_0}{\bar{m}H_1(i,j)}\left(S_j(\gamma)+\frac{\xi^2}{24}S_j''(\gamma)\right) \tag{18}$$

where

$$\Gamma_o=\frac{M}{\mu L} \tag{19}$$

Equation (18) is the transformed equation governing the motion of the beam resting on bi-parametric elastic foundation and subjected to travelling distributed masses. In what follows, a closed-form solution of equation (18) is sought, to this end, two special cases of equation (18) shall be discussed.

4.0 SOLUTION TO THE TRANSFORMED GOVERNING EQUATION

This section seeks to obtain the solution of equation (18). To do this, two special cases of equation (18) is considered: one, when only the force effect is considered and two when the inertia or gravitational effects of the travelling load is considered.

Case 1: The beam-load travelling force System

The model corresponding to the moving force of the governing equation (18) may be obtained by setting $\Gamma_o=0$. In this case, one obtains

$$\ddot{R}_i(t)+\gamma_{mf}^2 R_i(t)=\frac{P_0}{\bar{m}H_1(i,j)}\left(S_j(\gamma)+\frac{\xi^2}{24}S_j''(\gamma)\right) \tag{20}$$

equation (20), taking into account equation (6) can further be re-written as,

$$\ddot{R}_i(t)+\gamma_{mf}^2 R_i(t)=P_z\left[b_0\sin\frac{\lambda_i(vt)}{L}+b_1\cos\frac{\lambda_i(vt)}{L}+b_2\sinh\frac{\lambda_i(vt)}{L}+b_3\cosh\frac{\lambda_i(vt)}{L}\right] \tag{21}$$

Where

$$P_z=\frac{Mg}{\bar{m}H_1(i,j)},\quad a_0=\cos\frac{\lambda_i\xi}{2L}-A_i\sin\frac{\lambda_i\xi}{2L},\quad a_1=\sin\frac{\lambda_i\xi}{2L}+A_i\cos\frac{\lambda_i\xi}{2L},\quad a_2=B_i\cosh\frac{\lambda_i\xi}{2L}+C_i\sinh\frac{\lambda_i\xi}{2L}$$

$$a_3=B_i\sinh\frac{\lambda_i\xi}{2L}+C_i\cosh\frac{\lambda_i\xi}{2L},\quad b_0=a_0\left[1-\frac{1}{24}\left(\frac{\lambda_i\xi}{L}\right)^2\right],\quad b_1=a_1\left[1-\frac{1}{24}\left(\frac{\lambda_i\xi}{L}\right)^2\right],$$

$$b_2=a_2\left[1+\frac{1}{24}\left(\frac{\lambda_i\xi}{L}\right)^2\right],\quad b_3=a_3\left[1+\frac{1}{24}\left(\frac{\lambda_i\xi}{L}\right)^2\right] \tag{22}$$

Equation (20) is now the governing moving force model associated with this dynamical problem.
In what follows, an expression for $R_i(t)$ is sought. To do this, equation (20) is subjected to a Laplace transform defined as

$$(\&) = \int_0^\infty (\cdot)e^{-st}dt \tag{23}$$

where s is the Laplace parameter. Subjecting equation (21) to Laplace transform in conjunction with equation (2), yields

$$s^2R_i(s) + \gamma_{mf}^2 R_i(s) = P_z\left[\frac{b_0\omega_i}{s^2+\omega_i^2} + \frac{b_1 s}{s^2+\omega_i^2} + \frac{b_2\omega_i}{s^2-\omega_i^2} + \frac{b_3 s}{s^2-\omega_i^2}\right] \tag{24}$$

where

$$\omega_i = \frac{\lambda_i v}{L} \tag{25}$$

Equation (24) after some simplifications and rearrangements yields

$$R_i(s) = P_z\left[\frac{b_0\omega_i}{s^2+\omega_i^2}\cdot\frac{1}{s^2+\gamma_{mf}^2} + \frac{b_1 s}{s^2+\omega_i^2}\cdot\frac{1}{s^2+\gamma_{mf}^2} + \frac{b_2\omega_i}{s^2-\omega_i^2}\cdot\frac{1}{s^2+\gamma_{mf}^2} + \frac{b_3 s}{s^2-\omega_i^2}\cdot\frac{1}{s^2+\gamma_{mf}^2}\right] \tag{26}$$

To obtain the Laplace inversion of (26), use is made of the following representations.

$$f(s) = \frac{1}{s^2+\gamma_{mf}^2},\quad g_1(s) = \frac{b_0\omega_i}{s^2+\omega_i^2},\quad g_2(s) = \frac{b_1 s}{s^2+\omega_i^2},\quad g_3(s) = \frac{b_2\omega_i}{s^2-\omega_i^2},\quad g_4(s) = \frac{b_3 s}{s^2-\omega_i^2} \tag{27}$$

The Laplace inversion of equation (26) is defined as

$$f(s) * g_i(s) = \int_0^t f(t-u)g_i(u)du,\quad i = 1,2,3,4 \tag{28}$$

Thus, in view of equation (28), noting (27), the Laplace inversion of equation (26) is given as

$$R_i(t) = \frac{P_z}{\gamma_{mf}}\left[b_0Q_1 + b_1Q_2 + b_2Q_3 + b_3Q_4\right] \tag{29}$$

where

$$Q_1 = \int_0^t \sin\gamma_{mf}(t-u)\sin\omega_i u\,du,\ Q_2 = \int_0^t \sin\gamma_{mf}(t-u)\cos\omega_i u\,du,\ Q_3 = \int_0^t \sin\gamma_{mf}(t-u)\sinh\omega_i u\,du$$

$$Q_4 = \int_0^t \sin\gamma_{mf}(t-u)\cosh\omega_i u\,du \tag{30}$$

Evaluating integrals (30), above yields

$$Q_1 = \frac{1}{\gamma_{mf}^2-\omega_i^2}\left[\gamma_{mf}\sin\omega_i t - \omega_i\sin\gamma_{mf}t\right],\ Q_2 = \frac{\gamma_{mf}}{\gamma_{mf}^2-\omega_i^2}\left[\cos\omega_i t - \cos\gamma_{mf}t\right],$$

$$Q_3 = \frac{1}{\gamma_{mf}^2+\omega_i^2}\left[\gamma_{mf}\sinh\omega_i t - \omega_i\sin\gamma_{mf}t\right],\ Q_4 = \frac{1}{\gamma_{mf}^2+\omega_i^2}\left[\gamma_{mf}\cosh\omega_i t - \gamma_{mf}\cos\gamma_{mf}t\right] \tag{31}$$

Substituting equation (31) into (29) gives an expression for $R_i(t)$ as

$$R_i(t) = \frac{P_Z}{\gamma_{mf}(\gamma_{mf}^4-\omega_i^4)}\Big\{b_0(\gamma_{mf}^2+\omega_i^2)\left[\gamma_{mf}\sin\omega_i t - \omega_i\sin\gamma_{mf}t\right] + b_2(\gamma_{mf}^2-\omega_i^2)\left[\gamma_{mf}\sinh\omega_i t - \omega_i\sin\gamma_{mf}t\right]$$
$$+b_1\gamma_{mf}(\gamma_{mf}^2+\omega_i^2)\left[\cos\omega_i t - \cos\gamma_{mf}t\right] + b_3\gamma_{mf}(\gamma_{mf}^2-\omega_i^2)\left[\cosh\omega_i t - \cos\gamma_{mf}t\right]\Big\} \tag{32}$$

Thus, the beam deflection $W(x,t)$ in view of (10), gives

$$W(x,t) = \sum_{i=1}^{\infty}\left(\frac{P_z}{\gamma_{mf}(\gamma_{mf}^4-\omega_i^4)}\Big\{ b_0(\gamma_{mf}^2+\omega_i^2)\left[\gamma_{mf}\sin\omega_i t - \omega_i\sin\gamma_{mf}t\right] + b_2(\gamma_{mf}^2-\omega_i^2)\left[\gamma_{mf}\sinh\omega_i t - \omega_i\sin\gamma_{mf}t\right]\right.$$
$$\left.+b_1\gamma_{mf}(\gamma_{mf}^2+\omega_i^2)\left[\cos\omega_i t - \cos\gamma_{mf}t\right] + b_3\gamma_{mf}(\gamma_{mf}^2-\omega_i^2)\left[\cosh\omega_i t - \cos\gamma_{mf}t\right]\Big\}\right)\cdot\left(\sin\frac{\lambda_i x}{L} + A_i\cos\frac{\lambda_i x}{L} + B_i\sinh\frac{\lambda_i x}{L} + C_i\cosh\frac{\lambda_i x}{L}\right) \tag{33}$$

as the equation representing the response of an elastic thin beam to travelling distributed forces. Equation (33) holds for all boundary conditions of interest.

Case II: The Moving Mass System

This section seeks to obtain the solution of the governing equation (18) when $\Gamma_0 \neq 0$. It Γ_0 is set to zero in equation (18), the model corresponding to the moving mass of the governing equation (18) is obtained. Thus, the solution of the entire equation (18) when no term of the governing differential equation is neglected is required. In this case, obtaining an exact analytical solution of the governing equation becomes impossible. Thus, an asymptotic method of Struble discussed in [20, 21] is resorted to. The modified frequency of the free system due to the presence of the gravitational effects of the travelling distributed masses may be obtained by this asymptotic method. To this effect, equation (18) is rearranged to take the form

$$\begin{aligned}
&\ddot{R}_i(t) + \left(\frac{2\nu\Gamma_a H_1^*(i,j)\left\{N_4 - U_2(i,t) + \frac{\xi^2}{24}\left(N_5 - 3U_4(i,t) + N_6 - U_5(i,t)\right)\right\}}{1+\Gamma_a H_1^*(i,j)\left\{N_1 + U_1(i,t) + \frac{\xi^2}{12}\left(N_2 - U_2(i,t) + N_3 - U_3(i,t)\right)\right\}} \right)\dot{R}_i(t) \\
&+ \left(\frac{\gamma_{mf}^2 + \nu^2\Gamma_a H_1^*(i,j)\left\{N_7 - U_3(i,t) + \frac{\xi^2}{24}\left(N_8 - U_6(i,t) + N_9 + U_7(i,t) + N_{10} + U_8(i,t)\right)\right\}}{1+\Gamma_a H_1^*(i,j)\left\{N_1 + U_1(i,t) + \frac{\xi^2}{12}\left(N_2 - U_2(i,t) + N_3 - U_3(i,t)\right)\right\}} \right) R_i(t) \\
&+ \\
&\sum_{\substack{j=1\\ j\neq i}}^{\infty}\left(\frac{\Gamma_a H_1^*(i,j)\left\{U_9(i,j,t) + \frac{\xi^2}{12}\left(U_{10}(i,j,t) + U_{11}(i,j,t)\right)\right\}}{1+\Gamma_a H_1^*(i,j)\left\{N_1 + U_1(i,t) + \frac{\xi^2}{12}\left(N_2 - U_2(i,t) + N_3 - U_3(i,t)\right)\right\}} \right)\ddot{R}_j(t) \\
&+ \left(\frac{2\nu\Gamma_a H_1^*(i,j)\left\{U_{12}(i,j,t) + \frac{\xi^2}{24}\left(3U_{13}(i,j,t) + U_{14}(i,j,t)\right)\right\}}{1+\Gamma_a H_1^*(i,j)\left\{N_1 + U_1(i,t) + \frac{\xi^2}{12}\left(N_2 - U_2(i,t) + N_3 - U_3(i,t)\right)\right\}} \right)\dot{R}_j(t) \\
&+ \left(\frac{\nu^2\Gamma_a H_1^*(i,j)\left\{U_{15}(i,j,t) + \frac{\xi^2}{24}\left(2U_{16}(i,j,t) + U_{17}(i,j,t) + U_{18}(i,j,t)\right)\right\}}{1+\Gamma_a H_1^*(i,j)\left\{N_1 + U_1(i,t) + \frac{\xi^2}{12}\left(N_2 - U_2(i,t) + N_3 - U_3(i,t)\right)\right\}} \right) R_j(t) \\
&= \frac{\frac{\Gamma_a L g}{\bar{m} H_1(i,j)}\left(R_j(\gamma) + \frac{\xi^2}{24}R_j''(\gamma)\right)}{1+\Gamma_a H_1^*(i,j)\left\{N_1 + U_1(i,t) + \frac{\xi^2}{12}\left(N_2 - U_2(i,t) + N_3 - U_3(i,t)\right)\right\}}
\end{aligned} \tag{34}$$

where

$$N_1 = a_0^2 - a_2^2,\quad N_2 = \left(\frac{\lambda_i}{L}\right)\left[a_0 a_1 + a_2 a_3\right],\quad N_3 = \left(\frac{\lambda_i}{L}\right)^2\left[a_0^2 + a_2 a_3\right],\quad N_4 = \left(\frac{\lambda_i}{L}\right)\left[a_0 a_1 + a_2 a_3\right]$$

$$N_5 = 3\left(\frac{\lambda_i}{L}\right)^3\left[a_3^2 - a_0 a_1\right],\quad N_6 = \left(\frac{\lambda_i}{L}\right)^3\left[a_2 a_3 - a_0 a_1\right],\quad N_7 = \left(\frac{\lambda_i}{L}\right)^2\left[a_0^2 + a_2 a_3\right],\quad N_8 = 2\left(\frac{\lambda_i}{L}\right)^4\left[a_1^2 + a_3^2\right] \tag{35}$$

$$N_9 = \left(\frac{\lambda_i}{L}\right)^4\left[a_0^2 - a_3^2\right],\quad N_{10} = \left(\frac{\lambda_i}{L}\right)^4\left[a_0^2 + a_2^2\right]$$

Following the Struble's technique procedures extensively discussed in [20, 21, 35], the homogeneous part of equation (34) is simplified to take the form

$$\frac{d^2 R_i(t)}{dt^2} + \gamma_{mm}^2 R_i(t) = 0 \tag{36}$$

Where

$$\gamma_{mm} = \gamma_{mf}\left[1 - \frac{\Gamma^* H_1^*(i,j)}{2\gamma_{mf}^2}\left[\gamma_{mf}^2\left(N_1 + N_2\frac{\xi^2}{24} + N_3\frac{\xi^2}{12}\right) - \nu^2\left(N_7 + N_8\frac{\xi^2}{24} + N_9\frac{\xi^2}{24} + N_{10}\frac{\xi^2}{24}\right)\right]\right] \tag{37}$$

is the so-called modified frequency corresponding to the frequency of the dynamical system due to the gravitational effects of the travelling mass. Equation (34) in view of (36) reduces to

$$\frac{d^2R_i(t)}{dt^2}+\gamma_{mm}^2R_i(t)=\frac{\Gamma^*Lg}{H_1(i,j)}\left[b_0\sin\frac{\lambda_i(vt)}{L}+b_1\cos\frac{\lambda_i(vt)}{L}+b_2\sinh\frac{\lambda_i(vt)}{L}+b_3\cosh\frac{\lambda_i(vt)}{L}\right] \quad (38)$$

The solution of equation (38) is obtained following the previous arguments and procedures as

$$R_i(t)=\frac{P_F}{\gamma_{mm}(\gamma_{mm}^4-\omega_i^4)}\Big\{ b_0(\gamma_{mm}^2+\omega_i^2)\left[\gamma_{mm}\sin\omega_i t-\omega_i\sin\gamma_{mm}t\right]+b_2(\gamma_{mm}^2-\omega_i^2)\left[\gamma_{mm}\sinh\omega_i t-\omega_i\sin\gamma_{mm}t\right] \quad (39)$$

$$+b_1\gamma_{mm}(\gamma_{mm}^2+\omega_i^2)\left[\cos\omega_i t-\cos\gamma_{mm}t\right]+b_3\gamma_{mm}(\gamma_{mm}^2-\omega_i^2)\left[\cosh\omega_i t-\cos\gamma_{mm}t\right]\Big\}$$

which leads to

$$W(x,t)=\sum_{i=1}^{\infty}\left(\frac{P_F}{\gamma_{mm}(\gamma_{mm}^4-\omega_i^4)}\Big\{ b_0(\gamma_{mm}^2+\omega_i^2)\left[\gamma_{mm}\sin\omega_i t-\omega_i\sin\gamma_{mm}t\right]+b_2(\gamma_{mm}^2-\omega_i^2)\left[\gamma_{mm}\sinh\omega_i t-\omega_i\sin\gamma_{mm}t\right] \quad (40)$$

$$+b_1\gamma_{mm}(\gamma_{mm}^2+\omega_i^2)\left[\cos\omega_i t-\cos\gamma_{mm}t\right]+b_3\gamma_{mm}(\gamma_{mm}^2-\omega_i^2)\left[\cosh\omega_i t-\cos\gamma_{mm}t\right]\Big\}\right)\cdot\left(\sin\frac{\lambda_i x}{L}+A_i\cos\frac{\lambda_i x}{L}+B_i\sinh\frac{\lambda_i x}{L}+C_i\cosh\frac{\lambda_i x}{L}\right)$$

where

$$P_F=\frac{\Gamma^*Lg}{H_1(i,j)} \quad (41)$$

equation (40) represents the response of an elastic thin beam to travelling distributed masses. Equation (40) holds for all boundary conditions of interest.

5.0 COMMENTS ON THE CLOSED FORM SOLUTIONS

In any study that pertains to structural vibration, it is very crucial to pay particular attention to the occurrence of the phenomenon called resonance. This phenomenon may be experienced when the vibration of an elastic beam becomes unbounded. Conditions under which this may occur is established in this section. From equation (35), it is evident that the vibrating beam may experience resonance effects whenever

$$\gamma_{mf}=\frac{\lambda_i v}{L} \quad (42)$$

Similarly, equation (40) shows that the same beam under the actions of moving distributed masses will experience resonance effects whenever

$$\gamma_{mm}=\frac{\lambda_i v}{L} \quad (43)$$

Recall from (37) that

$$\gamma_{mm}=\gamma_{mf}\left[1-\frac{\Gamma^*H_1^*(i,j)}{2\gamma_{mf}^2}\left[\gamma_{mf}^2\left(N_1+N_2\frac{\xi^2}{24}+N_3\frac{\xi^2}{12}\right)-v^2\left(N_7+N_8\frac{\xi^2}{24}+N_9\frac{\xi^2}{24}+N_{10}\frac{\xi^2}{24}\right)\right]\right] \quad (44)$$

which implies

$$\gamma_{mf}=\frac{\frac{\lambda_i v}{L}}{1-\frac{\Gamma^*H_1^*(i,j)}{2\gamma_{mf}^2}\left[\gamma_{mf}^2\left(N_1+N_2\frac{\xi^2}{24}+N_3\frac{\xi^2}{12}\right)-v^2\left(N_7+N_8\frac{\xi^2}{24}+N_9\frac{\xi^2}{24}+N_{10}\frac{\xi^2}{24}\right)\right]} \quad (45)$$

It can be deduced that the critical speed for the system under the actions of travelling distributed force is greater than that of moving distributed mass. This implies that, for the same natural frequency, resonance is reached earlier in the moving distributed mass than in the moving distributed force system for all boundary conditions of interest.

6.0 NUMERICAL RESULT AND DISCUSSION

In this section, the analysis proposed in the previous sections is illustrated by considering a homogenous beam of modulus of elasticity $E=2.9012\times10^9 N/m^2$, the moment of inertia $I=2.87698\times10^{-3}kgm^2$, the beam span $L=35m$ and the mass per unit length of the beam $\overline{m}=2758.291kg/m$. The load is also assumed to travel with constant velocity $v=8.128m/s$. The values of foundation moduli varied between $0N/m^3$ and $40000N/m^3$, the values of axial force N varied between $0N$ and $2.0\times10^7 N$ and the values of Shear moduli G varied between $0N$ and $3.0\times10^6 N$.

6.1 Clamped-Clamped ends condition

Fig.1 displays the transverse displacement response of a clamped-clamped uniform beam under the action of uniform partially distributed forces moving at a constant velocity for the various values of axial force N and for fixed values of subgrade moduli K and shear modulus G. The Figure shows that as N increases, the response amplitude of the uniform beam decreases. Similar results are obtained when the fixed-fixed beam is subjected to partially distributed masses travelling at a constant velocity as shown in Fig.7. For a various travelling time t, the displacement response of the beam for various values of subgrade moduli K and for fixed values of axial force $N = 20000$ and shear modulus $G = 30000$ are shown in Fig.2. It is observed that higher values of subgrade moduli K reduce the deflection of the vibrating beam. The same behaviour characterizes the response of the clamped-clamped beam under the actions of uniform partially distributed masses moving at a constant velocity for various values of subgrade moduli K as shown in Fig.8. Also, Figs.3 and 9 display the deflection profile of the clamped-clamped uniform beam respectively to partially distributed forces and masses travelling at a constant velocity for various values of shear modulus and fixed values of axial force $N = 20000$ and subgrade moduli $K = 40000$. These figures clearly show that as the value of the shear moduli increases, the deflection of the clamped-clamped uniform beam under the action of both moving forces and masses travelling at constant velocity decreases.

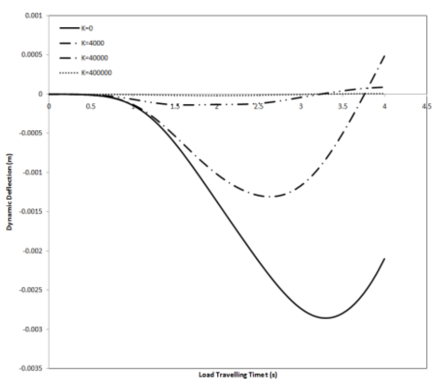

Fig.1: Displacement response of a clamped-clamped structural members resting on elastic foundation and under the actions of uniform partially distributed forces for various values of foundation modulus K and for fixed values of N = 20000, G = 30000.

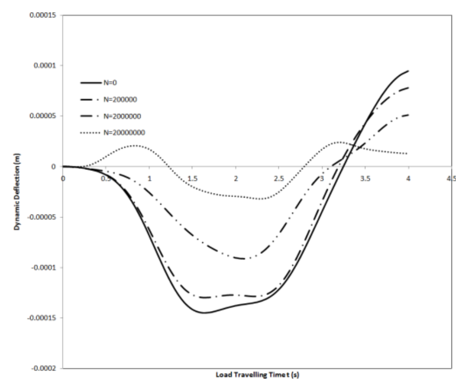

Fig.2: Displacement response of a clamped-clamped structural members resting on elastic foundation and under the actions of uniform partially distributed forces for various values of foundation modulus N and for fixed values of K = 40000, G = 30000.

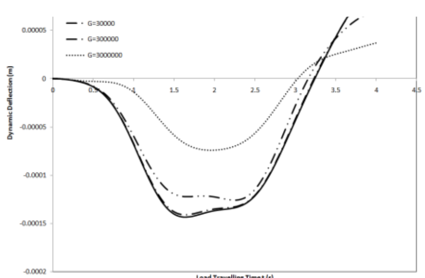

Fig.3:Deflection Profile of a clamped-clamped structural members resting on elastic foundation and under the actions of uniform partially distributed forces for various values of shear modulus G and for fixed values of K=40000 and N = 20000.

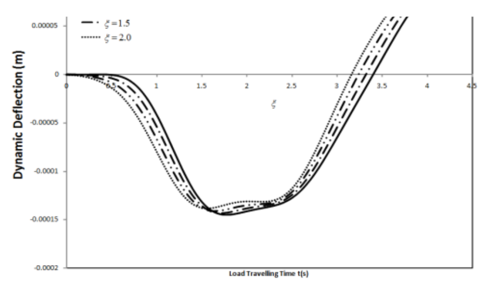

Fig.4: Response Amplitude of a clamped-clamped structural members resting on elastic foundation and under the actions of uniform partially distributed forces for various values of the load width ξ and for fixed values of G = 30000, K=40000 and $N = 20000$.

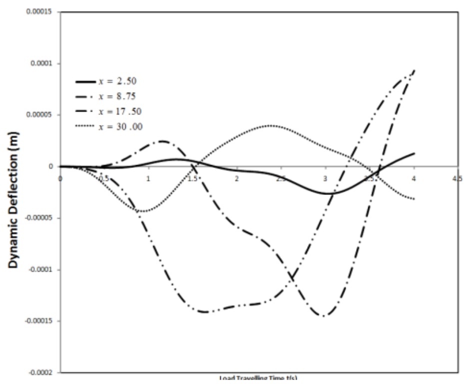

Fig.5:Transverse displacement response of a clamped-clamped structural members resting on elastic foundation and under the actions of uniform partially distributed forces for various values of axial force N and for fixed values of K = 40000, G = 30000.

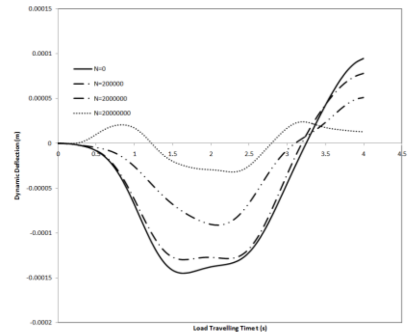

Fig.6: Displacement response of a clamped-clamped structural members resting on elastic foundation and under the actions of uniform partially distributed forces for various values of foundation modulus K and for fixed values of N = 20000$, G = 30000.

Fig.4 displays the response amplitude of a clamped-clamped uniform beam under the action of uniform partially distributed forces travelling at a constant velocity for various values of load width and for fixed values of subgrade moduli axial force N = 20000, subgrade moduli K=40000 and shear modulus G = 30000. The figures show that as the values of the load width increases, the effects of the width on the response amplitude of the uniform beam increases as the load progresses on the structure. Similar results are obtained when the clamped-clamped beam is subjected to partially distributed masses travelling at a constant velocity as shown in Fig.10. For a various travelling time t, the response of the beam for various values of travelling load positions x and for fixed values of axial force, subgrade modulus and shear modulus G = 30000 are shown in Fig 5. It is observed that the impact of the travelling load is greatest in the middle of this vibrating solid structure. The same behaviour characterizes the response of the clamped-clamped beam under the actions of uniform partially distributed masses moving at a constant velocity for different travelling load positions as shown in Fig.11.

The Dynamic deflections of the clamped-clamped uniform beam under distributed forces and masses for various values of the velocity v of the motion are respectively displayed in Figs. 6 and 12. These figures clearly show that as the value of the velocity of the motion increases, the deflection amplitude of the clamped-clamped uniform beam under the action of both moving force and mass respectively decreases. For a various travelling time t, the response amplitude of the clamped-clamped uniform beam under travelling masses is shown in Fig. 13. It is observed that the larger the value of the mass ratio, Γ^*, the larger the response amplitude of the beam.

Figures 14 and 15 depict the comparison of the response characteristics of the moving force and moving mass cases of a clamped-clamped uniform beam traversed by a moving distributed load travelling at a constant velocity for fixed values of $N = 0$, $K = 0$, $G = 0$ and $N = 20000$, $K = 40000$, and $G = 30000$. From these figures, it is seen that the dynamic deflection of the beam under the actions of the moving load is greatly affected when the structural parameters N, K and G are incorporated into the governing equation of motion. Figure 16 compares the deflection profiles of the moving force model of the beam for the two sets of values $K = 0$, $N = 0$, $G = 0$ and $N = 20000$, $K = 40000$ and $G = 30000$. It is deduced from this figure that the amplitude of deflection for the set of values $K = 0$, $N = 0$, $G = 0$ is much higher than that of the set of values $N = 20000$, $K = 40000$ and $G = 30000$. Similar result is obtained for a moving mass model of this structural member as shown in figure 17.

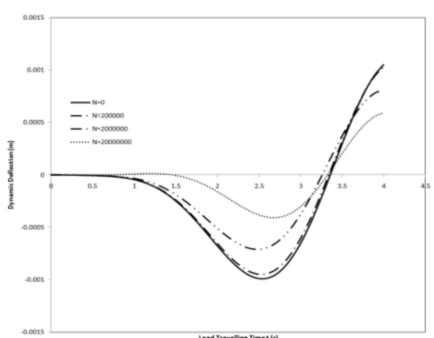

Fig.7: Transverse displacement response of a clamped-clamped structural members resting on elastic foundation and under the actions of uniform partially distributed masses for various values of axial force N and for fixed values of K = 40000, G = 30000.

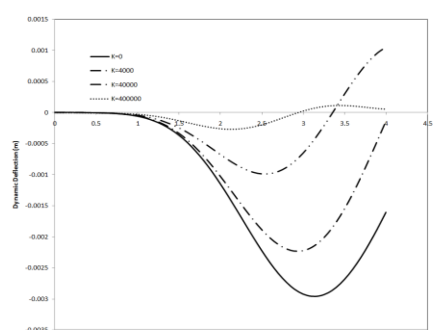

Fig.8: Displacement response of a clamped-clamped structural members resting on elastic foundation and under the actions of uniform partially distributed masses for various values of foundation modulus K and for fixed values of N = 20000, G = 30000.

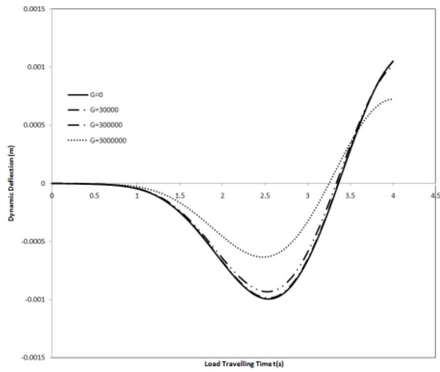

Fig.9: Deflection Profile of a clamped-clamped structural members resting on elastic foundation and under the actions of uniform partially distributed masses for various values of shear modulus G and for fixed values of K=40000 and N = 20000.

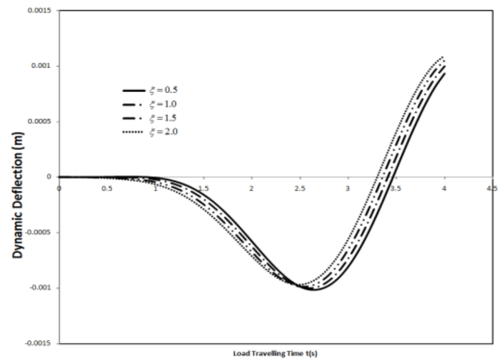

Fig.10: Response Amplitude of a clamped-clamped structural members resting on elastic foundation and under the actions of uniform partially distributed masses for various values of the load width ξ and for fixed values of G = 30000, K=40000 and N = 20000.

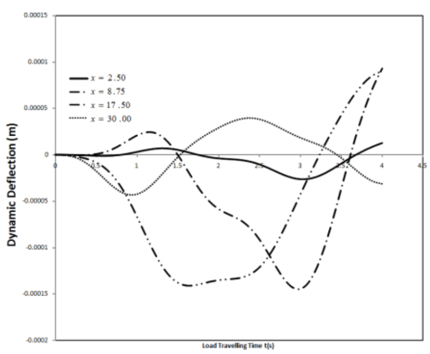

Fig.11: Response of a clamped-clamped structural members resting on elastic foundation to uniform partially distributed masses for various values of the load position X and for fixed values of G = 30000, K=40000 and N = 20000.

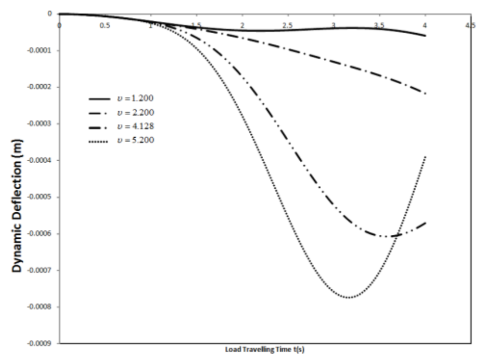

Fig.12: Response characteristics of a clamped-clamped structural members resting on elastic foundation to uniform partially distributed masses for various values of the travelling load velocities V and for fixed values of G=30000, K=40000 and N = 20000.

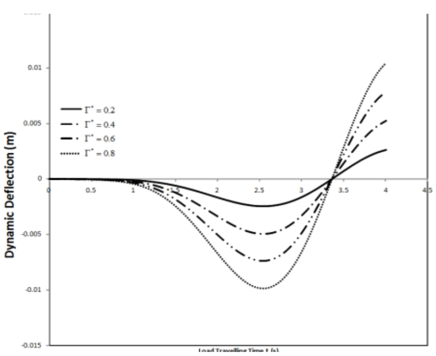

Fig.13: Response of a clamp-clamp structural members resting on elastic foundation to uniform partially distributed masses for various values of the mass ratio Γ^* and for fixed values of G = 30000, K=40000 and N=20000.

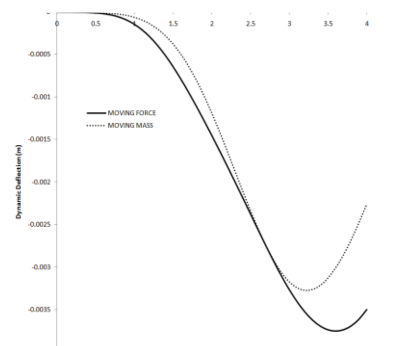

Fig.14: Comparison of the dynamic characteristic of moving force and moving mass cases of a uniform clamped-clamped beam for fixed values of G=0, K=0 and N=0.

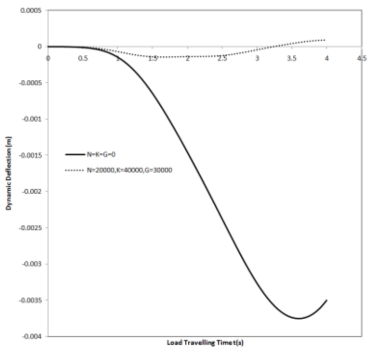

Fig.15: Comparison of the dynamic characteristic of moving force and moving mass cases of a uniform clamped-clamped beam for fixed values of G=30000, K=40000 and N=20000.

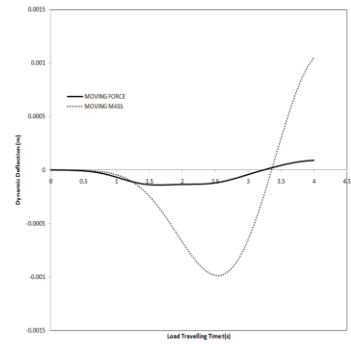

Fig.16: Comparison of the deflection profiles of the clamped-clamped moving force uniform beam for values K=G=N=0 versus G = 30000, K=40000 and N = 20000.

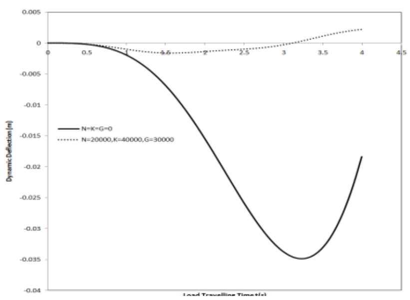

Fig.17: Comparison of the deflection profiles of the clamped-clamped moving mass uniform beam for values K=G=N=0 versus G = 30000, K=40000 and N=20000.

6.2 Clamped-Free ends condition

Fig.18 displays the transverse displacement response of a cantilever uniform beam under the actions of uniform partially distributed forces moving travelling at a constant velocity for the various values of axial force N and for fixed values of subgrade moduli $K = 40000$ and shear modulus $G = 30000$. The Figure depicts that as N increases, the response amplitude of the non-uniform beam decreases. Similar results are obtained when the cantilever beam is subjected to partially distributed mass travelling at a constant velocity as shown in Fig.24. For various travelling time t, the displacement response of the travelling for various values of subgrade moduli K and for fixed values of axial force $N = 20000$ and shear modulus $G = 30000$ is shown in Fig.19. It is observed that higher values of subgrade moduli K reduce the deflection of the vibrating beam. The same behaviour characterizes the response of the clamped-free beam under the actions of uniform partially distributed masses moving at a constant velocity for various values of subgrade moduli K as shown in Fig.25. Also, Fig.20 and 26 display the deflection profile of the cantilever beam respectively to partially distributed forces and masses travelling at a constant velocity for various values of shear modulus G and for fixed values of axial force $N = 20000$ and subgrade moduli $K = 40000$. These figures clearly show that as the value of the shear modulus increases, the deflection of the cantilever beam under the action of both moving forces and masses travelling at constant velocity decreases.

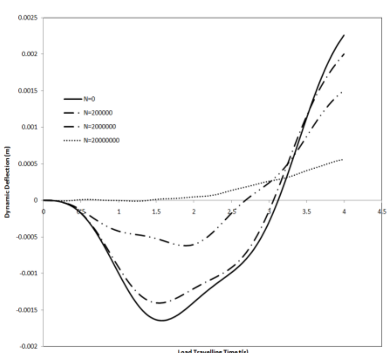

Fig.18: Response Amplitude of a cantilever structural members resting on elastic foundation and under the actions of uniform partially distributed forces for various values of the axial force N and for fixed values of G = 30000, K=40000 and N = 20000.

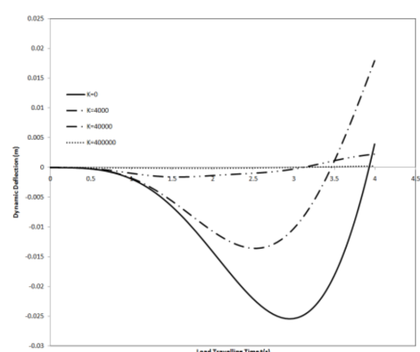

Fig.19: Displacement response of a cantilever structural members resting on elastic foundation and under the actions of uniform partially distributed forces for various values of foundation modulus K and for fixed values of N = 20000, G = 30000.

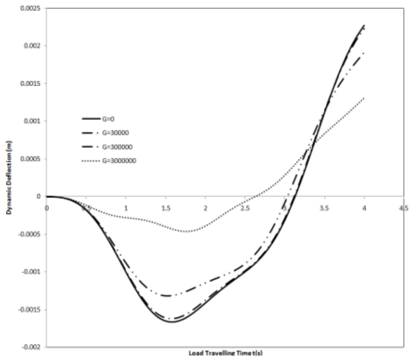

Fig.20: Deflection Profile of a cantilever structural members resting on elastic foundation and under the actions of uniform partially distributed forces for various values of shear modulus G and for fixed values of K=40000 and N = 20000.

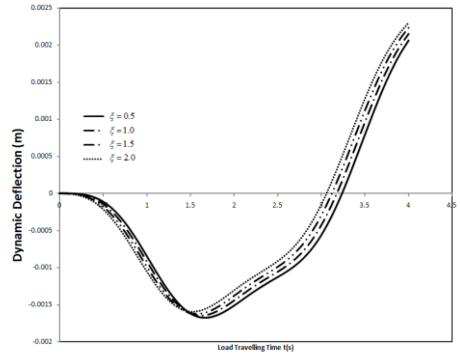

Fig.21: Response Amplitude of a cantilever structural members resting on elastic foundation and under the actions of uniform partially distributed forces for various values of the load width ξ and for fixed values of G = 30000, K=40000 and N = 20000.

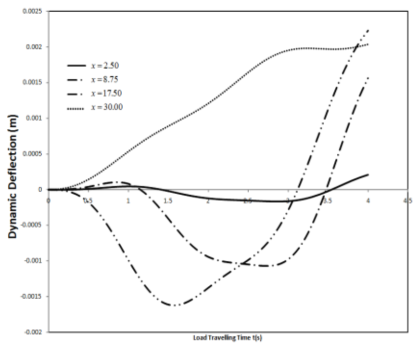

Fig.22: Response of a cantilever structural members resting on elastic foundation to uniform partially distributed forces for various values of the load position x and for fixed values of G = 30000, K=40000 and N = 20000.

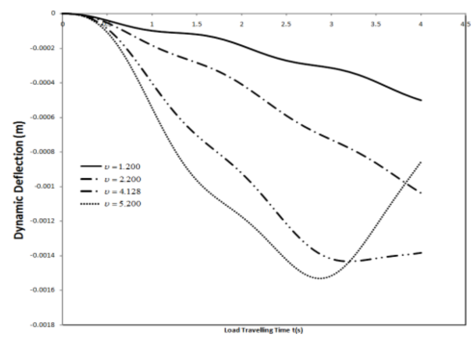

Fig.23: Response characteristics of a cantilever structural members resting on elastic foundation to uniform partially distributed forces for various values of the travelling load velocities v and for fixed values of G = 30000, K=40000 and N = 20000.

Fig.21 displays the response amplitude of a clamped-free uniform beam under the action of uniform partially distributed forces travelling at a constant velocity for various values of load width ξ and for fixed values of axial force $N = 20000$, $K = 40000$ and shear modulus $G = 30000$. The Figures show that as the width increases, the effects of the width on the response amplitude of the uniform beam increases as the load progresses on the structure. Similar results are obtained when the cantilever beam is subjected to partially distributed masses travelling at a constant velocity as shown in Fig.27. For a various travelling time t, the response of the beam for various values of travelling load positions ξ and for fixed values of axial force $N = 20000$, subgrade modulus $K = 40000$ and shear modulus $G = 30000$ are shown in Fig.22. It is observed that the impact of the travelling load is greatest in the middle of this vibrating solid structure. The same behaviour characterizes the response of the cantilever beam under the actions of uniform partially distributed masses moving at a constant velocity for different travelling load positions as shown in Fig.28.

The Dynamic deflections of the clamped-free uniform beam under distributed forces and masses for various values of the velocity v of the motion are respectively displayed in Figs. 23 and 29. These figures clearly show that as the value of the velocity of the motion increases, the deflection amplitude of the clamped-free uniform beam under the action of both moving force and mass respectively decreases. For various travelling time t, the response amplitude of the clamped-free uniform beam under travelling masses is shown in Fig. 30. It is observed that the larger the value of the mass ratio, Γ^*, the larger the response amplitude of the beam.

Figures 31 and 32 depict the comparison of the response characteristics of the moving force and moving mass cases of a clamped-free uniform beam traversed by a moving distributed load travelling at a constant velocity for fixed values of $N = 0$, $K = 0$, $G = 0$ and $N = 20000$, $K = 40000$ and $G = 30000$. From these figures, it is seen that the dynamic

deflection of the beam under the actions of the moving load is greatly affected when the structural parameters N , K and G are incorporated into the governing equation of motion. Figure 33 compares the deflection profiles of the moving force model of the beam for the two sets of values $N=0$, $K=0$, $G=0$ and $N=20000$, $K=40000$ and $G=30000$. It is deduced from this figure that the amplitude of deflection for the set of values $N=0$, $K=0$, $G=0$ is much higher than that of the set of values $N=20000$, $K=40000$ and $G=30000$. Similar result is obtained for a moving mass model of this structural member as shown in figure 34.
Figures 35 and 36 depict the comparison of the clamped-clamped and clamped-free uniform beam under moving load for both the moving force and the moving mass. It is deduced from the Figures that for both cases, the amplitude of the deflection for the clamped-free beam is much higher than that of the clamped-clamped beam. This implies that the clamped-clamped beam is more stable under a travelling uniform partially distributed masses than that of the clamped-free beam. As higher values of the structural parameters G , K , N , are required in the case of the clamped-free beam than in the case of a clamped-clamped beam.

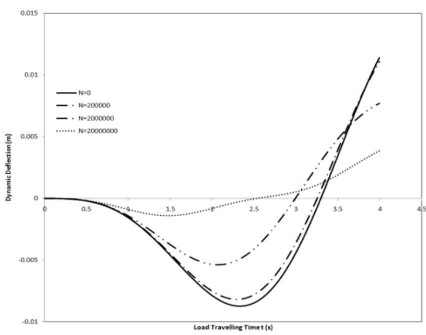

Fig.24:Transverse displacement response of a cantilever structural members resting on elastic foundation and under the actions of uniform partially distributed masses for various values of axial force N and for fixed values of K = 40000, G = 30000.

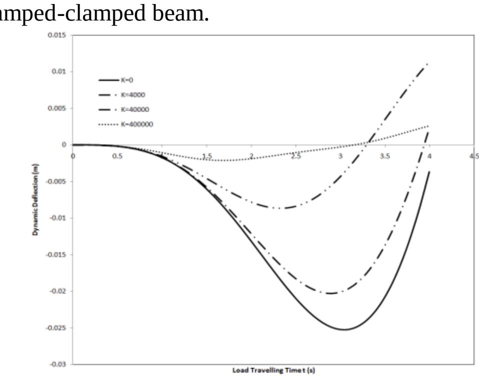

Fig.25: Displacement response of a cantilever structural members resting on elastic foundation and under the actions of uniform partially distributed masses for various values of foundation modulus K and for fixed values of N = 20000, G = 30000.

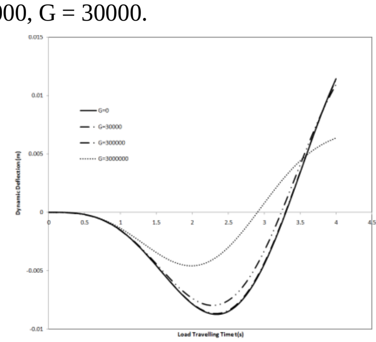

Fig.26: Deflection Profile of a cantilever structural members resting on elastic foundation and under the actions of uniform partially distributed masses for various values of shear modulus G and for fixed values of K=40000 and N = 20000.

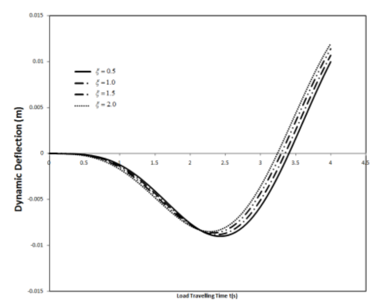

Fig.27: Response Amplitude of a cantilever structural members resting on elastic foundation and under the actions of uniform partially distributed masses for various values of the load width ξ and for fixed values of G = 30000, K=40000 and N=20000.

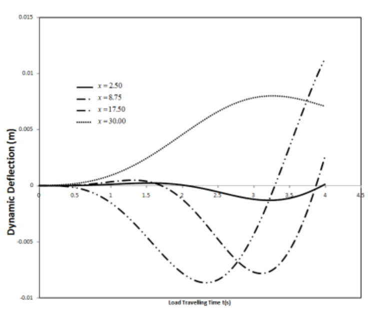

Fig.28: Response of a cantilever structural members resting on elastic foundation to uniform partially distributed masses for various values of the load position x and for fixed values of G = 30000, K=40000 and N=20000.

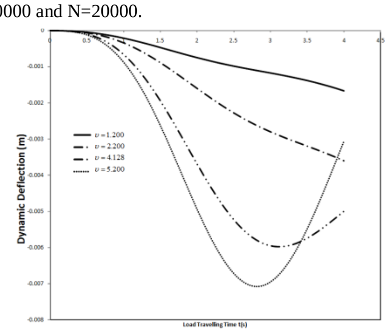

Fig.29: Response characteristics of a cantilever structural members resting on elastic foundation to uniform partially distributed masses for various values of the travelling load velocities v and for fixed values of G=30000, K=40000 and N = 20000

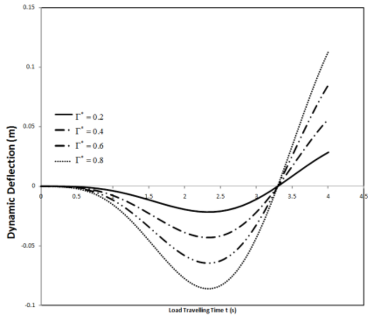

Fig.30: Response of a cantilever structural members resting on elastic foundation to uniform partially distributed masses for various values of the load position Γ^* and for fixed values of G = 30000, K=40000 and N=20000.

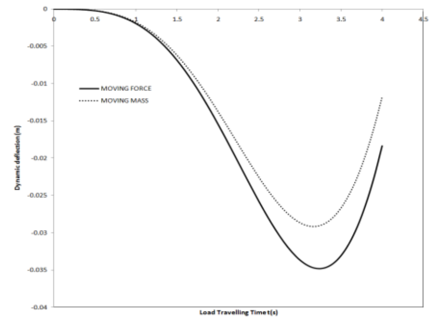

Fig.31: Comparison of the dynamic characteristic of moving force and moving mass cases of a uniform clamped-free beam for fixed values of G = 0, K=0 and N=0.

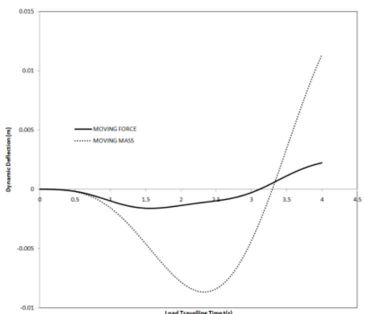

Fig.32: Comparison of the dynamic characteristic of moving force and moving mass cases of a uniform clamped-free beam for fixed values of G = 30000, K=40000 and N=20000.

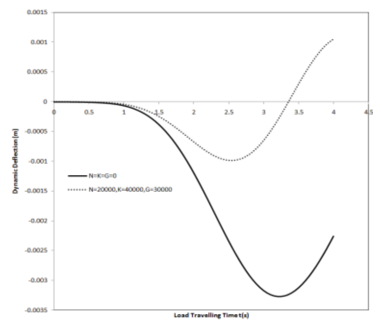

Fig.33: Comparison of the deflection profiles of the clamped-free moving force uniform beam for values K=G=N=0 versus G = 30000, K=40000 and N = 20000.

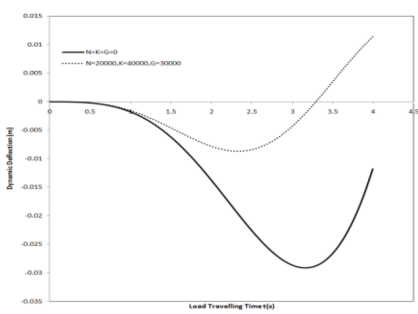

Fig.34: Comparison of the deflection profiles of the clamped-free moving mass uniform beam for values K=G=N=0 versus G = 30000, K=40000 and N = 20000.

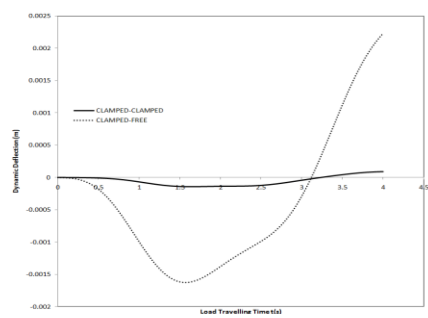

Fig.35: Comparison of the dynamic characteristic of moving force clamped-clamped and clamped-free uniform beam for fixed values of G = 30000, K=40000 and N=20000.

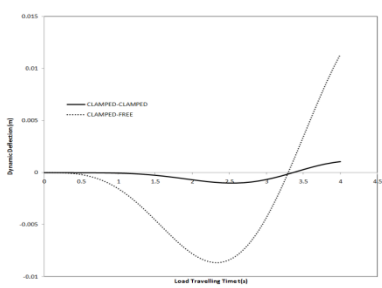

Fig.36: Comparison of the dynamic characteristic of moving mass clamped-clamped and clamped-free uniform beam for fixed values of G = 30000, K=40000 and N=20000.

7.0. CONCLUDING REMARKS

The problem of the complex dynamic interactions of elastic beams and the load traversing them is investigated in this study. Approximate analytical solutions for the moving force and moving mass models are presented.
Results and analysis establish the effects of some vital structural parameters on response characteristics and stability of thin beams traversed by travelling loads are established. Influence of the mass ratio and load distribution on the dynamic behaviour of the structural member carrying distributed moving masses is carefully scrutinized. Conditions under which the amplitude of vibration of this structure-mass system may grow without bound for both the moving force and moving mass models are clearly indicated. Results further revealed that, for the same natural frequency, the critical speed for the beam-force system is higher than that of the beam-mass system. Hence, the risk of resonance effects is higher in a beam-mass system. Finally, the proposed mathematical procedure and analysis in this present study is applicable to all class of these problems.

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