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ON THE LINEAR FORM OF FISHER EVOLUTION EQUATION

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Abstract

This study concerns the diffusive behavior that characterize a number of physical manifestations often observed. These manifestations had been adequately described using Fisher evolution equation. In this study, our analysis will be confined to diffusive process with moderate intensity and, thus, governed by the linear form of Fisher equation. Nevertheless, the solutions in this consideration is still diffuse. A number of boundary conditions are introduced and analyzed. The frequent appearance of the function (error/function) seems interesting but though unexpected in this study

1. Introduction

Fisher evolution equation [1,2] is usually associated with the analytical study of physical, chemical and biological processes. This is so because, these processes re-act accordingly while describing their physical manifestations. Fisher equation is a modification of a generalized model equation known as nonlinear reaction diffusion equation [1,3] being one of the most important equation of mathematical physics. The equation had played an essential role in the in-depth study of the evolution of some large-scale evolving wave phenomena globally [5,2].

The equation (1) in this study describes diffusive wave processes for which the advance of the advantageous gene is an example. Consequently, several researchers [2,4,5] have studied various identical problems and provided new ideas in this consideration. Obviously, the present re-action diffusion method of approach had radically changed the mathematical and physical features of the solutions of a significant numberof problem in the chemical and biological processes [2].

It is stated that the complete analytical solution of Fisher equation is yet to be satisfactorily established. Thus, in this study, the approach will be analytical. Furthermore, we assume that the diffusive process is still sufficiently moderate such that its form can be explained satisfactorily using linear model form of equation due to Fisher.

2. Specifications

The process propagates in x-direction, with $x > 0$ and $t > 0$ is the time duration of oscillation, $U(x,t)$ represent the process. The Fisher evolution linearized equation is of the form

$$U_t - CU_{xx} + R_oU = 0 \qquad (1)$$

$$R_0 = \frac{1}{K}(k-1) \qquad (2)$$

$R_0 > 0$

All the constants in (2) are clearly explained inFisher equation (1).

$R_0 > 0$ provides decaying process and $R_0 < 0$ is the reverse in growth intensity

The boundary condition states that

$U(0,t) = H(t)$, $U(x,0) = A_0$ (a constant as initial data) and $U(x,t) \to 0$ as $x \to \infty$

3. Introduce the $\mathcal{L}$ operator

Thus

$$\mathcal{L}\{U(x,t)\} = \overline{U}(x,s) = \int_0^\infty U(x,t)e^{-st}dt, s > 0 \qquad (3)$$

$U(x,t) = \mathcal{L}^{-1}\{\overline{U}(x,s)\}$ as the inverse transform

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$\mathcal{L}\{U_t(x,t)\} = s\bar{U}(x,s) - U(x,0) = s\bar{U}(x,s) - A_0$

$\mathcal{L}\{U_{xx}(x,t)\} = \frac{d^2}{dx^2}\mathcal{L}\{\bar{U}(x,t)\} = \bar{U}_{xx}(x,s)$

Thus, using (3) on (2)

$s\bar{U}(x,s) - A_0 - CU_{xx}(x,s) + R_0\bar{U}(x,s)$

$$\bar{U}_{xx} = \frac{(s+R_0)}{C}\bar{U} = -\frac{A_0}{C} \tag{4}$$

From (4) $\bar{U}(x,s) = \bar{U}_1 + \bar{U}_2$

$$\bar{U}_1(x,s) = B_1(s)e^{-x\frac{(s+R_0)^{1/2}}{\sqrt{C}}} + B_2(s)e^{x\frac{(s+R)^{1/2}}{\sqrt{C}}} \tag{5}$$

U(x, t) → 0 as x→ ∞, thus $\bar{U}(x,s) \to 0$, hence $B_2 = 0, as\ x \to 0$

$$\bar{U}_2(x,s) = \frac{1}{D^2 - \sigma^2}\left(-\frac{A_0}{C}\right) = \frac{1}{\sigma^2\left(1 - \frac{D^2}{\sigma^2}\right)}\frac{A_0}{C}$$

$$= \frac{1}{\sigma^2}\left(1 + \frac{D^2}{\sigma^2} + \right)^{-1}\frac{A_0}{C} = \frac{A_0}{C\sigma^2}, \sigma^2 = \frac{(s+R_0)}{C}$$

Because, $\bar{U}(x,s) = B_1\exp\left[-x\frac{(s+R_0)^{1/2}}{C}\right] + \frac{A_0}{C}\left(\frac{C}{s+R_0}\right)$

$$= B_1\exp\left[-\frac{x}{c}\sqrt{\frac{(s+R_0)}{c}}\right] + \frac{A_0C}{s+R_0}$$

$$U(0,t) = H(t), \mathcal{L}^{-1}\{U(0,t)\} = \bar{H}(s) \tag{6}$$

At the end x = 0

$$\bar{H}(s) = B_1 - \frac{A_0}{s+R_0}, B_1 = H(s) + \frac{A_0}{s+R_0}$$

And

$$\bar{U}(x,s) = \left(\bar{H}(s) + \frac{A_0}{s+R_0}\right)exp\left[-x\left(\frac{(s+R_0)^{1/2}}{\sqrt{C}}\right)\right] + \frac{A_0}{s+R_0} \tag{7}$$

Case (i) H(t) = 0, $\bar{H}(s) = 0$

$$\bar{U}(x,s) = \frac{A_0}{s+R_0}exp\left[-x\left(\frac{(s+R_0)^{1/2}}{\sqrt{C}}\right)\right] + \frac{A_0}{s+R_0}$$

$$= A_0\left[\frac{1}{s+R_0}exp\left[-x\left(\frac{(s+R_0)^{1/2}}{C}\right)\right] + \frac{1}{s+R_0}\right] \tag{8}$$

$$U(x,t) = A_0e^{-R_0t}[1 + \mathcal{L}^{-1}\left[\frac{1}{S}exp\left(-x\sqrt{\frac{S}{c}}\right)\right]$$

$$= A_0e^{-R_0t}\left[1 + erfc\left(\frac{x}{\sqrt{c}}\frac{1}{2\sqrt{t^3}}\right)\right]$$

$$= A_0e^{-R_0t}\left[1 + erfc\left(\frac{1}{2\sqrt{ct}}\right)\right] \tag{9}$$

Case (ii) $H(t) = \delta(t), \mathcal{L}\{\delta(t)\} = 1 = \bar{U}(0,s) = \bar{H}(s)$ (10)

From (7) and (10)

$$\bar{U}(x,s) = \left(1 + \frac{A_0}{s+R_0}\right)exp\left[-x\left(\frac{(s+R_0)^{1/2}}{\sqrt{C}}\right)\right] + \frac{A_0}{S+R_0} \tag{11}$$

$$\mathcal{L}^{-1}\left\{exp\left[-x\left(\frac{S+R_0}{C}\right)^{1/2}\right]\right\} = exp(-R_0t)\,\mathcal{L}^{-1}\left\{exp\left[-\frac{x}{\sqrt{C}}\sqrt{s}\right]\right\}$$

$$= exp(-R_0t)\frac{x}{2\sqrt{c\pi t^3}}exp\left[-\left(\frac{x}{\sqrt{c}}\right)^2\frac{1}{4t}\right] = \frac{x}{2\sqrt{c\pi t^3}}exp\left[-\left(R_0t + \frac{x^2}{4tc}\right)\right]$$

$$\mathcal{L}^{-1}\frac{A_0}{S+R_0} = A_0\exp(-R_0t)$$

$$A_0\,\mathcal{L}^{-1}\left\{\frac{1}{S+R_0}exp\left[-\frac{x}{\sqrt{c}}(S+R_0)^{\frac{1}{2}}\right]\right\} = e^{-R_0t}\,\mathcal{L}^{-1}\left\{\frac{1}{s}exp\left(-\frac{x}{\sqrt{c}}\sqrt{s}\right)\right\}$$

$A_0e^{-R_0t}\left[1-erf\left(\frac{x}{2\sqrt{c\pi t}}\right)\right] = A_0e^{-R_0t}erfc\left[\frac{x}{2\sqrt{ct}}\right]$

$U(x,t) = exp(-R_0t)\left[\exp\left(-\frac{x^2}{4ct}\right)+\left(2-erf\left(\frac{x}{2\sqrt{ct}}\right)\right)\right]$ (12)

Case (iii) $H(t) = 1$ (any constant may also serve in this section

$\bar{U}(0,t) = \bar{H}(s) = \frac{1}{s}, s > 0$ (13)

From (7)

$$\bar{U}(x,s) = \left(\frac{1}{s}+\frac{A_0}{s+R_0}\right)exp\left[-x\left(\frac{(s+R_0)^{1/2}}{\sqrt{C}}\right)\right]+\frac{A_0}{s+R_0}$$

$= \bar{U}_1+\bar{U}_2+\bar{U}_3(x,s)$ (14)

$\mathcal{L}^{-1}\{\bar{U}(x,s)\} = U_1(x,t)+U_2(x,t)+U_3(x,t) = U(x,t)$ (15)

$U_1(x,t) = \mathcal{L}^{-1}\left\{exp\left[-x\left(\frac{s+R_0}{c}\right)^{1/2}\right]\right\}$. Convolution theorem designed for Laplace's transform will be applied to evaluate $U_1(x, t)$, $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$\mathcal{L}^{-1}\left\{exp\left[-x\left(\frac{s+R_0}{c}\right)^{1/2}\right]\right\} = e^{-R_0t}\,\mathcal{L}^{-1}\left\{exp\left[-x\left(\frac{s}{c}\right)^{1/2}\right]\right\}$

$= \frac{x}{2\sqrt{c\pi t}}exp\left[-\left(R_0t+\frac{x^2}{4ct}\right)\right]$

Thus, $U_1(x, t) = \frac{x}{2\sqrt{\pi c}}\int_0^t\left[\tau^{-\frac{1}{2}}exp\left[-\left(R_0\tau+\frac{x^2}{4ct}\right)\right]\right]d\tau$ (16)

$U_2(x, t) = \mathcal{L}^{-1}\{\bar{U}_2(x,t)\} = \mathcal{L}^{-1}\left\{\frac{A_0}{s+R_0}exp\left[-x\left(\frac{s+R}{c}\right)^{1/2}\right]\right\}$

$A_0e^{R_0t}\,\mathcal{L}^{-1}\left\{\frac{1}{s}exp\left[\left(\frac{x}{\sqrt{c}}\right)\sqrt{s}\right]\right\} = A_0e^{-R_0t}erfc\left(\frac{x}{2\sqrt{ct}}\right)$

$U_3(x,t) = \mathcal{L}^{-1}\left\{\frac{A_0}{s+R_0}\right\} = A_0\exp(-R_0t)$ (17)

$U(x,t) = U_1(x,t)+U_2(x,t)+U_3(x,t)$

$=\frac{x}{2\sqrt{\pi c}}\int_0^t\tau^{-1/2}exp\left[-\left(R_0\tau+\frac{x^2}{4c\tau}\right)d\tau+A_0e^{-R_0t}\left(1+erfc\left(\frac{x}{2\sqrt{ct}}\right)\right)\right]$ (18)

4. Conclusion

A linear version of Fisher evolution equation is considered. Consequently, this procedure implies that the intensity of diffusivity is sufficiently moderate to justify this assumption. Nevertheless, the intensity of diffusive behavior is still quite substantial even though the effect of nonlinearity is ignored. For example, calculations involving tidal wave appeared to suggest that linear approximation may provide realistic conclusions. For example, wave height of A_0 = 3m may decay to 10cm if $R_0 > 0$ and may grow 13m if $R < 0$ and t = 10min in each case.

Further, in this consideration, three difference types of boundary conditions are considered. For $R_0>0$, each case is decaying following the profile described by error function which interestingly appears in each case of boundary condition

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