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ON FRATTINI SUBGROUP OF A FINITE GROUP

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Abstract

We study the concept of Frattini subgroup and extend some of its results. Notion of fully and non-fully Frattini group are proposed and some relevant results are presented. Also, concepts of maximal and minimal generating set are proposed, and some results are presented. Finally, some results in the literatures were extended.

Keywords: Frattini subgroup, fully and non-fully Frattini group, generators, non-generators of a group, minimal and maximal generating set.

1. Introduction.

In group theory, normal subgroups are used to study the structure of other groups such as factor groups, nilpotent groups, cyclic group, conjugacy classes, maximal subgroups etc., the maximal subgroups has been used to characterize a special type of group called Frattini subgroup in [1]. Frattini subgroup is the intersection of all the maximal subgroups of a group. Due to its interesting properties it has attracted the attention of many researchers see $[1-7]$.

Studies in [8] showed that for any cyclic group G, the Frattini subgroup of G is the collection of all the set of non-generators of the group G. Otal in [7], showed that the Frattini subgroup of G is cyclic whenever G is cyclic. Further, Beidleman and Seo in [9], generalized the notion of Frattini subgroup of G by exploiting a concept of sylow $p-$subgroup.

A key aspect of Frattini subgroups is the existence of maximal subgroups. So, groups which guarantee the existence of maximal subgroups are fundamental for studying Frattini subgroups. In general, the union of a group may not be a group. Our principal goal is to study some situation under which the union of all maximal subgroups is a group. When this happens, we called such group "Fully Frattini" otherwise, such group is called "non-fully Frattini". A concept of minimal generating set is introduced in connection with Frattini subgroups. Furthermore, other relevant concept such as generator and non-generator of a group are exploited to obtain some results.

2. Some basic definitions

a. $\boldsymbol{p-group}$**:** A group G is called a $p-$group if it is of prime power order.

b. **Normal subgroup:** A subgroup H of G is said to be a normal subgroup of G if for every $g \in G$ and $h \in H$, $ghg^{-1} \in H$.

c. **Center of a group**: The center of a group G is the set of elements that commute with every element of G. That is, $Z(G) = \{a \in G | \forall g \in G, ag = ga\}$.

d. **Coset:** Suppose G is a group and H is any subgroup of G. Let a be any element of G. Then the set $Ha = \{ha: h \in H\}$ is called a right coset of H in G generated by a.. Similarly the set $aH = \{ah: h \in H\}$ is called a left coset of H in G generated by a.

e. **Quotient group**: If G is a group and H is a normal subgroup of G, then the set G/H of all cosets of H in G is a group with respect to multiplication of cosets. It is called Quotient group or factor group of $G$by H.

f. **Maximal subgroup:** Anormal subgroup H of a group G is said to be maximal if there exist no normal subgroup K of G which properly contains H.

g. **Frattini subgroup:** Let G be a group. The Frattini subgroup of G, denoted as $\Phi(G)$or $Frat(G)$, is the intersection of all maximal subgroups of$G$. If G has no maximal subgroups, then$Frat(G) = G$.

h. **Minimal normal subgroup:** A non-trivial subgroup H of a group G is termed a minimal normal subgroup if there is no other proper normal subgroup K such that$K \trianglelefteq H$.

i. **Upper central series:** The upper central series (or ascending central series) of a group G is the sequence of subgroups $1 = Z_0(G) \trianglelefteq Z_1(G) \trianglelefteq \cdots \trianglelefteq Z_n(G) = G$such that$Z_{i+1}(G)/Z_i = (G/Z_i) \forall\, i \geq 0$.

j. **Nilpotent Group:** A group with an upper central series that terminates with$G$ is called a nilpotent group.

k. **Commutator or Derived subgroup of a group:** Let G be a group and$a, b \in G$. The element $aba^{-1}b^{-1}$ is called the commutator of the ordered pair (a, b).

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Let $U = \{aba^{-1}b^{-1}$: a, b ∈ G}. If G' is the subgroup of G generated by U, then G' is called the commutator subgroup of G.

l. **Generating set of a group**: A subset X of a group G is said to be a generating set for G if all elements of G can be expressed as the finite product of elements in X and their inverses.

m. **Cyclic group**: A group G is called cyclic if for some $a \in G$, every element $x \in G$ is of the form a^n where n is an integer. The element a is then called the generator of G otherwise a non-generator.

n. **Non generator of a group:** An element $x \in G$ is a non generator if whenever set Sgenerates G, $S\backslash\{x\}$ generate G.

o. **Covering**: a family A of non empty subsets of X whose union contains X is called a cover or covering of X.

p. **Internal direct product**: suppose H and K are subgroups of a group G. Then G is an internal direct product of H and K if,

i. every element of H commute with every element of K,

ii. every element of G is uniquely expressible as a product of an element of H and an element of K.

3. Some Existing Results

Theorem 3.1. Let G be a finite group. If $K < G$, then $Fratt(K) < G$[8].

Theorem 3.2. For everygroup , $G^1 \cap Z(G) \leq \Phi(G)$[6].

Theorem 3.3. For every group G, the Frattini subgroup of G is the set of all non-generators[8].

Theorem 3.4. If G is a finite group, then, $G^1 \leq FratG$ [7].

4. Main Results

Some definitions

a. **Fully Frattini group:** A group G is called fully Frattini if the union of all the maximal subgroups equals G. Otherwise, it is called non-fully Frattini.

If G has no maximal subgroups, then G is said to be trivial fully Frattini.

b. **Minimal Generating Set:** Let G be a cyclic group and X be a subset of G. Then X is said to be a minimal generating set if there exist no set $Y(\subset X)$ that generates G.

c. **Maximal Generating Set:** Let X be a generating set of a group G. $X$is maximal if there is no generating set of G that contains it.

Theorem 4.1. For every finite non-abelian group G, $Z(G) \trianglelefteq \Phi(G)$.

Proof. Let G be a finite non abelian group ,$\Phi(G)$ be the Frattini subgroup of G and $Z(G) = \{x \in G : xy = yx \ \forall \ y \in G\}$be the center of G. Suppose $\Phi(G) = G$ then the result hold trivially. If $\Phi(G) \neq G$. We show that $Z(G)$ is a subgroup of $\Phi(G)$.

Let$x_1, x_2 \in Z(G)$. Then $x_1y = yx_1$ and $x_2y = yx_2 \forall y \in \Phi(G)$.

$\Rightarrow {x_2}^{-1}(x_2y){x_2}^{-1} = {x_2}^{-1}(yx_2){x_2}^{-1}$,

$\Rightarrow y{x_2}^{-1} = {x_2}^{-1}y \forall y \in \Phi(G)$

$\Rightarrow {x_2}^{-1} \in Z(G)$.

Now, $(x_1{x_2}^{-1})y = x_1({x_2}^{-1}y) = x_1(y{x_2}^{-1}) = (x_1y){x_2}^{-1} = (yx_1){x_2}^{-1}$.

Therefore, $x_1{x_2}^{-1} \in Z(G)$.

Thus, $x_1, x_2 \in Z(G) \Rightarrow x_1{x_2}^{-1} \in Z(G)$. Therefore $Z(G)$ is a subgroup of $\Phi(G)$.

We now show that $Z(G) \trianglelefteq \Phi(G)$.

Let $x \in \Phi(G)$and $y \in Z(G)$. Then $xyx^{-1} = (xy)x^{-1} = (yx)x^{-1} = y \in Z(G)$.

Thus, $x \in \Phi(G)$, $y \in Z(G) \Rightarrow xyx^{-1} \in Z(G)$. Hence, $Z(G) \trianglelefteq \Phi(G)$.

Remark 4.1. For every finite non abelian group G, if K is a normal subgroup of G and M_i for each i are the maximal subgroups of G, then the maximal subgroups of K are subgroups of M_i.

Theorem 4.2. Let G be a finite non abelian group, if K is a normal subgroup of G, then$\Phi(K) \leq G$ and $\Phi(K) \leq \Phi(G)$.

Proof. Let G be a finite non abelian group and K be a normal subgroup of G. Let $\Phi(K)$ be the Frattini subgroup of K then it follows from [8], that$\Phi(K) \leq G$.

We show that $\Phi(K) \leq \Phi(G)$. Clearly, $\Phi(K)$ and $\Phi(G)$ are subgroups of G.

Let M_k's be the maximal subgroups of K and M_G's be the maximal subgroups of G. Then by *remark* 4.1 we have

M_k's $\leq M_G$'s

$Card(\cap M_k$'s$) \leq Card(\cap M_G$'s$)$

$\Longrightarrow Card(\Phi(K)) \leq Card(\Phi(G))$

Therefore, $\Phi(K) \subseteq \Phi(G)$.

To show that $\Phi(K) \leq \Phi(G)$, we consider two cases;

Case one: when K is maximal and has at least one non trivial maximal subgroup, if the intersection of K and all the M_G's is not identity then $\Phi(K) \leq \Phi(G)$ and if the intersection of K with all M_G's is identity the result holds trivially.

If K is maximal and does not have any other maximal subgroup then $\Phi(K) = K$. Since G is a non abelian group, all of its maximal subgroups intersect at a point distinct from the identity hence K is maximal and has a maximal subgroup which is a contradiction. The case that K is maximal and does not have any other maximal subgroup does not arise.

Case two: when K is not maximal then K is contained in some maximal subgroups of G. If K is not in all the maximal subgroups of G then $K \cap M_G$'s is its Frattini with $\Phi(K) = \{e\}$ a subgroup of $\Phi(G)$.
If K is contained in all of the M_G's, then $\Phi(K) \leq \Phi(G)$ and the result follows immediately

Remark 4.2. Let G be a finite abelian group, if K is a maximal subgroup of G,

i. $\Phi(K) \leq G$, however, $\Phi(K) \nleq \Phi(G)$.
ii. $\Phi(G) \leq \Phi(K)$iff $(G) = \{e\}$.

Theorem 4.3. If G is a finite group, then G' is a normal subgroup of $\Phi(G)$.

Proof. Let G be a finite group and $\Phi(G)$ be Frattini subgroup of G.
Let $X = \{aba^{-1}b^{-1} | a, b \in G\}$ and G' be the commutator subgroup of G.
Clearly, $G' \cap \Phi(G) \neq \emptyset$ (since $e \in \Phi(G)$and $e \in G'$). If e is the only element in G', then the result follows.
In [7],$G' \leq \Phi(G)$. To show the desired result, let $G' \neq e$. Since every maximal subgroup is normal in G by definition, implies $\Phi(G)$ is normal.
This implies that for any $g \in G'$, $ga = ag$, $\forall\, a \in \Phi(G)$.
Let $g \in G'$then, $ga = ag$ implies $ag = ga$and $aga^{-1} = g \in G'$. Hence, $G' \trianglelefteq \Phi(G)$.

Remark 4.3. For every finite non abelian group G, the center and the commutator subgroup of G are contained in the maximal subgroups of G.

Theorem 4.4. For every group G, the Frattini subgroup of G is the set of all non-generators[8].

Remark 4.4. Frattini subgroup of G is not the set of all non generators in a group.

Counter example; Let $G = (Z_{12}, +)$. Then the $\Phi(G) = \{0, 6\}$.
If $S = \{0, 2, 3, 4, 5, 6, 9, 10\}$ is the generating set of G then the set of non generators of G denoted as $NGEN(G) = \{0, 2, 3, 4, 6, 9, 10\} \neq \Phi(G)$.

Theorem 4.5. Let G be a cyclic group. Then$\Phi(G)$is contained in the set of all non-generators of G. In particular, $\Phi(G)$ is equal to the set of all non-generators if the group has only one maximal subgroup.

Proof. Let G be a cyclic group and $\Phi(G)$ denote the Frattini subgroup of G. Let $Gen(G)$ be the set of all generators of G and $M_{i=1,2,\dots,n}$ be the maximal subgroups of G, then for all $x \in Gen(G)$, $x \notin M_{i=1,2,\dots,n}$. In fact, all $y \in M_i$ is a non-generator.
Further, $G = Gen(G) \cup M_1 \cup M_2 \cup \dots \cup M_n$and $y \in G \backslash Gen(G) = \cup M_i$. Since $\Phi(G) = \cap M_i$, we have that for all $y \in \Phi(G)$, $y \in \cup M_i$. This implies that$\Phi(G) \subseteq \cup M_i$. But $\cup M_i$ is the largest set containing all non generators. Hence $\Phi(G)$ is contained in the set of all non-generators. Suppose G has only one nontrivial maximal subgroup say M then, $G = Gen(G) \cup M$ and $y \in M (y \notin Gen(G))$. Since $\Phi(G) = M$, therefore$y \in \Phi(G)$ for all non-generators y. Hence, $\Phi(G)$ is indeed the set of all non-generators.

Theorem 4.6. If G has two maximal subgroups, then the union of the generators is a non generating set of G.

Proof. Let M_1 and M_2 be the maximal subgroups of G and $Gen(G)$ be the collection of all generators of G. Clearly, $Gen(G) \nsubseteq M$ (since M does not contain any generator). Now, $[Gen(G)]^n$ can be expressed as $[Gen(G)]^n = M\, Gen(G)$ if n is odd and $[Gen(G)]^n = M$ if n is even with $M \cap Gen(G) = \emptyset$ for any maximal subgroup of G.
Also, $M_1 Gen(G) = G \backslash M_2$. That is, $[Gen(G)]^n = G \backslash M_2$ for odd values of n and for any M but since $= Gen(G) \cup M_1 \cup M_2$, for n even we have $[Gen(G)]^n = M_1 = G \backslash M_2 \neq G$ for any M. Hence the result.

Theorem 4.7. If G has two maximal subgroups, then the union of the non generators is a generating set of G.

Proof. Let M_1 and M_2 be the maximal subgroups of G and $NGen(G)$ be the collection of all non- generators of G. Clearly, $NGen(G) \subseteq M_1 \cup M_2$ and so $NGen(G)$generates G.
$NGen(G) = \{i, j, k, \dots, u\}$, where $i, j, k, \dots, u \in G$.
$[NGen(G)]^2 = \{i, j, k, \dots, u\}\{i, j, k, \dots, u\}$
Taking every $i, j \in NGen(G)$, $ij = k \in G$ (for some $k \in G$) if $i, k \neq \{e\}$ (i.e., ij generates distinct elements in G).
Since $e \in NGen(G)$, we have that $[NGen(G)]^n = NGen(G) \cup (G \backslash NGen(G))$ for some $n \in \mathbb{N}$. More explicitly,
$[NGen(G)]^n = \{i, j, k, \dots, u\}\{i, j, k, \dots, u\} \dots for\ n\ time$
$= G = \{i, j, k, \dots u, p, r, q, \dots, t\}$where $p, r, q, \dots, t \in Gen(G)$.
This yields $[NGen(G)]^n = NGen(G) \cup Gen(G) = G$ for some $n \in \mathbb{N}$.

Theorem 4.8. If G has two maximal subgroups and $Gen(G)$is the set of generators of G, then $[NGen(G)]^n$ form one of the maximal subgroup for some $n \in \mathbb{N}$.

Proof. Let G be a cyclic group, M_1, M_2 be the maximal subgroups of G and $i, j, k, \dots, u$ be the generators of G.
Then,$[Gen(G)]^n = \{i, j, k, \dots, u\}\{i, j, k, \dots, u\} \dots for\ n\ time$.
Since $e \notin Gen(G)$, for all $s, t \in Gen(G)$, $s \cdot t \notin Gen(G)$, $s \cdot t \in [Gen(G)]^2$ and $s, t \notin M_{i=1,2}$.
But, $G = Gen(G) \cup M_1 \cup M_2$. Therefore, $s \cdot t \in G \backslash Gen(G)$.
In particular, $[Gen(G)]^2 \subseteq G \backslash Gen(G)$ and$[Gen(G)]^2 = M_i$ for any i.

Remark 4.5.

a. A generator of any group G is not contained in any of its maximal subgroups.
b. The set of non-generators of any group G may not be a subgroup of group G.

Theorem 4. 9. Every cyclic group is not fully Frattini.

Proof Let G be a cyclic group and $Gen(G) = \{x_j | j \geq 1\} \forall x_j$ where x_j is a generator of the group then $G = Gen(G) \cup M_i$, where $M_i's$ are the maximal subgroups of G and $x_j \notin M_i$.

Since, $\cup M_i = G \backslash Gen(G) \neq G$. Hence G is not fully Frattini.

Theorem 4. 10.Every non trivial fully Frattini group has at least three maximal subgroups.

Proof. Let G be a non trivial fully Frattini group and $|G| = n$. Suppose H_1 and H_2 are the only maximal subgroups of $G$and $|H_1| = p$, $|H_2| = q$. Then by Lagrange theorem, p/n and q/n and so $p + q \neq n$. This implies that there exist $\in G \ni y \notin H_1 \cup H_2$. Therefore, $H_1 \cup H_2 \neq G$ that is, there exist other subgroups H_i for $i > 2$ such that $\cup H_i = G$.

Definition: A group is said to be simple if it has only trivial normal subgroups.

Remark 4.6. Every group of prime order is a simple group.

Remark 4.7. Every simple group is a trivial fully Frattini group.

Remark 4.8. A non-fully Frattini group$(\mathbb{Z}_n, +)$ has at most three maximal subgroups.

Theorem 4. 11. Let G be a cyclic group.Then the union of all the minimal generating sets of G is equal to G.

Proof. Let G be a cyclic group and $F = \{X_1, X_2, \dots, X_n\}$ be the family of all minimal generating sets of G. Suppose $G \neq \bigcup_{i=1}^{n} X_i$, then there exist a set X' which contains non generators such that $G = \bigcup_{i=1}^{n} X_i \cup X'$. Since X_i are the minimal generating sets of G, then every $a \in X'$ is in one of the X_i and so $\bigcup_{i=1}^{n} X_i = G$. If there exist $b \in X'$ and $b \notin X_i$ then X_i does not generates G which contradict the fact that X_i are minimal generating sets of G.

Remark 4.9. Every minimal generating set contains a non-generator.

Theorem 4. 12. Let X be a minimal generating set of a cyclic group$G$, then $X \nsubseteq \Phi(G)$.

Proof. Suppose $X \subseteq \Phi(G)$, then $X \subseteq M_i$, $\forall\ i$ where M_i is a maximal subgroup of G. Now, since X contains at least one generator of G, then it implies that every M_i contains at least one generator of G which is a contradiction. Hence, $X \nsubseteq \Phi(G)$.

Remark 4.10. Given a cyclic group G with order k, If X is a minimal generating set of G then X^m gives $m + 1$ elements of G.

Theorem 4. 13. Let G be a cyclic group of order k and Xbe a minimal generating set of G.Then $X^{k-1} = G$.

Proof. Let G be a cyclic group of order k and X be a minimal generating set of G.

If $k = 1$and 2, the result holds trivially.

By remark 4.10, $|X^m| = m + 1$ elements of G. Now given that $|G| = k$ for $|X^m| = k$implies $m = k - 1$.

Therefore, $|X^{k-1}| = k = |G|$. Hence, $X^{k-1} = G$.

Remark 4.11.

a. Any minimal generating set of G is not a subgroup of G.

b. If $|G| = n$ then the total number of minimal generating sets of G divides n.

c. The collection of all the minimal generating sets of G forms a covering.

d. Minimal generating sets of a cyclic group are mutually exclusive.

5. Conclusion

In this paper we discussed the condition for which a group is fully and non-fully Frattini.It was observed that the Frattini subgroup of the group is not always equal to the set of all non- generators. However, it is always contained in the set of all non-generators. We underline that the set of non-generators of a group is not always a subgroup of the group. Further, our future direction is to investigate Frattini subgroup, fully and non-fully Frattini subgroup, non-generator and generator of fuzzy group as well as minimal and maximal generating set of fuzzy group.

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