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SOME CARDINALITIES OF SEMI-GROUP OF CONTRACTION MAPPINGS

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Abstract

Let T_n be the set of full transformations and P_n be the set of partial transformations. It is shown that T_n forms a semi-group of order n^n and P_n forms a semi-group of order $(n+1)^n$. Furthermore, we obtain that $|\alpha S| = \sum_{k=}^{n} \binom{n+2p}{3p}$, $|\alpha S| = 5n+1 = a+(n-1)d$, $|ORCP_n| = \binom{n}{m}\binom{n+2}{(m-1)+2}$, $|\alpha S| = \binom{n}{k}\binom{n-1}{k-1}$ and $|ODCP| = \binom{n+m}{2m}\binom{(n+1)+1}{m+1}$, $|\alpha S| = 3n+3$, if OCP_n , $ORCP_n$ and $ODCP_n$ represent the sub-semi-groups of order-preserving, order-reversing and order-decreasing partial contraction mappings respectively on $X_n = \{1,2,3\ldots\}$ while $|Q|$ denotes the order of Q.

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1. Introduction

A binary relation is any subset of the Cartesian product $A * B$ (say), where A and B are non-empty sets. A transformation is a relation between A and B in which all the elements of A are involve in the relationship and two elements of B have one pre-image in A. Given any non-empty sets A and B with a sub-set S of $A * B$, if for $a \in A$, $b \in B$. $(ab) \in S$, then "a is related to b by the relation S" which is written by aSb. Given $S \subset A * B$, set $\{x | x \in A$ and $(x,y) \in S$ for some $y \in B\}$ is called the domain while set $\{y | y \in B$ and $(x,y) \in S$ for some $x \in A\}$ is called the co-domain (range). Transformation is another name of mapping. Refer to [1] for an introduction to semi-group of mappings. Since empty set is a sub-set of every set (A_i say); for it is not, it means there is an element of empty set that is not in A_i. This is contrary to the definition of empty set. In the current article, we can conclude the empty set $\emptyset$ in the mappings. Let $(-)$ stands for $\emptyset$ and let $\binom{a\ b\ c}{x\ y\ z}$ be represented by $(x\ y\ z)$ not (x,y,z) and not complicating the cycle notation. The mappings that included the empty set $\emptyset$ are called partial transformation.

Let T_n be the set of full transformations (mappings) and P_n be the set of partial transformations. Then T_n forms a semi-group of order n^n and P_n forms a semi-group of order $(n+1)^n$. In [2] contraction mapping in P_n is defined as : for all $x, y \in dom(\alpha)$, $|\alpha x - \alpha y| \le |x - y|$. The breadth of α is denoted by $b(\alpha) = |dom(\alpha)|$. The height of α is denoted by $h(\alpha) = |Im(\alpha)|$. The fix of α is given by $f(\alpha) = |f(\alpha)|$ and the collapse is denoted by $c(\alpha)$. Let S_n be the symmetric sub-group of the transformation semi-group T_n. Then $|S_n| = n!$. The n! is always less than n^n, when $n > 1$. Refer to [3, 4] for order symmetric semi-group $|S_n|$ and for the order of the alternating sub-group of the symmetric group $|A_n|$.

Let $X_n = \{1,2\ldots\}$ be the set of n number of ordered elements of counting numbers representing symbols of mathematical intuitions and thought. For basics counting principles and combinatorics rules refer to [5, 6]. Let α be a mapping from sub-sets of X_n to sub-sets X_n. Then the set of all αS if equipped with the binary operation of composition of mappings forms the partial transformation semi-group.

A transformation $\alpha \in P_n$ is said to be order-preserving if for all $x, y \in dom(\alpha)$: $x \le y$ implies $\alpha x \le \alpha y$. It is order-reversing if $x \le y$ implies $\alpha x \ge \alpha y$. It is order-decreasing if $\alpha x \le x$. It is order-preserving or reversing if $x \le y$ implies $\alpha x \le \alpha y$ union $x \le y$ implies $\alpha x \ge \alpha y$ and it is order-preserving and decreasing if $x \le y$ implies $\alpha x \le \alpha y$ intersection $x \le y$ implies $\alpha x \le x$.

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For example; Let $|CP_n|$ be the order of set of all contraction mappings of P_n , for a (partial) $\alpha \in P_n$: $\alpha = \binom{1\,2\,3}{1\,1\,2}$ where $dom(\alpha) = (1\ 2\ 3)$ and $Im(\alpha) = (1\ 1\ 2)$ then we need to show that $|\alpha x - \alpha y| \le |x - y|$ whenever $x, y \in dom(\alpha)$:

$|1 - 1| \le |1 - 2|$ implies $0 \le 1$

$|1 - 2| \le |2 - 3|$ implies $1 \le 1$

$|1 - 2| \le |1 - 3|$ implies $1 \le 2$

Therefore, if for all $x, y \in dom(\alpha)$, α satisfy contraction inequalities. Hence, α is a contraction mapping.

Let $\alpha \in P_n$. Then the elements of P_n can be represented as $\alpha(P_n)$. For economy of size, space and time viz:

P_0	P_1
$(-)$	(1)

Stand for

P_0	P_1
$\binom{1}{\emptyset}$	$\binom{1}{1}$

$(2\ 2)$ stands for $\binom{1\,2}{2\,2}$ and $(-\ 2)$ stands for $\binom{1\,2}{\emptyset\,2}$ and down the road in. Authors in [5] said transformation semi-group were the most promising class of semi-groups for future study. We used the structure above to generate $\alpha(CP_2)$ as follows:

P_0
$(-)$

is 1, that is $0 \cap 0$,

P_0	P_1
$(-)$	(1)

is 1 on $1 \cap 0$ and 1 on $1 \cap 1$. Then

P_0	P_1	P_2
$(-\ -)$	$(1\ 1)\ (2\ 2)$	$(1\ 2)\ (2\ 1)$
	$(-\ 1)\ (-\ 2)$	
	$(1\ -)\ (2\ -)$	

is 1 on $2 \cap 0$, is 6 on $2 \cap 1$ and 2 on $2 \cap 2$.

The purpose of the current article is to present the combinatorial properties of the above mentioned semi-groups $S \subseteq P_n$. Combinatorics could be described as the act of arranging objects (elements) according to specified rule. Refer to [7] for the elementary knowledge on semi-group theory and algebraic structures. For a binary relation in any subset of the Cartesian product $A * B$ (say), where A and B are non-empty sets author in [8] categorized it to be two. The article is organized: In section 2, we define some basic preliminaries. In section 3, we consider $|\alpha_{ij}(S)|$ and thereby obtain explicit formula for $|S|$ in each case. The formula obtained in this way are in closed form when S is one of T_n , P_n or some of its sub-semi-groups and are expressed as sum involving binomial coefficient. In section 4, we have concluding remarks.

2. Basic Preliminaries

We define the following terms which features in the proofs of our results.

Definition 2.1 [Partial Transformation P_n]: Let $X_n = \{1,2 \dots\}$ be a natural ordering of numbers and $\alpha: dom(\alpha) \subseteq X_n$implies $Im(\alpha) \subseteq X_n$. Then partial transformation P_n is the set of all functions $\alpha: dom(\alpha) \subseteq X_n$ on X_n.

Definition 2.2 [Partial Contraction Mapping CP_n]: Let $X_n = \{1,2 \dots\}$. Then a transformation $\alpha \in P_n$ is said to be partial contraction mapping if for all $x, y \in dom(\alpha)$: $|\alpha x - \alpha y| \le |x - y|$.

Definition 2.3 [Breadth / Width, (α)]: This is the number of elements in the domain of α. That is $|dom(\alpha)| = b(\alpha)$.

Definition 2.4 [Height / Length, $h(\alpha)$]: This is the number of elements in the image sets of α. That is $h(\alpha) = |Im(\alpha)|$ and it is denoted by $h(\alpha) = p$.

Definition 2.5 [Collapse, $c(\alpha)$]: This is the order of the union of image sets of α which is greater than 2.

That is $c(\alpha) = |\cup \{t(\alpha)^{-1}: t \in Im(\alpha) and\ |t(\alpha)^{-1}| \ge 2\}|$. It is denoted by $c(\alpha) = q$.

Definition 2.6 [Right waist, $\omega^+(\alpha)$]: This is the maximum element in the image sets of $(Im(\alpha))$ of α. That is $\omega^+(\alpha) = \max(Im(\alpha))$. It is denoted by $\omega^+(\alpha) = k$.

Definition 2.7 [Left waist, $\omega^{-}(\alpha)$]: This is the minimum element in the image sets of $(Im(\alpha))$ of α. That is $\omega^{-}(\alpha) = \min(Im(\alpha))$. It is denoted by $\omega^{-}(\alpha) = k^{-}$.
Definition 2.8 [Fix, $f(\alpha)$]: This is the order of the only element that maps itself. That is, $f(\alpha) = |f(x)| = \{x \in dom(\alpha): \alpha x = x\}$. It is denoted by $f(\alpha) = m$.

3. Results and Findings

The results of combinatorial study of any algebraic structure are theorems. The following theorems with their proofs are the outcome of the research in the current article.
Theorem 3.1: Let $= OCP_n$, then $|\alpha S| = \sum_{r=0}^{n} \binom{n+2p}{3p}$
Proof: Let $\alpha \in S$, and $X_n = \{1,2 \dots n\}$, then $\alpha \in OCP_n$. The semi-group OCP_n contains an empty map {}, since it is a partial transformation and α is a bijection; h element of the domain can be chosen from X_n in $\binom{n}{p}$ ways. Let $Im(\alpha) \subseteq X_n$, if $|Im(\alpha)| = 0$, then $|h(\alpha)| = 1$ and if $n = p$, $|h(\alpha)| = 1$ (i; identity), then $|h(\alpha)|$ is also 1 for each n and h; $h = \{1,2 \dots\}$. For $f(n,p) = |\alpha \in S: h(\alpha)| = |Im(\alpha)| = p$, where $x \in dom(\alpha)$ implies $\alpha x \leq x$. If $\alpha x = i$, then $\alpha \in \{i, i-1, i-2 \dots n\}$ also x has $n - i + 1$ degree of freedom. Hence, $\sum_{i=0}^{n}(n - i + 1) = 1$ which is equivalent to $|\alpha S| = \sum_{r=0}^{n} \binom{n+2p}{3p}$ $\forall\, n \geq 1$. The result follows immediately.

Table 3.2: Order of Element of OCP_n

$\frac{n}{\lvert Im(\alpha)\rvert} = h$	0	1	2	3	4	5	$\lvert\alpha S\rvert = \sum_{r=0}^{n} \binom{n+2p}{3p}$
0	1						1
1	1	1					2
2	1	6	1				8
3	1	21	11	1			34
4	1	60	62	16	1		140
5	1	155	258	127	21	1	563

Theorem 3.3: Let $S = |ORCP_n|$, then $f(n,p) = \binom{5n-4}{i} = 5n + i$.
Proof: Let $X_n = \{1,2 \dots n\}$, given any $\alpha \in S$, then $dom(\alpha) \subseteq X_n$ such that $h(\alpha) = |Im(\alpha)|$. The image set of $ORCP_n$ can be chosen in $\binom{n}{p}$ ways such that $f(n, p_{n-1}) = 5n + 1\ \forall\, n \geq 2$. It is equivalent to the set of $X_n = a + (n-1)d$ for $a = 6$, $d = 5$ then $5n + 1 = \binom{5n-4}{i}$. Since for $p = 0,1$. The concept of contraction coincides, but distinct otherwise. However, there is a bijection between n and p for $n \geq 2$. The result follows immediately.

Table 3.4: Order of Element of $ORCP_n$

$\frac{n}{\lvert Im(\alpha)\rvert} = h$	0	1	2	3	4	5	$\sum_{r=0}^{n} f(n,p) = \binom{5n-4}{i}$
0	1						1
1	1	1					2
2	1	6	1				8
3	1	21	11	1			34
4	1	60	62	16	1		140
5	1	155	258	127	21	1	563

Theorem 3.5: Let $= ODCP_n$, then $|\alpha S| = 2^{n-1}$
Proof: Let $X_n = \{1,2 \dots n\}$, then if a (partial) transformation $\alpha \in S$ such that $dom(\alpha)X_n$, there exist 2^n elements having the property of $|Im(\alpha)| = 2$. Since k is the maximum of elements in the image set of α, that is $|\omega^{+}(\alpha)| = k$, then implies that $\alpha \in ODCP_n$ is a bijection. $|\alpha S| = 1$ if $|Im(\alpha)| = 0$. For $|Im(\alpha)| = 2$ when $n = 2$, we have 2 elements, and 2^2 elements when $|Im(\alpha)| = 3$ when $n = 3$, then $|Im(\alpha)| = k$ when $k \geq n \geq 2$ we have 2^n elements. Since the mapping was defined from $X_n \to X_n$, and $Im(\alpha)$ is $= 0,1,2 \dots n$. Then $|ODCP_n|$ occurs exactly in 2^{n-1} ways. Hence, the proof is complete.

Table 3.6: Order of Element of $ODCP_n$

$\frac{n}{\lvert\omega^+(\alpha)\rvert} = k$	0	1	2	3	4	5	$\lvert ODCP_n\rvert = 2^{n-1}$
0	1						1
1	1	1					2
2	1	3	2				6
3	1	7	10	4			22
4	1	15	38	24	8		86
5	1	31	129	116	56	16	349

Theorem 3.7: Let $S = |ORCP_n|$, then $|\alpha S| = \binom{n}{m}\binom{n+2}{(m-1)+2}$

Proof: Let $\alpha: X_n \to X_n$ and $Im(\alpha)$ is such that $i = 0,1,2 \ldots$ since m elements of domain in a set X_n can be chosen from X_n in $\binom{n}{m}$ ways and each partial bijection $\alpha: dom(\alpha) \to Im(\alpha)$ which can be done in $\binom{n}{m}$ ways. Then if $\alpha \in S$, and $\binom{n+2}{(m-1)+2}$ element when $|Im(\alpha)| = 2$. We observed that $\binom{n}{m}$ and $\binom{n+2}{(m-1)+2}$ are equivalent to $n+2$ which yield $|\alpha S|$ to occur in $\binom{n+2}{(m-1)+2}$ ways. The result follows immediately.

Table 3.8: Order of Element of $ORCP_n$

$\frac{n}{\lvert f(\alpha)\rvert} = m$	0	1	2	3	4	5	$\lvert\alpha S\rvert = \binom{n}{m}\binom{n+2}{(m-1)+2}$
0	1						1
1	1	1					2
2	3	4	1				8
3	13	15	5	1			34
4	48	64	21	6	1		140
5	193	249	86	27	7	1	563

Theorem 3.9: Let $S = OCP_n$, then $|\alpha S| = \binom{n}{k}\binom{n-1}{k-1}$ and $2^n - 1$.

Proof: Let : $dom(\alpha) \subseteq X_n \to Im(\alpha) \subseteq X_n$. Since contraction elements yield zero mapping in composition of mapping and domain can be empty in partial transformation, then if $|Im(\alpha)| = 0$, we observed that $|\alpha S| = 1$ for each n. For the second statement Let $Im(\alpha)\{\}$ denotes an empty map, if $|Im(\alpha)| = k_i$ (where $i = 1$) the identity element in OCP_n, since k is the maximal elements in the image set of α, then $n(k_i)$ element of $Im(\alpha)$ can occurs from X_n in $2^n - 1$ ways for each value of $n = \{1,2, \ldots\}$. The result follows immediately.

Table 3.10: Order of Element of OCP_n

$\frac{n}{\lvert\omega^+(\alpha)\rvert} = k$	0	1	2	3	4	5	$\lvert\alpha S\rvert = 2^n - 1$
0	1						1
1	1	1					2
2	1	3	4				8
3	1	7	12	14			34
4	1	15	32	44	48		140
5	1	31	80	129	157	165	563

Theorem 3.11: Let $S = ODCP_n$, then $|\alpha S| = \binom{n+m}{2m}\binom{(n+1)+1}{m+1}$

Proof: Let $\alpha \in ODCP_n$, and let $dom(\alpha) \subseteq X_n$ such that $Im(\alpha) \subseteq X_n$. Since the empty map is the sub-set of $ODCP_n$ and $\alpha \in S$ is bijection then $|\alpha S| = 1$, if $|Im(\alpha)| = 0$ and if $|Im(\alpha)| = 1$, $|\alpha S| = 1$. Similarly, $|Im(\alpha)| = 2$ when $n = 2$, then we observed that $|Im(\alpha)| = m$ when $m = 1,2 \ldots$ and $n = 1,2 \ldots$ we have $\binom{n+m}{2m}$ element. It follows from theorem 3.7 that if $\alpha \in S$ and m elements of domain in a set X_n can be chosen from X_n and α is a bijection such that $dom(\alpha) \to Im(\alpha)$, then $\binom{(n+1)+1}{m+1}$ elements when $|Im(\alpha)| = 3$ and $n = 3$ can be chosen from X_n which implies that $|\alpha S|$ for $n \geq m \geq 3$ can occur in $\binom{n+m}{2m}\binom{(n+1)+1}{m+1}$ ways. Hence the result.

Table 3.12: Order of Element of $ODCP_n$

$\frac{n}{\lvert f(\alpha)\rvert} = m$	0	1	2	3	4	5	$\lvert\alpha S\rvert = \binom{n+m}{2m}\binom{(n+1)+1}{m+1}$
0	1						1
1	1	1					2
2	2	3	1				6
3	6	10	5	1			22
4	22	37	20	6	1		86
5	85	146	82	28	7	1	349

Theorem 3.13: Let $= ODCP_n$, then $|\alpha S| = 3n + 3$
Proof: Let $X_n = \{1,2 \dots\}$, then if $X_n \to X_n$ and the image of α $Im(\alpha)$ is such that $i = 1,2, \dots$ let $\alpha \in S$, then we observed that n elements of $dom(\alpha)$ can be chosen from X_n in $n(3)$ ways for $n = 3,4 \dots$ and $p = 2,3 \dots$ Then the number of order of $|S|$ for $\alpha \in ODCP_n$ is $3n + 3$, when $n = 3$ and $p = n - 1$. Since empty map is an element of $ODCP_n$ and p elements of domain of α can be chosen from X_n. If $\alpha \in S$ $|Im(\alpha)| = 0$ when $|\alpha S| = 1$ and when $|Im(\alpha)| = P_k$ where $k = \{0,1 \dots\}$ and $n = \{0,1 \dots\}$ we have that $|Im(\alpha)| = |\alpha S| = i$ (identity element). The result follows immediately.

Table 3.14: Order of Element of $ODCP_n$

$\frac{n}{\lvert Im(\alpha)\rvert} = h$	0	1	2	3	4	5	$\lvert\alpha S\rvert = 3n + 3$
0	1						1
1	1	1					2
2	1	4	1				6
3	1	11	9	1			22
4	1	26	46	12	1		86
5	1	57	185	90	15	1	349

Table 3.15: Calculated values of $\alpha(S)$ for small value of n

n	1	2	3	4	5	6 …..
CP_n	1	2	2	2	2	2 ……
OCP_n	1	3	7	15	31	63 …..
$ORCP_n$	1	4	5	6	7	8 …..
$ODCP_n$	3	9	27	81	245	729 …..

Table 3.16 Formula for the $\alpha(S)$ generated by a semi-group $S \subseteq P$

S	Formula
CP_n	$\binom{n-1}{p-1}$
OCP_n	$\sum_{r=0}^{n} \binom{n+2p}{3p}, \binom{n}{k}\binom{n-1}{k-1}$
$ORCP_n$	$5n + 1, \binom{n}{m}\binom{n+2}{(m-1)+2}$
$ODCP_n$	$3^n, 3n + 3, \binom{n+m}{2m}\binom{(n+1)+1}{m+1}$

4. Concluding Remarks

Remark 4.1: For all the semi-group presented, $f(n,m) = 1$ whenever $n = m$. Similarly, $f(n,k) = 1$ whenever $k = 0$ and $|OCP_n| = |OCP_n| = 2^n$. However, there exist bijection between the sets of the semi-groups of $P_n\ for\ n \geq 2$.

Remark 4.2: The combinatorial nature of integer sequences and their triangular arrangement arise naturally and thus make it essentially important to find their cardinalities, general formula, hence make it applicable to mathematics and science as a whole. (OIES)

Remark 4.3: The transformation semi-group was the most promising class of semi-groups for future study [7]. The forecast was justified because many researchers have worked extensively on the subject. Such as [8, 9] and their supervisor [2]. But few have worked on contraction mapping among them are: [9, 13].

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