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STRUCTURE OF THE SEMIGROUP OF FULL TRANSFORMATION RESTRICTED BY AN EQUIVALENCE

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Abstract

Let Xbe a nonempty set. The full transformation, $T(X)$ on a set X is the mapping from Xinto itself with a composition operation. This study is sequel to the work of Mendes-Goncalves and Sullivan (2011), who studied the semigroup of transformations restricted by an equivalence E(X,σ). We considered the structures of the semigroup of transformations restricted by an equivalence, E(X,σ) on two classes of semigroups, the Complete regular semigroup and Inverse Semigroup, thus we showthat E(X,σ) is completely regular but not an inverse semigroup on its largest regular subsemigroup for any non-trivial $|X| \geq 4$and characterize the Starred Ideals of E(X,σ).

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1 Introduction

Let Xbe an arbitrary nonempty set. The semigroup of full transformation $T(X)$ on a set X is the mapping from X into itself under composition. The Semigroup of Transformations restricted by an equivalence is an offshoot of the semigroup of transformations with restricted range $T(X,Y)$ which has been widely studied by various authors in [1 – 14] and a host of others. In [4], the author considered a subsemigroup of $T(X)$ determined by an equivalence relation *Y*on *X* defined thus:

$T(X,Y) = \{\alpha \in T(X): \forall\, x, y \in X, (x,y) \in Y \Rightarrow (x\alpha, y\alpha) \in Y\}$

stating *T*(*X,Y*) is regular if $\sigma = \{id_x, X \times X\}$ holds, where $id_x$is the identity relation on *X*, then *T*(*X,σ*) = *T*(*X*). Also [9], studied another subsemigroup of *T*(*X*) called the *Semigroups of Transformations Restricted by an equivalence, E*(*X,σ*) defined thus:

$E(X,\sigma) = \{\alpha \in T(X): \forall\, x, y \in X, (x,y) \in \sigma \Rightarrow x\alpha = y\alpha\}$

Unlike the full transformations semigroup *T*(*X*) which is known to be regular, *E*(*X,σ*) was characterised on its regular part which is known as *Largest Regular Subsemigroup*, $\mathbb{E}$defined as:

$\mathbb{E} = \{\alpha \in E(X,\sigma): Im(\alpha\ is\ a\ partial\ cross-section\ of\ {}^{X}/_{\sigma}\}$

in order to characterize its Green's relations and ideals. In [12] it was shown that the semigroup,$E(X,\sigma)$ is right abundant but not left regular whenever the equivalence σ on a set $X$is nontrivial. A result which is the dual of a similar result by [11] on $T(X,Y)$stating that $T(X,Y)$ is left abundant but not right abundant whenever $Y = X$and $|Y| \geq 2$.The cardinality of $E(X,\sigma)$as determined in [12] is stated thus that, if $|X| = n$and $|X/\sigma| = m$, where $n, m$are positive integers, then $|E(X,\sigma)| = n^m$. Recently, [13] stated that two semigroups of $E(X,\sigma)$, $E(X,\sigma_1)\ and\ E(X,\sigma_2)$ are isomorphic if and only if there exists a bijection $\theta : X \Rightarrow Y$such that $(x\sigma_1)\theta = (x\sigma_2)\theta$for all $x \in X$.In this paper, we extend the structure of $E(X,\sigma)$ by determining its completely regular semigroup, inverse semigroup and starred ideals.

This paper is arranged thus: In Section 2, we gave some basic definitions and examples to explain the semigroupof transformation restricted by an equivalence, (X,σ) , Section 3, we showed that the semigroup of transformation restricted

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by an equivalence, $E(X,\sigma)$is completely regular for all nontrivial $|X| \geq 4$. We build on the last result insection 4 by showing that $E(X,\sigma)$is not an inverse semigroup for all non-trivial, $|X| \geq 4$. In section 5, using the left-right characterization in [3], we discussed the Green's Starred relations of $E(X,\sigma)$and finally in section 6, we characterize the starred-ideals of $E(X,\sigma)$.

2 Preliminaries

Let X be a nonempty set and $T(X)$ is the semigroup (under composition) of the full transformation from X is to X(i.e. $\alpha{:}X \longrightarrow X$). Suppose σ is an equivalence a set X, we consider a subsemigroup of $T(X)$ defined as:

$E(X,\sigma) = \{\alpha \in T(X) : \forall\, x, y \in X, (x,y) \in \sigma \Rightarrow x\alpha = y\alpha\}$

We adopt a convention introduced in [16] which states that if $\alpha \in T(X)$ we write:

$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}$

where the subscript i belongs to some (unspecified) index set I, that the abbreviation $\{x\} = \{x_i : i \in I\}$, the image of $\alpha$is denoted by $X\alpha = im(\alpha) = \{x_i\}$and the partition of the set X is denoted as $x_i\alpha^{-1} = A_i$ also referred to as the $= dom(\alpha) = \cup\, x\alpha^{-1}$, which is the disjoint union of α for any $x \in dom(\alpha)$ and for any $\alpha \in dom(\alpha)$, the image of $x$under $\alpha$is denoted as $x\alpha$. If we fix an equivalence on σ on X, we can write $X/\sigma = \{S_i\}$ for the partition induced σ on X. In a partition each cells are disjoint (i.e. no overlapping/intersection) so $A = A_1 \cup A_2, \ldots, A_n$ where $A_1 \cap A_2 = \emptyset$.

Definition 2.1: *A subset Y of X is a* **Partial cross-section** *of X/σ if every σ-class in X/σ contains atmost one element of Y i.e. $\forall A \in X/\sigma$, $|A \cap Y| \leq 1$ (i.e. the cardinality can either be 0 or 1). But if $|A \cap Y| = 1$, we have a* **Cross-section**. *Equivalently, partial cross-section is when no two elements of the image come from the same equivalence class of σ. Thus partial cross-section has nothing to do with the domain or kernel but with the image.*

Example 2.1:*Considering a finite case. Let $X = \{1,2,3,4,5,6\}$ and*
$X/\sigma = \{\{1,2\},\{3,4\},\{5,6\}\}$
We define a map

$\alpha = \begin{pmatrix} \{123\} & \{456\} \\ 4 & 2 \end{pmatrix}$

where $A = \{1,2\}$, $B = \{3,4\}$, $C = \{5,6\}$ and $Im(\alpha_2) = \{2,4\} = Z$
$A \cap Z = 2$ *and* $|A \cap Z| = 1$
$B \cap Z = 4$ *and* $|B \cap Z| = 1$
$C \cap Z = 0$ *and* $|C \cap Z| = 0$
Thus $|A \cap Y| = |B \cap Y| = 1$ *and* $|C \cap Y| = 0$. *So this a*Partial Cross-Section.

Example 2.2:*Consider a finite case. Let $X = \{1,2,3,4,5,6\}$ and the partition of the set X given as*
$X/\sigma = \{\{1,2\},\{3,4\},\{5,6\}\}$
We define a map α_1 thus

$\alpha_1 = \begin{pmatrix} \{1234\} & \{56\} \\ 2 & 3 \end{pmatrix}$

Suppose we fix an equivalence σ on X thus
$\sigma = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4),(5,5),(5,6),(6,5),(6,6)\}$
Recall that only the elements in the same kernel can be paired. Thus the kernel of α_1:
$ker(\alpha_1) = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),$
$(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4),(5,5),(5,6),(6,5),(6,6)\}$
Comparing the elements in the fixed equivalence σ with $ker(\alpha_1)$. We see that since $\sigma \subseteq ker(\alpha_1)$, then $\alpha_1 \in E(X,\sigma)$ ■

Example 2.3:*We define a set α_2 thus*

$\alpha_2 = \begin{pmatrix} \{1\} & \{23456\} \\ 2 & 5 \end{pmatrix}$

The equivalence σ is
$\sigma = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4),(5,5),(5,6),(6,5),(6,6)\}$
Let the kernel of α_2 be given as:
$ker(\alpha_2) = \{(1,1),(2,2),(2,3),(2,4),(2,5),(2,6),(3,2),(3,3),(3,4),$
$(3,5),(3,6),(4,2),(4,3),(4,4),(4,5),(4,6),(5,2),(5,3),$
$(5,4),(5,5),(5,6),(6,2),(6,3),(6,4),(6,5),(6,6)\}$
Thus, comparing the elements in the fixed equivalence σ with the elements in $ker(\alpha_2)$, we see that $\sigma \nsubseteq ker(\alpha_2)$ since $(1,2)$ *and* $(2,1)$ *are in σ but not in $ker(\alpha_2)$. Therefore since $\sigma \nsubseteq ker(\alpha_2)$ and $\alpha_2 \notin E(X,\sigma)$* ■

Example 2.4: *We define a set α_3 thus*

$\alpha_3 = \begin{pmatrix} \{123\} & \{456\} \\ 2 & 3 \end{pmatrix}$

Thus the fixed equivalence σ = {(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4),(5,5),(5,6),(6,5),(6,6)}

Let the kernel of α_3 be given as:

$ker(\alpha_3)$ ={(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),
(4,4),(4,5),(4,6),(5,4),(5,5),(5,6),(6,4),(6,5),(6,6)}

Thus, comparing the elements in the fixed equivalence σ with the elements in $ker(\alpha_3)$, we see that since $\sigma \subseteq ker(\alpha_3)$ then $\alpha_3 \in E(X,\sigma)$ ■

Example 2.5: *We define a set α_4 thus*

$\alpha_4 = \begin{pmatrix} \{12345\} & \{6\} \\ 4 & 5 \end{pmatrix}$

Thus, σ = {(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4),(5,5),(5,6),(6,5),(6,6)}

We define the kernel of α_4 as:

$ker(\alpha_4)$ ={(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(2,4),
(2,5),(3,1),(3,2),(3,3),(3,4),(3,5),(4,1),(4,2),(4,3),
(4,4),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5),(6,6)

Comparing the elements in the fixed equivalence σ with the elements in $ker(\alpha_4)$, we see that $\sigma_4 \not\subseteq ker(\alpha_4)$ since (5,6) *and* (6,5) *are in σ but not in $ker(\alpha_4)$. Therefore since $\sigma \not\subseteq ker(\alpha_4)$ and $\alpha_4 \notin E(X,\sigma)$* ■

So far we have considered five (4) examples. Summarily, we have some α_i for i = {1,...,4} of the set X which make up the equivalence set of the semigroup of transformations restricted by an equivalence, $E(X,\sigma)$ ■

The followings are the characterization of regularity of $E(X,\sigma)$:

Theorem 2.7[9]**:** *Let σ be an equivalence relation on a set X. Then the following statements hold:*

(i) *If $\sigma = 1_x = Id_x$, the identity equivalence on X, then $E(X,\sigma) = T(X)$. Where $\sigma = 1_x$, then σ contains a constant map X_a with range a given as $\alpha = \begin{pmatrix} X_a \\ a \end{pmatrix}$.*

(ii) *If $\sigma = X \times X$ (The Universal Relation), then $E(X,\sigma) = K(X)$, where $K(X)$ is the set of all constant mapping in $T(X)$.*

Also [9], characterized $E(X,\sigma)$ on its largest regular subsemigroup, $\mathbb{E}$ thus

Theorem 2.8 [9]:*Let $\alpha,\beta \in E(X,\sigma)$, then*

(i) $\alpha \mathcal{L} \beta$ *if and only if ($\alpha,\beta \in \mathbb{E}, X\alpha = X\beta$);*

(ii) $\alpha \mathcal{R} \beta$ *if and only if ($\alpha,\beta \in \mathbb{E}, ker(\alpha) = ker(\beta)$), where $ker(\alpha) = x\alpha^{-1}; x \in X\alpha$;*

(iii) $\alpha \mathcal{D} \beta$ *if and only if ($\alpha,\beta \in \mathbb{E}, |X\alpha| = |X\beta|$);*

(iv) $\alpha \mathcal{J} \beta$ *if and only if ($ker(\alpha) = ker(\beta)$ or $|X\alpha| = |X\beta|$);*

(v) $\mathcal{D} = \mathcal{J}$.

3 Completely Regular Semigroup of $E(X,\sigma)$

A semigroup is said to be *Completely Regular* if every element in *S* is in some subgroup of the semigroup, thus it is referred to as *"Union of Groups"*. This is an important subclass of the class of regular semigroups and the class of inverse semigroups. The work of [17] laid the ground work in his paper using the term "*Semigroups Admitting Relative Inverses"* to refer the term *"Completely Regular Semigroup"*. Historically, the term *"Completely Regular Semigroup"* stems from the Russian literature written by, [21] titled *"Semigroup"* in which the author often refer to completely regular semigroup as *"Clifford Semigroup"*. It is observed that in a completely regular semigroup, each $\mathcal{H}-class$ is a group and the semigroup is the union of these groups. Thus, if e is the identity of G, a subgroup of S within G, we have $ea = ae = a$, $aa^{-1} = a^{-1}a = e$, hence $a \mathcal{H} e \in S$. Thus the $\mathcal{H}-class$, $\mathcal{H}_a$ which coincides with $\mathcal{H}_e$ is a group.

Next is the have the characterization of a completely regular semigroup.

Theorem 3.1[20]:

Let S be a semigroup. Then the following statements are equivalent:

(1) S is completely regular.

(ii) *Every element in S in a subgroup of S.*

(iii) *Every $\mathcal{H}-class$ in S is a group.*

Remark 3.2: *It is well known that $E(X,\sigma)$ is not regular except on its regular part called* largest regular subsemigroup *which was defined by* [9] *as* $\mathbb{E}=\{\alpha\in(X,\sigma) : Im(\alpha)$ *is a partial cross − section of X/σ*}.Thus, we consider a nontrivial domain for a set of X, where $|X|\geq 4$.

Theorem 3.3: *For any non-trivial X, where* $|X|\geq 4$. *The largest regular subsemigroup* $\mathbb{E}$ *of* $E(X,\sigma)$ *is a completely regular semigroup.*

Proof: It is clear that

$$E(X,\sigma) = \{\alpha \in T(X) : \sigma \in ker(\alpha) = \alpha \circ \alpha^{-1}\}$$

with count $|E(X,\sigma)| = n^m$ in ([21]). Thus for a set $X = \{1,2,3,4\}$, with two partition classes as

$X/\sigma = \{\{1,2\}\{3,4\}\}$

we have that $|X| = n = 4$ and $|X/\sigma| = m = 2$, thus

$|E(X_4,\sigma)| = n^m = 4^2 = 16$

We write all the 16 elements of $|E(X_4,\sigma)|$ as:

$$E(X,\sigma) = \left\{\begin{matrix} \begin{pmatrix}1234\\1\end{pmatrix}\begin{pmatrix}1234\\2\end{pmatrix}\begin{pmatrix}1234\\3\end{pmatrix}\begin{pmatrix}1234\\4\end{pmatrix} \\ \begin{pmatrix}12 & 34\\1 & 2\end{pmatrix}\begin{pmatrix}12 & 34\\2 & 1\end{pmatrix}\begin{pmatrix}12 & 34\\1 & 3\end{pmatrix}\begin{pmatrix}12 & 34\\3 & 1\end{pmatrix} \\ \begin{pmatrix}12 & 34\\1 & 4\end{pmatrix}\begin{pmatrix}12 & 34\\4 & 1\end{pmatrix}\begin{pmatrix}12 & 34\\2 & 3\end{pmatrix}\begin{pmatrix}12 & 34\\3 & 2\end{pmatrix} \\ \begin{pmatrix}12 & 34\\2 & 4\end{pmatrix}\begin{pmatrix}12 & 34\\4 & 2\end{pmatrix}\begin{pmatrix}12 & 34\\3 & 4\end{pmatrix}\begin{pmatrix}12 & 34\\4 & 3\end{pmatrix} \end{matrix}\right\}$$

Therefore, we select all the elements that satisfies the condition of the largest regular subsemigroup of $E(X,\sigma)$, $\mathbb{E}$which are partial cross section of the kernel classes as defined above which are twelve (12) out of the 16. From these we have elements of height 1 and 2 which forms the $\mathcal{H} -$ *classes* thus:

$\begin{pmatrix}1234\\1\end{pmatrix}$	$\begin{pmatrix}1234\\2\end{pmatrix}$	$\begin{pmatrix}1234\\3\end{pmatrix}$	$\begin{pmatrix}1234\\4\end{pmatrix}$

$\begin{pmatrix}12 & 34\\1 & 3\end{pmatrix}$ $\begin{pmatrix}12 & 34\\3 & 1\end{pmatrix}$	$\begin{pmatrix}12 & 34\\2 & 3\end{pmatrix}$ $\begin{pmatrix}12 & 34\\3 & 2\end{pmatrix}$	$\begin{pmatrix}12 & 34\\1 & 4\end{pmatrix}$ $\begin{pmatrix}12 & 34\\4 & 1\end{pmatrix}$	$\begin{pmatrix}12 & 34\\2 & 4\end{pmatrix}$ $\begin{pmatrix}12 & 34\\4 & 2\end{pmatrix}$

It is clear from this that, all the elements are idempotent and the largest regular subsemigroup $\mathbb{E}$is completely regular ■

4 Inverse Semigroup of *E*(*X*,*σ*)

The semigroup *S* is said to be an *Inverse Semigroup* if every element of *S*, has exactly one inverse. Equivalently, a semigroup is an inverse semigroup if it is regular and its idempotent commute. Thus an inverse semigroup is an example of a regular semigroup. Examples of Inverse Semigroup are Groups, Semilattices, Clifford Semigroup and Symmetric Inverse Semigroup.

This result is the characterization of inverse semigroup

Theorem 4.1 [20]: *Let S be a semigroup. Then the following statements are equivalent:*

(1) S is an inverse semigroup.

(ii) S is regular and its idempotent commute.

(iii) Every $\mathcal{L}-$ *class and every* $\mathcal{R}-$ *class contains exactly one idempotent.*

(iv) Every element of S has a unique inverse.

The next results considers the non-trivial domain which defies the regularity conditions of $E(X,\sigma)$, since it is well known that its largest regular subsemigroup is regular, we prove if its idempotents commute.

Theorem 4.2: *For any nontrivial X, where* $|X|\geq 3$. *The largest subsemigroup* $\mathbb{E}$ *of* $E(X,\sigma)$, *is not an inverse semigroup.*

Proof: This proof is straight forward from (Theorem 3.4). Here we show that the largest subsemigroup, $\mathbb{E}$of $E(X,\sigma)$ is an inverse semigroup. It is known that $\mathbb{E}$is regular and have also shown that it is completely regular, here we only need to verify if any two idempotents in its largest regular subsemigroupE commute. For any two idempotent, *e* and *f* we define

$e = \begin{pmatrix}12 & 34\\1 & 3\end{pmatrix}$and $f = \begin{pmatrix}12 & 34\\2 & 3\end{pmatrix}$

$ef = \begin{pmatrix} 12 & 34 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 12 & 34 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 12 & 34 \\ 2 & 3 \end{pmatrix}$
and
$fe = \begin{pmatrix} 12 & 34 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 12 & 34 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 12 & 34 \\ 1 & 3 \end{pmatrix}$

Thus, it is obvious that $ef \neq fe$thus any two idempotents in the largest regular subsemigroupE does not commute. So the largest regular subsemigroup$\mathbb{E}$of $E(X,\sigma)$ is not an inverse semigroup ■

5 Abundant Semigroup of $E(X,\sigma)$

The relations $\mathcal{L}^*$and $\mathcal{R}^*$on a semigroup S are generalization of the classical Green's relations L and R. Two elements *a* and*b* are said to be $\mathcal{L}^* - related$ if and only if they are $\mathcal{L} - related$ in some oversemigroup of *S*. The relation $\mathcal{R}^*$is defined dually. The join of the equivalence relations of $\mathcal{L}^*$and $\mathcal{R}^*$is denoted as $\mathcal{D}^*$and the meet is denoted as $\mathcal{H}^{**}$.

Definition 5.1 *Let S be a semigroup. Two elements α,β∈S are said to be:*

- $\mathcal{L}^* - related$*if and only if they are* $\mathcal{L} - related$ *in some oversemigroup of S.*
- $\mathcal{R}^* - related$*if and only if they are* $\mathcal{R} - related$ *in some oversemigroup of S.*

Definition 5.2 *A semigroup S is called* Abundant *if any* $\mathcal{L}^* - class$*and* $\mathcal{R}^* - class$ *contains an idempotent of*S.

It is well known that a regular semigroup is abundant but the converse is not true. For example, [23] showed that the semigroup of order-decreasing finite full transformations is abundant but not regular.

Definition 5.3:*The* $\mathcal{L}^*$*containing the element a of the semigroup S will be denoted by*${\mathcal{L}^*}_a$. *The corresponding notation is used for the other classes relations.*

Next, we present the characterization of the starred-Green's relations by [19]

Theorem 5.1 [19]: *Let S be a semigroup. Then*

(a) $\mathcal{L}^* = \{(a,b) \in S \times S : (\forall s,t \in S^1) as = at \Leftrightarrow bs = bt\}$

(b) $\mathcal{R}^* = \{(a,b) \in S \times S : (\forall s,t \in S^1) sa = ta \Leftrightarrow sb = tb\}$

Let $Y \subseteq X$ and denote$\overline{Y} = \{A \in X/\sigma : A \cap Y \neq \emptyset\}$ as the collection of equivalence with non-zero intersection.

Definition 5.4: *A transformation* $\alpha \in E(X,\sigma)$ *is* **Discrete** *on X if* $|A \cap X\alpha| \leq 1$ *for every* $A \in X/\sigma$*.We see that partial cross-section and discreteness coincides.*

Hence the characterization of the starred-Green's relation on $E(X,\sigma)$.

Theorem 5.3:*Let* $\alpha,\beta \in E(X,\sigma)$*. If* $(\alpha,\beta) \in \mathcal{L}^*$*, then* $\overline{Im(\alpha)} = \overline{Im(\beta)}$.

Proof:Let $\rho = \{A \in X/\sigma : A \in \overline{Im(\alpha)} - \overline{Im(\beta)}\}$.We show that $\rho = \emptyset$. For any nonempty ρ, we take some distinct elements $a,b \in X$ and prove that for all $p,q \in E(X,\sigma)$ with the property that for each $A \in X/\sigma$, we have that

$$p(A) = \begin{cases} q(a), & if\ A \in \rho \\ \{a\}, & if\ x \notin \rho \end{cases}$$

And

$$q(A) = \{b\}\ if\ A \in \rho$$

Clearly, $p \neq q$. By this, we prove two cases when $p\alpha = q\alpha$ and $p\alpha \neq q\alpha$. Let $A \in X/\sigma$ since $Im(A) \cap \rho$, if $B \in X/\sigma$ such that $Im(A) \subseteq B$, by this we have that $p\beta = q\beta$ and so for $p\alpha(A) = q\alpha(A)$. Thus, $p\alpha = q\alpha$. For any fixed $A \in \rho$ $Im(B) \cap A \neq \emptyset$ there exist $B \in X/\sigma$ such that $Im(B) \subseteq A$. Hence, $pIm(B) = p(A) = \{a\}$ and $qIm(B) = q(A) = \{b\}$. Then $p\beta \neq q\beta$, which is a contradiction when $(\alpha,\beta) \in \mathcal{L}^*$. Therefore, $\rho = \emptyset$ and $\overline{Im(\beta)} \subseteq \overline{Im(\alpha)}$. Also, by symmetry we have the converse where$\overline{Im(\alpha)} \subseteq \overline{Im(\beta)}$ and equality hold as $\overline{Im(\alpha)} = \overline{Im(\beta)}$■

Next we state the characterizations of Green's Starred Relations on $E(X,\sigma)$

Theorem 5.4 *Let* $\alpha,\beta \in E(X,\sigma)$*. Then* $(\alpha,\beta) \in \mathcal{L}^*$*if and only if either*

i. $Im(\alpha) = Im(\beta)$ *or*

ii. α,β *are not discrete on X and* $\overline{Im(\alpha)} = \overline{Im(\beta)}$.

Proof. Suppose, that α,β satisfy (i). Then α,β are $\mathcal{L}^*$*−related* in the full transformation $T(X)$, as such α and β are $\mathcal{L}^*$−related in $E(X,\sigma)$.

For the "if" part, suppose $(\alpha,\beta) \in \mathcal{L}^*$, it follows from (Theorem 5.1(a)) that$\overline{Im(\alpha)} = \overline{Im(\beta)}$. Clearly, there two possibilities, either α and β are discrete on X or α and β are not discrete on X ([12], Lemma 2.6). Thus we define the sets of α and β as $Im(\alpha) = \{ai : i \in I\}$and $Im(\beta) = \{bi : i \in I\}$, where I is some index set, $a_i \neq a_j$, $b_i \neq b_j$for any distinct, i,j

$\in I$ and $(a_i,b_i) \in \sigma$ for any $i \in I$. Now we show that $a_i = b_i$ for any $i \in I$. Take $c \in X$, for any $d \in E(X,\sigma)$, $d \neq id_x$ and $d\alpha = id_x\alpha$. Thus, for characterization of $\mathcal{L}^*$ in (Theorem 5.1 (a)) that $d\beta = id_x\beta$. Thus $d(b_i) = b_i$, for any $i \in I$. Since it is clear that $d(a_i) = b_i$ for any $i \in I$, then $a_i = b_i$ for any $i \in I$ and $Im(\alpha) = Im(\beta)$.

For the "only if " part.Let α,β satisfy (ii). We show that $(\alpha,\beta) \in \mathcal{L}^*$ by (Theorem 5.1 (a)). Now suppose $p\alpha = q\alpha$ for any $p,q \in E(X,\sigma) \cup \{id_x\}$. We assume that $p = id_x$ and $q \in E(X,\sigma)$. Then $\alpha = q\alpha$, since α is not discrete on X, we have that $|Im(\alpha) \cap A| \geq 2$ for some $A \in X/\sigma$ ([12], Lemma 2.6). So $|q(Im(\alpha) \cap A)| \geq 2$. But for $q \in E(X,\sigma)$, we have that $|q(Im(\alpha) \cap A)| = 1$ which is a contradiction. Thus by symmetry, we have two cases: either p and q equals an identity relation on X, id_x or $p,q \in E(X,\sigma)$. By the first case, we have that $p\beta = q\beta$, so we assume that $p,q \in E(X,\sigma)$. Let $A \in X/\sigma$, we take an element $B \in X/\sigma$ suchthat $Im(\alpha) = \{B\}$.Then $B \cap Im(\alpha) \neq \emptyset$,since $\overline{Im(\alpha)} = \overline{Im(\beta)}$.Let $x \in X$ such that $Im(x) \in B$. Then
$p\beta(A) = p(B) = \{p(Im(x))\} = \{q(Im(x))\} = q(B) = q\beta(A)$.
Hence, $p\beta = q\beta$ and $(\alpha,\beta) \in \mathcal{L}^*$ ■

Theorem 5.5 *Let $\alpha,\beta \in E(X,\sigma)$. Then $(\alpha,\beta) \in \mathcal{R}^*$ if and only if $ker(\alpha) = ker(\beta)$.*
Proof. For the "if" part, suppose $ker(\alpha) = ker(\beta)$, then α,β are $\mathcal{R}^* - related$ in the full transformation, $T(X)$ in (Theorem 5.1 (b)), hence R^*-related in $E(X,\sigma)$.

For the "only if" part. Let $(\alpha,\beta) \in \mathcal{R}^*$. For any $x \in X$, we define a constant map $\varphi = \binom{A}{a} \in E(X,\sigma)$.
Take $(a,b) \in E(X,\sigma)$. Then $\alpha<\varphi(a)> = <\alpha\varphi(a)> = <\varphi(b))> = \alpha<\varphi(b)>$ and by characterization of $\mathcal{R}^*$in (Theorem 5.1 (b)) we have that $\beta < \phi(a) > = \beta < \phi(b) >$. Thus $\beta(a) = \beta(b)$ and $(a,b) \in ker(\beta)$.
Thus $ker(\alpha) \subseteq ker(\beta)$ and by symmetry $ker(\beta) \subseteq ker(\alpha)$. Hence, equality holds and $ker(\alpha) = ker(\beta)$ ■
Definition 5.6: *Let σ be an equivalence relation on the set X. Let α,β be any two subsets of X and ψ be a map from α into β i.e. ($\psi : \alpha \longrightarrow \beta$). ψ is a $\sigma-$Preserving if for any $x,y \in \alpha, (x,y) \in \sigma$ implies $(x\psi, y\psi) \in \sigma$. ψ is said to be σ^*- Preserving if for any $x,y \in \alpha, (x,y) \in \sigma$ if and only if $(x\psi, y\psi)$.*
Theorem 5.7: *Let $\alpha,\beta \in E(X,\sigma)$. Then $(a,b) \in \mathcal{D}^*$ if and only if there exists a σ^*-preserving bijection on the map $\rho : X\alpha \longrightarrow X\beta$.*
Proof: Suppose, we define a relation τ on $E(X,\sigma)$ such that if and only if there exists an σ^*- *preserving bijection*, ρ: $X\alpha \longrightarrow X\beta$. Assume $(\alpha,\beta) \in \mathcal{L}^*$ on $E(X,\sigma)$,then $X\alpha = X\beta$. Clearly, $(\alpha,\beta) \in \tau$ and so $\mathcal{L}^* \subseteq \tau$.Now suppose that $(\alpha,\beta) \in \mathcal{R}^*$, then $ker(\alpha) = ker(\beta)$. Clearly, $|X\alpha| = |X\beta|$. We define a map $\rho : X\alpha \longrightarrow X\beta$ by $x\rho = x\alpha^{-1}\beta$. Thus, it is evident from the foregoing that the map $\rho : X\alpha \longrightarrow X\beta$ is a σ^*- preserving bijection. Dually, for any $(\alpha,\beta) \in \tau$ and so $\mathcal{R}^* \subseteq \tau$. Therefore, $\mathcal{D}^* \subseteq \tau$.
Conversely, suppose that $(\alpha,\beta) \in \tau$, then there exist an σ^*-preserving bijection $\rho : X\alpha \longrightarrow X\beta$. We define a map $\gamma : X \longrightarrow X$ by $x\gamma = a\rho$, where $x \in a\alpha^{-1}$ and $a \in X\alpha$. It is easy to see that $\gamma \in E(X,\sigma), ker(\gamma) = ker(\alpha)$ *and* $X\gamma = X\beta$. So that $(\alpha,\gamma) \in \mathcal{R}^*$,and $(\gamma,\beta) \in \mathcal{L}^*$. Thus $(\alpha,\beta) \in \mathcal{D}^*$and so $\tau \subseteq \mathcal{D}^*$and consequently equality holds and $\mathcal{D}^* = \tau$ ■
Remark 5.8: *We recall from [23] that two elements are $\mathcal{J}^*$- related if there exist a*starredideal *between them. So we require the starred-ideal to generate the $\mathcal{J}^*$analogue of the classical Green's relations on $E(X,\sigma)$, which is still dependent on the previous result on the $\mathcal{D}^*$relations.*

Theorem 5.9: *Let $\alpha,\beta \in E(X,\sigma), (\alpha,\beta) \in \mathcal{J}^*$, then $|X\alpha| = |X\beta|$.*
Proof: Suppose that $(\alpha,\beta) \in \mathcal{J}^*$, then $\mathcal{J}^*(\alpha) = \mathcal{J}^*(\beta)$. Let
$I(X,\beta) = \{\gamma \in (X,\sigma)(X) : |X\gamma| \leq |X\beta|\}$.
Hence, it is easy to see that $I(X,\beta)$ is a starred-ideal of $E(X,\sigma)$ to which βbelongs. Since $\alpha \in \mathcal{J}^*$,
$\mathcal{J}^*(\alpha) = \mathcal{J}^*(\beta) \subseteq I(X,\beta)$, then $|X\alpha| \leq |X\beta|$. Dually, we also obtain the similar result for β.
Hence, $|X\alpha| = |X\beta|$ ■

Theorem 5.10: *Let X be a finite set, then on the semigroup $E(X,\sigma)$, $\mathcal{D}^* = \mathcal{J}^*$.*
Proof: Suppose that $(\alpha,\beta) \in \mathcal{J}^*$, then $\mathcal{J}^*(\alpha) = \mathcal{J}^*(\beta)$. Let
$I(X,\beta) = \{\gamma \in E(X,\sigma) : |X\gamma| < |X\beta|\}$.
It is easy to show that $I(X,\beta)$, is a starred-ideal of $E(X,\sigma)$ to which β belongs. Since $\alpha \in \mathcal{J}^*(\alpha) = \mathcal{J}^*(\beta) \subseteq I(X,\beta)$, then $|X\alpha| < |X\beta|$, or there exists an σ^*- preserving bijection ρ: $X\alpha \longrightarrow X\beta$. Dually, we obtain the similar results for β. Hence there exists an σ^*-preserving bijection ρ: $X\alpha \longrightarrow X\beta$ consequent to (Theorem 5.7), so that $(\alpha,\beta) \in \mathcal{D}^*$and $\mathcal{J}^*.\subseteq \mathcal{D}^*$. We recall that in the characterization of the full transformation semigroup $T(X)$, that $\mathcal{D}^* \subseteq \mathcal{J}^*$. Thus equality holds as $\mathcal{D}^* = \mathcal{J}^*$.as required ■

Theorem 5.11 *Suppose $\alpha,\beta \in E(X,\sigma)$such that $(\alpha\beta)^2 = \alpha\beta$. Then α and β are regular.*
Proof: Suppose $A \in X/\sigma$. It is clear that for any $\beta \in E(X,\sigma)$, $A\beta \subseteq B$ for some $B \in X/\sigma$. Thus we can we see that $B\alpha\beta\alpha = B\alpha\beta$, which implies that $B\alpha\beta$ is contained in B and consequently, $B\alpha \subseteq A$. Thus $A \cap X\alpha \neq \emptyset$ implies for each $A \in X/\sigma$. So we conclude that α is regular.

Let $A \in X/\sigma$, suppose $\alpha \in E(X,\sigma)$, $A\beta \subseteq B$ for some $B \in X/\sigma$. Then we have that $A\alpha\beta\alpha = A\alpha\beta$, implying that $A\alpha\beta$ is contained in A and $B\beta \subseteq A$. Thus $A\cap X\beta \neq \emptyset$ for any $A \in X/\sigma$. Thus, β is regular ■

Theorem 5.12 *Let $\alpha \in E(X,\sigma)$. Then α is regular if and only if $A\cap X\alpha \neq \emptyset$ for any $A \in X/\sigma$*
Proof: We assume that α is regular, so that $\alpha\beta\alpha = \alpha$ for some $\beta \in E(X,\sigma)$, $A\beta \subseteq B$ for some $B \in X/\sigma$. Thus by $B\alpha\beta\alpha = B\alpha$, therefore $B\alpha\beta$ is contained in B and consequently $B\alpha \subseteq A$, which is a contradiction to our previous assumption that $A\cap X\alpha$ is empty. So we can prove that for any $A \in X/\sigma, A\cap X\alpha \neq \emptyset$. Let

$$x\beta = \begin{cases} a, & if\ x \in X\alpha\ where\ a \in x\alpha^{-1} \\ b, & if\ x \in A \setminus X\alpha\ where\ b \in (A\cap X\alpha)\alpha^{-1} \end{cases}$$

It is trivial to show that $\beta \in E(X,\sigma)$. Suppose $x \in X$, then $x\alpha\beta\alpha \in x\alpha\alpha^{-1} = \{x\alpha\}$. Thus the cardinality of $\{x\alpha\}$ gives us a constant map i.e. $|\{x\alpha\}| = 1$ and $\alpha\beta\alpha = \alpha$ which proves regularity ■

Theorem 5.13 *$E(X,\sigma)$ is abundant if and only if $|X/\sigma|$ is finite.*
Proof: Suppose $|X/\sigma|$ is infinite, for any partitions set of X with A_i (where i= 1,2,3,4,...) as
$X/\sigma = \{A_1,A_2,A_3,A_4...\}$. We define a map $\xi : X \longrightarrow X$ by $x \in A_i$, $x\xi = a_{i+1}$, where $a_{i+1} = A_{i+1}$, i = 1,2,3,4,.... Thus by this we see that $\xi \in E(X,\sigma)$ and $A \cap X\alpha = \emptyset$.So we see that all the elements in A_i generates distinct integral powers which is not periodic since no two power repeat. Thus $E(X,\sigma)$ is not abundant, which contradicts our assumption. Thus, $|X/\sigma|$ is finite.
Conversely, Suppose $|X/\sigma|$ is finite, it is clear that $A \cap X\alpha \neq \emptyset$ for any $\xi \in E(X,\sigma)$, and $A \in X/\sigma$. We see, by (Theorem 5.1.) that any $\mathcal{L}^*-$ *class* and $\mathcal{R}^*-$ *class* contains an idempotent, and hence $E(X,\sigma)$ is abundant ■

6 The Starred-Ideal of $E(X,\sigma)$

Analogous to the work of [19], we introduce the starred-ideal to obtain the starred analogue of the classical Green's relations J. The which $\mathcal{L}^*-$ *class* containing the element a is denoted as $\mathcal{L}_a^*$. We can also adopt this corresponding notation for the relations.Thus, a *Left(Right) starred -ideal* of a semigroup S to be the *Left(Right) Ideal I of S* for which $\mathcal{L}_a^* \subseteq I$ ($\mathcal{R}_a^* \subseteq I$) for all $a \in I$. A subset I of S ($I \subseteq S$) is a starred-ideal if it is both a Left-Starred Ideal and a Right-Starred Ideal. *The Principal Starred Ideal*, $\mathcal{J}^*(a)$ generated by the element a of S is the intersection of all *Starred−Ideals* of S to which a belongs. The relations J^* is defined by the rule if and only if $\mathcal{J}^*(a) = \mathcal{J}^*(b)$.It is also important to note in this section that this is where we get to understand the role of $\sigma -$ *Preserving* and σ^*- *Preserving Ideal* to $E(X,\sigma)$ previously mentioned, we state this for emphasis:

Definition 6.1: *Let σ be an equivalence relation on the set X. Let $\alpha,\beta \subseteq X$ and φ be a mapping from α into β i.e. ($\varphi : \alpha$ 7−→ β). φ is a sigma−preserving if for any $x,y \in \alpha, (x,y) \in \sigma$ implies $(x\varphi, y\varphi) \in \sigma$. φ is said to be a σ^*- Preserving if for any $x,y \in \alpha, (x,y) \in$ sigma if and only if $(x\varphi, y\varphi) \in \sigma$. In otherwords, it is σ^*- preserving if φ is both $\sigma -$ preserving and bijective.*

We define our starred ideal for starred ideal for $E(X,\sigma)$ similar to [18] thus,
Let X/σ be the partition of X into equivalence classes of σ. For any $\alpha \in E(X,\sigma)$, we define an infinite collection of non-intersecting equivalence classes as
$Z(\alpha) = \{A \in X/\sigma : A \cap X\alpha = \emptyset\}$
For any non-negative integer r, let
$Q^*(X,r) = \{\alpha \in E(X,\sigma) : r \leq |Z(\alpha)| < +\infty\}$
be the starred ideal of $E(X,\sigma)$.
Theorem 6.1: *The followings were stated that:*
(i) *if r = 0, then $Q^*(X,r)$ is a starred-ideal of $E(X,\sigma)$.*
(ii) *if r >0, then $Q^*(X,r)$ is a Left starred-ideal of $E(X,\sigma)$.*
(iii) *if r >0, then $Q^*(X,r)$ is not a right starred-ideal of $E(X,\sigma)$ for any $\alpha \in Q^*(X,r)$ such that $\mathcal{R}_\alpha^* Q^*(X,r)$* ■

Remark 6.2: According [18], *if r >0, then all the Green's relations are trivial in $Q^*(X,r)$(r >0). Thus we denote σ_α as the restriction of σ to the $X\alpha$.*
$r_\alpha = \{(a,b) \in r : a,b \in X\alpha\}$
In proving this theorem we consider the finite and infinite case.

Theorem 6.3: *Suppose r >0 and $\alpha,\beta \in Q^*(X,r)$ Then $(\alpha,\beta) \in \mathcal{L}^*$ if and only if $Im(\alpha) = Im(\beta)$.*
Proof: If $X\alpha = X\beta$ thus $(\alpha,\beta) \in T(X)$. Hence, $(\alpha,\beta) \in \mathcal{L}^*$. Conversely, suppose that $(\alpha,\beta) \in \mathcal{L}^*$ for all $v,\mu \in Q^*(X,r), \alpha v = \alpha\mu$ if and only if $\beta v = \beta\mu$. It is clear if $X\alpha \neq X\beta$, we assume that $X\beta \setminus X\alpha \neq \emptyset$ such that there exist $\alpha \in X\beta \setminus X\alpha$ and $b\beta = a$ for some $b \in X$. By this we have 2 cases to consider:

Case 1:
Considering the finite case where $a \in A \in X/\sigma$ and $A\cap X\alpha \neq \emptyset$. Now suppose that there exists $c \in A\cap X\alpha$. We see by $\alpha \in E(X,\sigma)$, $|X/\sigma| = |X\alpha/\sigma_\alpha|$, we have a σ^*- *preserving* mapping given as
$\varphi: X \setminus A \longrightarrow X\alpha \setminus A$. We define a map $x\,\varphi: X \longrightarrow X$ by

$$x\eta = \begin{cases} x, & x \in A \\ x\varphi, & x \notin A \end{cases}$$

We also define another map thus: $\mu: Z \longrightarrow Z$ by

$$z\mu = \begin{cases} c, & z = a \\ x\varphi, & if\ z \in A\{a\} \\ z\varphi, & else \end{cases}$$

But it is obvious that $\eta,\mu \in Q^*(X,r)$ and $\alpha\eta = \alpha\mu$. However,
$b\beta\eta = \alpha\eta = a \neq c = a\mu = b\beta\mu$
which is in contradiction to our assumption with $\beta\eta = \beta\mu$.

Case 2:
Considering the infinite case $a \in A \in X/\sigma$ and $A \cap X\alpha \neq \emptyset$. We define a map $\mu: X \longrightarrow X$ thus: for an element $a \in A$, $x\mu = x\alpha$. It is then obvious that $\mu \in E(X,\sigma)$ and $\alpha^2 = \alpha\mu$. However, $b\beta\alpha = a\alpha \neq a = a\mu = b\beta\mu$ which is in contradiction with our assumption $\beta\alpha = \beta\mu$. Thus our proof is complete since $X\alpha = X\beta$ ■

Theorem 6.4 *Suppose* $r > 0$ *and* $\alpha,\beta \in Q^*(X,r)$. *Then* $(\alpha,\beta) \in R^*$ *if and only if* $\ker(\alpha) = \ker(\beta)$.

Proof: Suppose $\ker(\alpha) = \ker(\beta)$, thus by [Theorem 5.1 (b)], $(\alpha,\beta) \in \mathcal{R} \in T(X)$. Thus if and only if $(\alpha,\beta) \in \mathcal{R}^*$. We show the converse that if $(\alpha,\beta) \in \mathcal{R}^*$ by [Theorem 5.1 (b)] $\forall x,y \in Q^*(X,r)$, $\eta\alpha = \mu\alpha$ if and only if $\eta\beta = \mu\beta$. Thus, suppose, $\ker(\alpha) \neq \ker(\beta)$, then there exist arbitrary element $y_1, y_2 \in A \in X/\sigma$ such that $y_1 \neq y_2$, $y_1\alpha\alpha^{-1} = y_2\alpha\alpha^{-1}$ and $y_1 \neq y_2$, $y_1\beta\beta^{-1} = y_2\beta\beta^{-1}$. With this, two (2) cases arise for consideration.

Case 1:
We consider the finite case where $A\cap X\alpha \neq \emptyset$. Since $\alpha \in E(X,\sigma)$, then
$|X/\sigma| = |X\alpha/\sigma_\alpha|$, we see that there exist an σ^*- *Preserving* mapping given by
$\varphi: X \setminus A \longrightarrow X\alpha \setminus A$

We define a map $\eta: Y \longrightarrow Y$ by

$$y\eta = \begin{cases} y_1, & if\ y \in A \\ y\varphi, & if\ y \notin A \end{cases}$$

Also we define another map $\mu: Y \longrightarrow Y$ by

$$y\mu = \begin{cases} y_2, & if\ y \in A \\ y\varphi, & if\ y \notin A \end{cases}$$

It is thus clear that $\eta,\mu \in Q^*(X,r)$ and $\eta\alpha = \mu\alpha$. However,
$A\eta\beta = y_1\beta \neq y_2\beta = A\mu\beta$
which is in contradiction to our assumption with $\eta\beta = \mu\beta$.

Case 2: Considering the infinite case where $A\cap X\alpha = \emptyset$.

We define a map $\eta: Y \longrightarrow Y$ by

$$y\eta = \begin{cases} y_1, & if\ y \in A \\ y\varphi, & if\ y \notin A \end{cases}$$

Also we define another map $\mu: Y \longrightarrow Y$ by

$$y\mu = \begin{cases} y_2, & if\ y \in A \\ y\varphi, & if\ y \notin A \end{cases}$$

Thus it is easy to see that $\eta,\mu \in Q^*(X,r)$ and $\eta\alpha = \mu\alpha$. Thus,
$A\eta\beta = y_1\beta \neq y_2\beta = A\mu\beta$
Contradicts our assumption that $\eta\beta = \mu\beta$, from which we conclude that $\ker(\alpha) = \ker(\beta)$ ■

Theorem 6.5 *Suppose* $r > 0$ *and* $\alpha,\beta \in Q^*(X,r)$. *Then* $(\alpha,\beta) \in \mathcal{D}^*$ *if and only if there exists a* σ^*- *Preserving bijection* $\sigma: Im(\alpha) = Im(\beta)$.

Proof: Let σ be a relation on $Q^*(X,r)$. such that $(\alpha,\beta) \in \sigma$ if and only if there exists a σ^**−preserving* bijection: $\sigma : X\alpha \longrightarrow X\beta$. Assume $(\alpha,\beta) \in \mathcal{L}^*$on $Q^*(X,r)$, then $X\alpha = X\beta$. Thus $(\alpha,\beta) \in \sigma$and so $\mathcal{L}^* \subseteq \sigma$. Next, suppose that $(\alpha,\beta) \in \mathcal{R}^*$, then $ker(\alpha) = ker(\beta)$. Clearly, $|X\alpha| = |X\beta|$. We define a map $\sigma : X\alpha \longrightarrow X\beta$ by $x\sigma = x\alpha^{-1}\beta$., From the foregoing, the map $\sigma : X\alpha \longrightarrow X\beta$ is a σ^**−preserving* bijection. Dually, for any $(\alpha,\beta) \in \sigma$and so $\mathcal{R}^* \subseteq \sigma$.

Conversely, suppose that $(\alpha,\beta) \in \sigma$, then there exist a σ^**− preserving* bijection$\sigma : X\alpha \longrightarrow X\beta$. We define a map γ: $X \longrightarrow X$by $x\gamma = a\sigma$, where $x \in a\alpha^{-1}$ and $a \in X\alpha$. It is very easy to see that $\gamma \in E(X,\sigma), ker(\gamma) = ker(\alpha)$ and $X\gamma = X\beta$.So that $(\alpha,\gamma) \in \mathcal{R}^*$and $(\gamma,\beta) \in \mathcal{R}^*$and $(\gamma,\beta) \in \mathcal{L}^*$. Thus $(\alpha,\beta) \in \mathcal{D}^*$and $\xi \subseteq \mathcal{D}^*$. Thus equality holds and $\mathcal{D}^* = \sigma$■

Theorem 6.6 *Suppose* $r > 0$ *and* $\alpha,\beta \in Q^*(X,r)$. *Then* $(\alpha,\beta) \in \mathcal{J}^*$*if and only if* $|Im(\alpha)| = |Im(\beta)|$.
Proof: It is obvious that two elements are $\mathcal{J}^*$*− related* if there exists a starred-ideal between them.
Now suppose $(\alpha,\beta) \in \mathcal{J}^*$, then $\mathcal{J}^*(\alpha) = \mathcal{J}^*(\beta)$. Let

$Q^*(X,r) = \{\alpha \in E(X,\sigma) \leq |Z(\alpha)| < +\infty\}$

It is not difficult to see that $Q^*(X,r)$is a starred-ideal of $E(X,\sigma)$ to which βbelongs. Since $\alpha \in \mathcal{J}^*, \mathcal{J}^*(\alpha) = \mathcal{J}^*(\beta) \subseteq Q^*(X,r)$then $|X\alpha| \leq |X\beta|$. Dually, we have that since$\in \mathcal{J}^*, \mathcal{J}^*(\beta) = \mathcal{J}^*(\alpha) \subseteq Q^*(X,r)$, then $|X\alpha| \leq |X\beta|$. Hence, $|X\alpha| = |X\beta|$■

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