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ON THE PROPERTIES OF A NILPOTENT GROUP

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Abstract

We study properties of nilpotent group and extend some of the existing results in the existing literature.

***Keywords*:** p-group, maximal subgroup, nilpotency class, nilpotent group,

1. Introduction

The concept of nilpotent group arose in Galois theorem, as well as in the classification of groups and it was credited to the work of a Russian mathematician Sergei Chernikov in 1930s. In group theory, nilpotent group is said to be "almost abelian" and the idea is motivated by the fact that, the quotient of every proper subgroup of a nilpotent group is abelian [1]. A group is nilpotent ifit has a central series.

In 1930's S. N Chernikov studied groups that satisfies minimal condition and its subgroup which descends and terminates in finitely many steps. Every finite group that satisfies normalizer condition is characterized to be nilpotent and normalizer condition is also equivalent to minimal condition for a group. An infinite group that exhibits the character of nilpotent group is known as hypercentral group. In [2], finite nilpotent group and its generalization was studied by considering its properties; chain of ascending central series and the group is a direct product of its Sylow p-subgroups. The subgroup of a nilpotent group satisfies the properties of the group which make such a group to be locally nilpotent. Examples are abelian and p-groups. An observation was made by [6] that the nilpotency class of a finite groups is at most 2.

2. Some Basic Definitions

i. **Center of a Group:** Let Gbe a group. Center of G, $Z(G)$ is the set of element that commute with other element of the group.
$Z(G) = \{x \in G \mid xy = yx, \forall y \in G\}$

ii. **Normalizer of subgroup:** Let G be a group and A be a subgroup of G, then normalizer of A in G is defined as $N_G(A) = \{x \in G \mid xy = yx, \forall y \in A\}$

iii. **Maximal subgroup:** A normal subgroup H of a group G is said to be a maximal if there is exists no proper normal subgroup K of G which properly contains H

iv. **Commutator:** Let G be a group and $a, b \in G$, then an element $aba^{-1}b^{-1} \in G$, is the commutator of a and b denoted as $[a, b]$. Then the subgroup of G, generated by all commutators in G, is called commutator subgroup of G denoted by $\gamma(G)$.

v. **Central Series:** Let G be a group, the upper central series of G is a normal series
$1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$
such that$G_{i+1}/G_i \leq Z(G/G_i)$, $0 \leq i \leq n-1$.

vi. **Nilpotent group:** A group, G is said to be nilpotent if it has a central series and the shortest length of the central series of G is called the nilpotency class.

Remark 2.1 Let G be a nilpotent group and p be a prime number then if G is;

i. trivial then the nilpotency class of G is 0,
ii. abelian then the nilpotency class is 1,
iii. non abelian group the nilpotency class is 2.

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3. Some Existing Results

Theorem 3.1. A group G is nilpotent if and only if G has a central series[7].

Theorem 3.2. Everynontrivial nilpotent group has a nontrivial center [8].

Theorem 3.3. Every finite p group is nilpotent, where p is any prime number[5].

Theorem 3.4. Everynilpotent group satisfies the normalizer condition [8].

Theorem 3.5. Let G be a nilpotent group.If H is a nontrivial normal subgroup of G,then the intersection of H and the center of G is nontrivial [8].

Theorem 3.6. Let G be a finite group. The following conditions on G are equivalent:

i. G is nilpotent

ii. Every Sylow p-subgroups of G is normal

iii. G is a direct product of Sylow p-subgroups of G[5].

Remark 3.1. Everymaximal subgroup of a group is nilpotent.

Proof. Let M be a maximal subgroup of a group G,since every maximal subgroup is normal then

$\implies g \in G, gmg^{-1} = m, m \in M$

$\implies mn = nm, \quad \forall\, n, m \in M$

Hence, $Z(M) = M$ from Theorem 2.2, M has a central series.

Therefore, M is nilpotent.

Theorem 3.7. Every abelian group is a nilpotent group.

Remark 3.2. The converse of Theorem 3.6 is not true in general.

Proof. Suppose G is nilpotent and $Z(G) \neq G$ such that $\exists\, x \neq e \in Z(G)$then from Theorem 2.2there exists $g \in G$ such that $gx \neq xg, \forall\, x \in G \implies x \notin Z(G)$.Hence, since $Z(G) \neq e$ it implies that G has a central series but is not abelian.

4. Main Results

Theorem 4.1. Let G be a p-group of nilpotency class 2 then $\gamma(G) = Z(G)$

Proof. Let G be a group with a normal series $e = G_0 \leq G_1 \leq \cdots \leq G_n = G$ such that $G_{i+1}/G_i \leq Z(G/G_i),\ 0 \leq i \leq n$.

Let G be a group of order p^n where p is prime and $Z(G) \neq G$, then the nilpotency class of G is 2. From Theorem 2.1,$G_i \leq Z_i(G)$, for $i \geq 0$. Since $\gamma_i(G) \leq G$, then$\exists\, G_i = \gamma(G)$ for some i. Since $|G| = p^n$,and $Z(G) \neq e$ then there exists $e \neq x \in \gamma(G)$ such that for $a, b \in G$, we have$x = aba^{-1}b^{-1}$. Then $\forall g \in G$, $gx = xg$

$\implies g(aba^{-1}b^{-1}) = (aba^{-1}b^{-1})g$.

Hence, $\forall x \in \gamma(G),\ x \in Z(G) \implies \gamma(G) = Z(G)$.

Theorem 4.2. Let H be a nontrivial subgroup of an abelian nilpotent group, then $H \cap \gamma(G) = e$.

Proof. Let$G$be an abelian group and $\gamma(G)$ be the commutator subgroup of G. Now, since for all $x, y \in G$, we have $\gamma(G) = \{\langle [x,y] \rangle : x, y \in G\}$,

where $[x,y] = xyx^{-1}y^{-1} = (xy)(xy)^{-1} = xyy^{-1}x^{-1} = xex^{-1} = xx^{-1} = e$.

Therefore, for any nontrivial subgroup H of $G, H \cap \gamma(G) = e$.

Theorem 4.3. Let G be a p-group. Then for any nontrivial normal subgroup$H$ of G,$H \cap \gamma(G) \neq e$ if and only if G has nilpotency class 2.

Proof. Let G be a group such that $Z(G) \neq G$ and $\gamma(G)$ be the commutator subgroup of G. Then from Theorem 4.1 we have $\gamma(G) = Z(G)$, and by Theorem 2.4 for any nontrivial normal subgroup H of $G, H \cap \gamma(G) \neq e$ as required.

Conversely, suppose $H \cap \gamma(G) \neq e$, then $\exists\, a \neq e \in H \cap \gamma(G)$. This implies $\gamma(G) \neq e$ thus G is a non-abelian group. Hence G has nilpotency class 2.

Theorem 4.4. Let H be a proper subgroup of a nilpotent group G. Let c and t be the nilpotency class of G and H respectively,then $c > t$ if $Z(G) \neq G$.

Proof. Let G be a nilpotent and $H < G$. Let c and t be the nilpotency class of G and H respectively. Since$G$ is nilpotent, from Theorem 2.5 $H$is nilpotent then $Z(H) = H$. By Remark 2.1 the nilpotency class of H is 1. Let G be a nilpotent group such that for

$g \in G, gx \neq xg, x \in G \implies x \notin Z(G)$

Hence, $Z(G) \neq G$. By Remark2.1, $c > t$.

Theorem 4.5. Let G be a p-groupwith nilpotency class2.Then G is not a direct product of its subgroups.

Proof. Let G be a group with a normal series $e = G_0 \leq G_1 \leq \cdots \leq G_n = G$ such that
$G_{i+1}/G_i \leq Z(G/G_i), 0 \leq i \leq n$.
Then from Theorem 4.1,we have $Z(G) = \gamma(G)$ for groups with nilpotency class 2.
Since G is a p-group with nilpotency class 2 then by Theorem3.3, for every subgroup K and H of $G$where$|K| = p^n / |G|$and$p^{n+1}/|G|$, similarly the result holds for H. Hence,
$K \cap H \neq e \implies G \neq KH$.

Theorem 4.6. Let G be a finite group. Then G is nilpotent if and only if $Z(G) \neq e$.

Proof. Let G be a finite group and $Z(G)$ be the center of G. Suppose G is nilpotent then
$e = G_0 \leq G_1 \leq \cdots G_n = G$ is a central series for G, therefore from Theorem 2.2, $Z(G) \neq e$.Conversely, suppose $Z(G) \neq e$ then G has a central series
$e = Z_0(G) \leq Z_1(G) \leq \cdots \leq Z_c(G) = G$,
such that $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G)), \quad 0 \leq i \leq c$. Hence by Theorem 2.1 G is nilpotent.

Theorem 4.7. Let G be a non-nilpotentgroup. Then the quotient of maximal subgroup in G is nilpotent.

Proof. Let G be a group and M be the maximal subgroup of G, which implies that for any
$g \in G, gmg^{-1} \in M, m \in M \implies gm = mg$. Hence, $M \trianglelefteq G$.Let $G/M = \{Mg : g \in G\}$ and$g_1, g_2 \in G$, then we have $(Mg_1)(Mg_2) = M(Mg_1)g_2$.Since$M$ is normal in $G, Mg_1 = g_1M$, then we have$M\big((g_1M)\big)g_2 = MMg_1g_2 = Mg_1g_2$.This shows that G/M is abelian which implies that $Z(G/M) \neq M$. Hence, G/M is nilpotent.

Theorem 4.8. Let G be a nontrivial nilpotent group. Then the order of commutator subgroup of G is equal to the nilpotency class of G.

Proof. Let G be a finite group with $e = Z_0(G) \leq Z_1(G) \leq \cdots \leq Z_C(G) = G$as the centrals series of G and c is the nilpotency class of G and $\gamma(G)$ be the commutator subgroup of G.Suppose $\forall\, x, y \in G$, and $xy = yx$ then we have $Z_0(G) = e, Z_1(G) = G$.This implies that $c = 1$. Hence, $\gamma(G) = \{\langle [x, y] \rangle, \forall x, y \in G\} = e$since$\forall\, x, y \in G, xy = yx$ by Remark 2.1, $|\gamma(G)| = 1 = c$. Therefore the result holds.
Also, if $Z(G) \neq G$which implies that$\exists\, z \in G$ such that $\exists\, z \notin Z(G)$we have
$Z_0(G) = e,\ Z_1(G) = \gamma(G), \quad \ldots, Z_n(G) = G$.
Hence from Remark 2.1,$c = 2 = |\gamma(G)|$.

Remark 4.1. The order of center of a group G is equal to the nilpotency class of G if $Z(G) \neq G$.

5. Conclusion

In this paper, we extended the intersection of normal subgroup and the center of the group, the quotient of all the subgroup in a nilpotent is satisfies nilpotent property, every abelian group is nilpotent, sylow$p$-subgroup is the direct product of nilpotent group and the nilpotency class of a group is at most 2. The group is nilpotent if the center is non-trivial.

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