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A COROLLARY OF THE OBCT

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Abstract

For $n \in \mathbb{N}$, $n \geq 2$, if MetricTop($\mathbb{R}^n$) is the metric topo- logy of $\mathbb{R}^n$ induced by the Euclidean metric $d_{\|\ \|}$ on $\mathbb{R}^n$, and ProductTop($\mathbb{R}^n$) = $\prod_n \tau_{\mathrm{R}}$ is the product topology on $\mathbb{R}^n = \mathbb{R}\times\mathbb{R}\times \ldots \times\mathbb{R}$ of n copies of $\tau_{\mathbb{R}}$ ($\equiv$ the usual topology of $\mathbb{R}$), then, MetricTop($\mathbb{R}^n$) = ProductTop($\mathbb{R}^n$). We invoke the OBCT to obtain this equality.

***Keywords*:** Usual topology of $\mathbb{R}$, metric topology of $\mathbb{R}^n$, product topology of $\mathbb{R}^n$.

1. LANGUAGE AND NOTATION

Our language and notation shall be pretty standard as found in standard texts of Undergraduate General Topology; for an instance, as found in [1, 2, 3, 4] Of course, we refer to terminologies already defined in [5]. For an instance, the *real numbers* $\mathbb{R}$, the *Euclidean norm*, $\|\ \|$, on $\mathbb{R}^n$, an *interval* in $\mathbb{R}^n$, a *cell* in $\mathbb{R}^n$, the *kth side of* a cell, an *open* interval */open* cell, the ball of radius r centred on $a \in \mathbb{R}^n$, $B(a, r)$, an *interior* point of A, $\varnothing \neq A \subseteq \mathbb{R}^n$. We signify the end or absence of a proof by. /// As pointed out in the Abstract, our task is to prove, using the OBCT[5], that MetricTop($\mathbb{R}^n$) = ProductTop($\mathbb{R}^n$). We first briefly review

(1) The theory of the concept of a *subbase* for a topology, and
(2) The theory of the concept of a product topology.

2 SUBBASE Let $X \neq \varnothing$ and consider a non-empty subfamily $\varnothing \neq \mathcal{L} \subseteq 2^X$ of subsets of X. Form the family

$B_{\mathcal{L}} = \{A \subseteq X : A$ is a finite intersection of members of $\mathcal{L}\}\cup\{X, \varnothing\}$ and then the family

$\tau_{\mathcal{L}} = \{G \subseteq X : G$ is a union of members of $B_{\mathcal{L}}\}$

$= \{\varnothing, X\}\cup\{$unions of finite intersections of members of $\mathcal{L}\}$.

$\tau_{\mathcal{L}}$ is a topology, and it is the unique smallest topology on X in which the members of $\mathcal{L}$ are open sets. $\tau_{\mathcal{L}}$ is called the *topology generated by* $\mathcal{L}$. The family $\mathcal{L}$ is called a *subbase* for $\tau_{\mathcal{L}}$.

3 PRODUCT TOPOLOGY If $I \neq \varnothing$ and the cardinality of I, $|\,I\,| \geq 2$ and $X_k \neq \varnothing$ for each $k \in I$, we denote the *Cartesian product* of the X_k's by $\prod_{k\in I} X_k$. Now suppose that for $k \in I$ τ_k is a topology on X_k, and so we have an indexed family of topological spaces $(X_k, \tau_k)_{k\in I}$. Fix $i \in I$ and suppose $\varnothing \neq G_i \in \tau_i$. Call the set

$G_i^{opstr} = \{(x_k)_{k\in I} \in \prod_{k\in I} X_k : x_i \in G_i$ and $x_k \in X_k$ for all $k \neq i\}$

$= \prod_{k\in I} Y_k$

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where $Y_i = G_i$ and $Y_k = X_k$ for all $k \neq i$, an *open strip* of the product set $\prod_{k\in I} X_k$. The topology on $\prod_{k\in I} X_k$ generated by the open strips is called the *product topology* on $\prod_{k\in I} X_k$, denoted $\prod_{k\in I} \tau_k$. The pair $\left(\prod_{k\in I} X_k, \prod_{k\in I} \tau_k\right)$ is called a *product topological space* or simply a *product*
space. The topological spaces (X_k, τ_k), $k \in I$, are called the *factor spaces* of the product space

$$\left(\prod_{k\in I} X_k, \prod_{k\in I} \tau_k\right) \qquad \text{….(ProdSpa)}$$

In particular, for $\alpha \in I$, (X_α, τ_α) is called the α*th factor space* of (ProdSpa). Let $i \in I$ and
$p_i : \prod_{k\in I} X_k \to X_i$, $(x_k)_{k\in I} \mapsto x_i$, $(x_k)_{k\in I} \in \prod_{k\in I} X_k$.
the projection of the Cartesian product $\prod_{k\in I} X_k$ onto its ith factor X_i. Then, clearly, the open strip.

$$G_i^{opstr} = \{(x_k)_{k\in I} \in \prod_{k\in I} X_k : x_i \in G_i \text{ and } x_k \in X_k \text{ for all } k \neq i\}$$
$$= p_i^{-1}(G_i).$$

Clearly, by the description of the topology $\tau_{\mathscr{L}}$ generated by $\mathscr{L} \subseteq 2^X$, the product topology $\prod_{k\in I} \tau_k$ is the family of subsets of $\prod_{k\in I} X_k$ with members $\varnothing$, $\prod_{k\in I} X_k$ and unions of finite intersections of open strips. That is, $\prod_{k\in I} \tau_k$ is the family with members $\varnothing$, $\prod_{k\in I} X_k$ and unions of sets of the form

$$p_{i_1}^{-1}(G_{i_1}) \cap p_{i_2}^{-1}(G_{i_2}) \cap \ldots \cap p_{i_n}^{-1}(G_{i_n})$$

where $n \in \mathbb{N}$, $\varnothing \neq G_{i_r} \in \tau_{i_r}$, $i_r \in I$, and of course, p_{i_t} is the projection of $\prod_{k\in I} X_k$ onto the i_tth factor X_{i_t}. Hence, $\prod_{k\in I} \tau_k$ is the family with members $\varnothing$, $\prod_{k\in I} X_k$ and unions of sets of the form $\prod_{k\in I} G_k$, where $\varnothing \neq G_k \in \tau_k$ for k running over a non-empty finite set $\{\alpha, \beta, \ldots, \gamma\}$, say, of indices, and $G_k = X_k$ for $k \notin \{\alpha, \beta, \ldots, \gamma\}$.

4 METRIC TOPOLOGY Let $\varnothing \neq X$ and d *a metric* on X. If $a \in X$ and $r \in \mathbb{R}$, $r > 0$, the set $B(a, r) = \{x \in X : d(a, x) < r\}$ is called a *ball of radius r centred on a*. If $a \in A \subseteq X$ and there exists a ball of some radius $r > 0$ centered on a, $B(a, r)$, contained in A, $\varnothing \neq A \subseteq X$, then we say that a is an *interior point* of A. If all the points of A are interior to A we say that A is an *open set* of the *metric space* (X, d), and the family $\tau_d = \{\varnothing, X\} \cup \{\varnothing \neq A \subseteq X : A \text{ is an open set of } (X, d)\} = \{\varnothing, X\} \cup \{\varnothing \neq A \subseteq X : A \text{ is a union balls}\}$.
called the *metric topology* of (X, d) on X induced by the metric d. *Note*: $\varnothing$ is also called an open set of (X, d). And observe trivially that X is also an open set of (X, d).

5 MetricTop($\mathbb{R}^n$) Let $n \in \mathbb{N}$, $n \geq 2$. The function

$$\|\ \| = \|\ \|_n : \mathbb{R}^n \to \mathbb{R}$$
$$x = (x_1, x_2, \ldots, x_n) \mapsto \sqrt{x_1^2 + x_2^2 + \ldots.. + x_t^2}$$

is called the *Euclidean norm* on $\mathbb{R}^n$, and the positive function

$$d_{\|\ \|} : \mathbb{R}^n \mathrm{x} \mathbb{R}^n \to \mathbb{R}$$
$$(x, y) \mapsto \|x - y\|$$

called the *Euclidean metric* on $\mathbb{R}^n$. We refer to [1], [2], [3] and [4] for details on $d_{\|\ \|}$. The topology on $\mathbb{R}^n$ induced by the Euclidean metric $d_{\|\ \|}$ is called the metric topology of $\mathbb{R}^n$ which we denote by MetricTop($\mathbb{R}^n$). So, by the description in the preceding paragraph,

$\text{MetricTop}(\mathbb{R}^n) = \{\varnothing, \mathbb{R}^n\} \cup \{\varnothing \neq G \subseteq \mathbb{R}^n :$
G is a union of $d_{\|\ \|}$-balls$\}$ ….(Δ)

6 ProductTop($\mathbb{R}^n$) Let $n \in \mathbb{N}$, $n \geq 2$. $\mathbb{R}^n = \mathbb{R}\times\mathbb{R}\times\ldots\times\mathbb{R}$ (n factors). Let $\tau_{\mathbb{R}}$ be the *usual topology* of $\mathbb{R}$. The product topology $\prod_{k\in\{1,2,\ldots,n\}} \tau_k$ where $\tau_k = \tau_{\mathbb{R}}$ for all $k \in \{1, 2, \ldots, n\}$, is of course, the product topology of $\mathbb{R}^n$. We denote this topology by Product- Top($\mathbb{R}^n$). So, by the description of the product topology in Section 3, ProductTop($\mathbb{R}^n$) = $\{\varnothing, \mathbb{R}^n\} \cup \{$unions of sets of the form
$\prod_{k\in\{1,2,\ldots,n\}} G_k$ where $\varnothing \neq G_k \in \tau_{\mathbb{R}}\}$ …(ΔΔ)
Finally,

7 Proof of the *COROLLARY*

THE *COROLLARY* Let $n \in \mathbb{N}$, $n \geq 2$. Then, MetricTop($\mathbb{R}^n$) = ProiductTop($\mathbb{R}^n$).

Proof We employ the OBCT[5] to show that MetricTop($\mathbb{R}^n$) $\subseteq$ ProductTop($\mathbb{R}^n$), and, that ProductTop($\mathbb{R}^n$) $\subseteq$ MetricTop($\mathbb{R}^n$).

ProductTop($\mathbb{R}^n$) $\subseteq$ *MetricTop*($\mathbb{R}^n$): Both topologies contain $\varnothing$ and $\mathbb{R}^n$. So, here it suffices to show that a non-empty member of ProductTop($\mathbb{R}^n$) belongs also to MetrictTop($\mathbb{R}^n$). From (ΔΔ) in Section 6, it therefore further suffices to show that

$$\prod_{k\in\{1,2,\ldots,n\}} G_k \in \text{MetricTop}(\mathbb{R}^n) \qquad ….(\rho)$$

where $\varnothing \neq G_k \in \tau_{\mathbb{R}}$. By a popular result of *Elementary Real Anal- ysis*, if $\varnothing \neq G \in \tau_{\mathbb{R}}$, then G is a union of open intervals. Hence, if $\varnothing \neq G_k \in \tau_{\mathbb{R}}$, then G_k is a union of open intervals of $\mathbb{R}$, and so $\prod_{k\in\{1,2,\ldots,n\}} G_k$ is a union of open intervals of $\mathbb{R}^n$.

But by Immediate 3.2(i) of [5], an open interval of $\mathbb{R}^n$ is a union of $d_{\|\ \|}$-balls of $\mathbb{R}^n$.

Hence, $\prod_{k\in\{1,2,\ldots,n\}} G_k$, where $\varnothing \neq G_k \in \tau_{\mathbb{R}}$, is a union of $d_{\|\ \|}$-balls. By the popular result that a non-empty set of a metric space is open in the space if and only if it is a union of balls, we have therefore proved (ρ).

MetricTop ($\mathbb{R}^n$) $\subseteq$ *ProductTop*($\mathbb{R}^n$): Again, both topologies contain $\varnothing$ and $\mathbb{R}^n$. By (Δ) of Section 5 it suffices to show that a $d_{\|\ \|}$-ball
of $\mathbb{R}^n$ belongs to ProductTop($\mathbb{R}^n$). By Immediate 3.2(iii) of [4], a $d_{\|\ \|}$-ball is a union of open intervals of $\mathbb{R}^n$. But by (ΔΔ) of Section 6, an open interval of $\mathbb{R}^n$ belongs to ProducTop($\mathbb{R}^n$), and so a $d_{\|\ \|}$-ball belongs to the topology ProductTop($\mathbb{R}^n$). ///

8 Remark It is hoped that new textbooks of General topology will present this proof.

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