Transactions of the Nigerian Association of Mathematical Physics
***Volume 10, (July and November, 2019*), pp1 – 12**
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OPEN BALL OPEN CELL TOPOLOGY THEOREM IN EUCLIDEAN SPACES (THE OBCT)

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Abstract

This paper states and proves what this author calls the Open Ball Open Cell Topology Theorem in Euclidean Spaces (the OBCT). A contribution of this paper, amidst several, is employing this theorem to justify the puritanical definition of the partial derivative at an interior point. Its Interval Characterization of continuity is also worthy of note.

Keywords: Euclidean *n*-space, open set, partial derivative, continuity.

1. INTRODUCTION

At the risk of sounding immodest this author makes bold to say that the literature is *subconscious* but **not** *fully conscious* of the Open Ball Open Cell Topology Theorem in Euclidean Spaces, as the literature fails to *explicitly state* it and *explicitly evoke* it in several places where it (the literature) needs it; at best the literature "stumbles" over the theorem in these places; the arguments in these places do not justify the claims in these places. A contribution of this paper is its offer of *clarity of idea* – by (i) explicitly pointing out this theorem and (ii) pointing out of some of the places [1, last paragraph of page 244 spilling into p.245][2, page 6-Definition of an interior point, and page10-Problem 1-15][3, page 79, claim of *openness* of A_1 and B_1 there][4, the claim in the last paragraph of p.288 spilling into p.289 that with P belonging to the open set $U \subseteq \mathbb{R}^n$, then, for some *open set* of values of t, $P + tH$ lie in U][5, 6.8.9, p.349] etc.etc., where this theorem far simply and clearly proves the assertion being claimed, compared to the "complicated and inconclusive" arguments of the respective author. A second contribution of this paper, and significantly too, are

(iii) applying the theorem to justify the *puritanical* definition of the partial derivative at an interior point, and

(iv) applying the theorem to give the *Interval Characterization of Continuity*

There are other applications of the OBCT given in and outside this paper. For an instance, outside this paper we prove the equality of the metric topology, $\tau_{d_{\|\,\|^n}}$, of $\mathbb{R}^n$ ($n \geq 2$), and the pro- duct topology, $\prod_n \tau^{\mathbb{R}}$, of $\mathbb{R}^n$.

The proof of our Open Ball Open Cell Topology Theorem in Euclidean Spaces is an adaptation of several simple arguments found in the literature. It is hoped that authors of new books on *Calculus in* $\mathbb{R}^n$ shall record this theorem and furnish in addition several other applications, e.g., in the proofs of the Implicit / Inverse Function Theorem, etc.

By $\mathbb{R}$ we shall mean the *real numbers* and by $\mathbb{N}$ the natural numbers 1, 2, 3, …. If $n \in \mathbb{N}$ and $n \geq 2$, by $\mathbb{R}^n$ we shall mean the Cartesian space $\mathbb{R}$x$\mathbb{R}$x……x$\mathbb{R}$ (n factors). Of course, $\mathbb{R}^n$ with the *Euclidean norm* $\|\;\|$ [6, p.206] is called the *Euclidean n-space*[7, 2.19, p.51]. If $I_1, I_2, \ldots, I_n$ are intervals in $\mathbb{R}$, the Cartesian product

$$I_1 \mathrm{x} I_2 \mathrm{x} \ldots \mathrm{x} I_n \; (\subseteq \mathbb{R}^n) \qquad \ldots(\Delta)$$

is called an *interval* in $\mathbb{R}^n$; an interval (Δ) in which all the *sides* $I_1, I_2, \ldots, I_n$ are finite intervals in $\mathbb{R}$ is called a *cell*[7, first paragraph, p.52]. I_k ($k = 1, 2, \ldots, n$) in (Δ) is called the *kth side of the interval*. An *open interval /open cell* is one with all its sides open intervals of $\mathbb{R}$.

Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. By the *Euclidean norm of a*, $\|a\|$, is meant $\sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$ [6, p.206], and if $r \in \mathbb{R}$,

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$r > 0$, by a *ball of radius r centered on a* denoted $B(a, r)$ is meant the set $\{x \in \mathbb{R}^n : \|x - a\| < r\}$[4, 3.3, p.49] referred to in [6, Definition 59.1, p.211] as an *r-neighbourhood of a*, and denoted $N_r(a)$.We adhere to $B(a, r)$. If $\varnothing \neq A \subseteq \mathbb{R}^n$ and $a \in A$, a is called an *interior point of A*[6, Definition 59.2,p.211][4, Definition 3.5, p.49] if there exists $r > 0$ such that $B(a, r) \subseteq A$. We state the

OPEN BALL OPEN CELL TOPOLOGY

THEOREM IN EUCLIDEAN SPACES (OBCT) 1 Let $n \in \mathbb{N}$, $n \geq 2$.

(i) Let I be an open interval, in particular an open cell of $\mathbb{R}^n$, and $x \in I$. Then, there exists $r > 0$ such that the ball $B(x, r) \subseteq I$. See Figure 1 below.

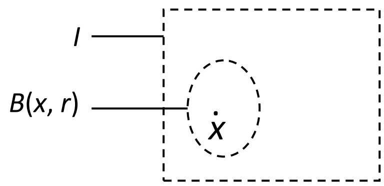

Fig.1

(ii) Let $r > 0$, $x \in \mathbb{R}^n$. and consider the ball $B(x, r)$. Then, there exists an open cell I of $\mathbb{R}^n$ such that $x \in I \subseteq B(x, r)$. See Figure 2 below.

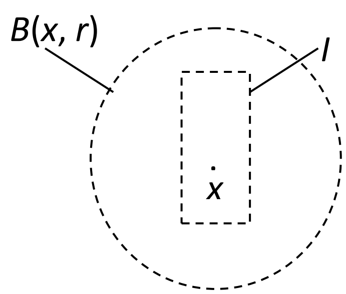

Fig. 2

(iii) Let $r > 0$, $x \in \mathbb{R}^n$ and $a \in B(x, r)$. Then, there exists an open cell I such that $a \in I \subseteq B(x, r)$. See figure 3 below.

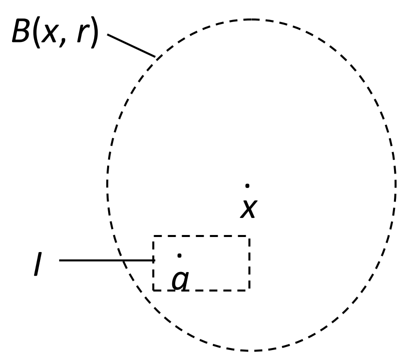

Fig.3

Clearly, (iii) generalizes (ii), while (i) and (iii) reverse the roles of the open cell I and the ball $B(x, r)$

Example 1 In $\mathbb{R}^2$, the plane, a ball $B(a, r)$ of radius $r > 0$ centered on $a \in \mathbb{R}^2$ is simply a circular disc with centre a but without its circular edge. See Figure 4 below.

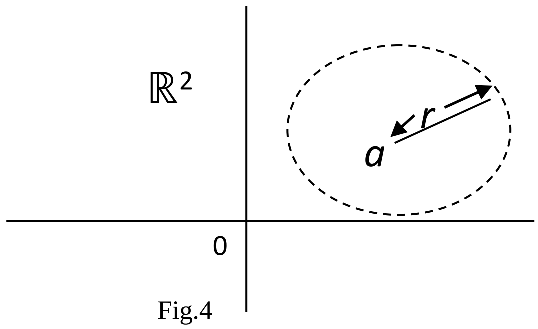

Fig.4

While the point (0, 0) = the origin of the plane, is clearly an interior point to the cell $I = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq 1 , |x_2| < 1\}$, since it is geometrically evident, even can be shown analytically, that (See Figure 5 below)

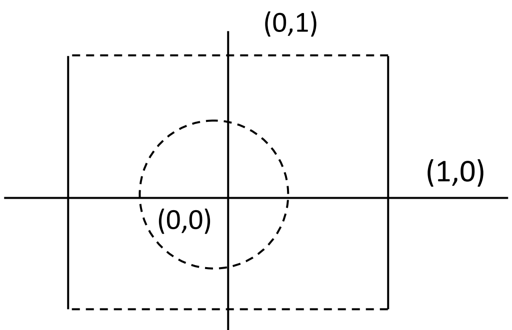

Fig. 5

$B((0, 0), ½) \subseteq I$, the point $(1, 0) \in I$ is *not* an interior point of I since any ball $B((1, 0), r)$ of whatever radius r (however small) will contain the point $(1+\frac{r}{2}, 0) \notin I$ [| *Proof* : That $(1+\frac{r}{2}, 0) \notin I$ is clear. And, $\| (1+\frac{r}{2}, 0) - (1, 0)\| = \|(1+\frac{r}{2} -1, 0 - 0)\| = \|(\frac{r}{2}, 0)\| =$
$\sqrt{(\frac{r}{2})^2 + 0} = \frac{r}{2} < r$. Hence , $(1 + \frac{r}{2}, 0) \in B((1, 0), r)$.///|]. And so for no $r > 0$ do we have $B((1, 0), r) \subseteq I$. ///

Let $\emptyset \neq A \subseteq \mathbb{R}^n$. A is called an *open set* of $\mathbb{R}^n$ if every point $a \in A$ is an interior point of A.
We signify by /// the end or absence of a proof.

2 A PROOF OF THE OBCT We need the
FUNDAMENTAL INEQUALITY (FIE)[6, Lemma 61.1, p.219][3, Theorem 7.11, p.65] For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$,
$| x_k | \leq \|x\| \leq \sqrt{n} \max\{|x_1|, |x_2|, ..., |x_n|\}$, $k = 1, 2, \ldots, n$. ///

Let $I = I_1 \mathrm{x} I_2 \mathrm{x} \ldots \mathrm{x} I_n$ be an *open* cell in $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, and suppose $x = (x_1, x_2, \ldots , x_n) \in I$. Hence, $x_k \in (a_k, b_k) = I_k$ for some $a_k, b_k \in \mathbb{R}$, $a_k < b_k$, $k = 1, 2, \ldots, n$, and so there exists, for each k, $\varepsilon_k > 0$ such that (See Figure 6 below)

$$a_k < x_k - \varepsilon_k < x_k + \varepsilon_k < b_k \qquad \ldots(\Delta)$$

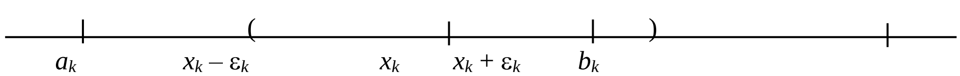

Fig. 6.

Let $r = \frac{1}{2} \min\{\varepsilon_1, \varepsilon_2, \ldots , \varepsilon_n\}$. We show that $B(x, r) \subseteq I$. Therefore, let $y = (y_1, y_2, ..., y_n) \in B(x, r)$, and so $\|y - x \| < r$. By the FIE, therefore,
$|y_k - x_k| \leq \|y - x\| < r \leq \frac{1}{2} \varepsilon_k < \varepsilon_k$, $k = 1, 2, \ldots, n$.

And so, by a property of $\mathbb{R}$ [8, Exercise 8.24: For $x, a, r \in \mathbb{R}$,
$r > 0$,
$|x - a| < r \Leftrightarrow a - r < x < a + r$]
$x_k - \varepsilon_k < y_k < x_k + \varepsilon_k,\ k = 1, 2, ..., n$ ….(ΔΔ)
(Δ) and (ΔΔ) now give
$a_k < x_k - \varepsilon_k < y_k < x_k + \varepsilon_k < b_k,\ k = 1, 2, …., n.$
And so, $y_k \in (a_k, b_k) = I_k$, $k = 1, 2, ..., n$. Hence, $y = (y_1, y_2, …, y_n) \in I_1 \text{x} I_2 \text{x} … \text{x} I_n = I$. We have thus proved (i) of the Open Ball Open

Cell Topology Theorem in Euclidean Spaces. Of course, if I is an open interval of $\mathbb{R}^n$ and $x \in I$, then there exists an open cell I' such that $x \in I' \subseteq I$.

What we have shown is geometrically evident in $\mathbb{R}^2$, the plane, as shown below in Figure 7.

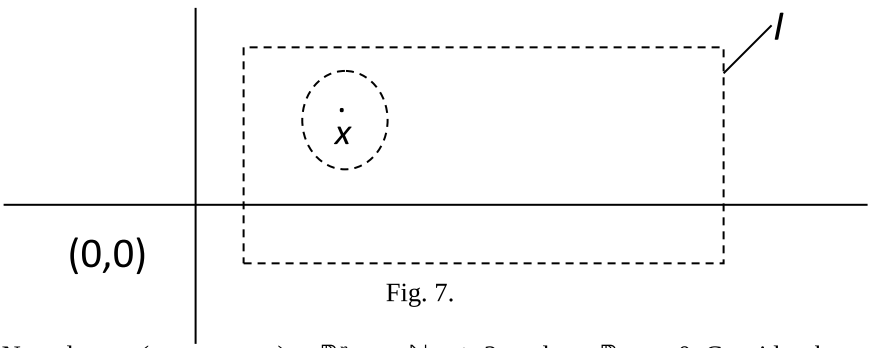

Fig. 7.

Now, let $x = (x_1, x_2, …, x_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, and $r \in \mathbb{R}$, $r > 0$. Consider the open cell
$I = (x_1 - \frac{r}{n}, x_1 + \frac{r}{n}) \text{x} (x_2 - \frac{r}{n}, x_2 + \frac{r}{n}) \text{x} … \text{x} (x_n - \frac{r}{n}, x_n + \frac{r}{n})$.
We shall show that $I \subseteq B(x, r)$ to prove (ii) of the Open Ball Open Cell Topology Theorem in Euclidean Spaces. So, let $y = (y_1, y_2, …, y_n) \in I$. Then, by the property of $\mathbb{R}$ cited before now

$|y_k - x_k| < \frac{r}{n}\quad k = 1, 2, …, n$

$\max\{|y_1 - x_1|\}, \{|y_2 - x_2|\}, …., \{|y_n - x_n|\} < \frac{r}{n}$ …(∇)

By the FIE,
$\|y - x\| \leq \sqrt{n}\ \max\{|y_1 - x_1|, |y_2 - x_2|, …, |y_n - x_n|\}$
and so by (∇),

$\| y - x\| < \sqrt{n} \cdot \frac{r}{n} = \frac{\sqrt{n}}{n} r,$

which by [8, Properties 8.16(iv), p.151: $0 < a < 1 \Leftrightarrow 0 < a < \sqrt{a}$, and $1 < a \Leftrightarrow \sqrt{a} < a$], since $1 < 2 \leq n$, $\sqrt{n} < n$, from which follows that $\frac{\sqrt{n}}{n} < 1$,

$< r$.
That is,
$\| y - x \| < r,$
from which follows that $y \in B(x, r)$. Since y was arbitrary, we have shown that $I \subseteq B(x, r)$.
Any book on Metric Space Theory records the

OPENBALL THEOREM [9, Proposition and Figure of page 40] [4, Figure 3.2 of page 57][10, Theorem 6.5(a) and Figure 6.3,p.139] A ball $B(x, r)$ in a metric space (X, d) is an open set of (X, d). ///

OPEN BALL THEOREM for $\mathbb{R}^n$ [6, Lemma 59.1, p.212] For $a \in \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, and $r > 0$, $B(a, r)$ is an open set. ///
And the proof (See Fig. 8 below) is essentially showing that

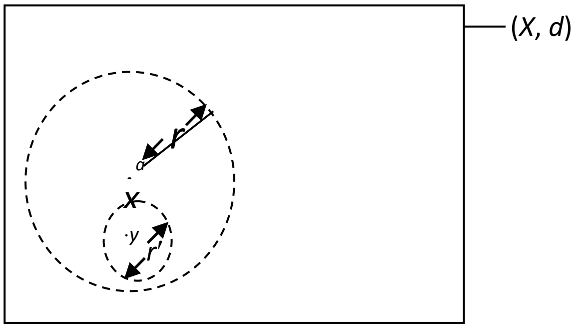

Fig.8

if $y \in B(a, r)$, there exists $r' > 0$ such that $B(y, r') \subseteq B(a, r)$. ///

So suppose $r > 0$, $x \in \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, and $a \in B(x, r)$. Then, by the OPEN BALL THEOREM for the metric space ($\mathbb{R}^n$, $d_{\|\|}$), the Euclidean n-space, there exists $r' > 0$ such that $B(a, r') \subseteq$
$B(x, r)$. By (ii) of our Open Ball Open Cell Topology Theorem in Euclidean Spaces there exists an open cell I such that
$a \in I \subseteq B(a, r')$.
Hence, $a \in I \subseteq B(a, r') \subseteq B(x, r)$ from which follows that
$a \in I \subseteq B(x, r)$.

And we have proved (iii) of our Open Ball Open Cell Topology Theorem in Euclidean Spaces, thus completing the proof. ///.

3. ILLUSTRATIONS AND CONSEQUENCES

We now move to discuss those places in the literature that need the Open Ball Open Cell Topology Theorem in Euclidean Spaces (OBCT) invoked to give a correct proof. First we recall the definitions of an interior point and of a non-empty *open* set in $\mathbb{R}^n$, and we come up with some immediate consequences of the OBCT. Suppose $\emptyset \neq A \subseteq \mathbb{R}^n$, and $a \in A$. a is called an *interior point of A* if there exists a ball $B(a, r)$, say, such that $B(a, r) \subseteq A$. $\emptyset \neq A \subseteq \mathbb{R}^n$ is called an *open set of* $\mathbb{R}^n$ if each $a \in A$ is interior to A. It follows from the OBCT that

IMMEDIATE 1 Let $n \in \mathbb{N}$ and $n \geq 2$.

(i) $a \in A \subseteq \mathbb{R}^n$ is interior to $A \Leftrightarrow$ there exists an open interval I such that $a \in I \subseteq A$. And so A is open $\Leftrightarrow$ for every $a \in A$ there exists an open interval I_a of $\mathbb{R}^n$ such that $a \in I_a \subseteq A$. And so a non-empty open set of $\mathbb{R}^n$ is a union of open intervals.

(ii) An open interval I in $\mathbb{R}^n$ (i.e., $I = I_1 \text{x} I_2 \text{x} \ldots \text{x} I_n$, each I_k, $k = 1, 2, \ldots, n$, is an open interval in $\mathbb{R}$) is an open set of $\mathbb{R}^n$.

(iii) $\emptyset \neq A \subseteq \mathbb{R}^n$ is an open set of $\mathbb{R}^n \Leftrightarrow A$ is a union of open intervals. ///.

Well-known is : $\emptyset \neq A \subseteq \mathbb{R}^n$ is open $\Leftrightarrow A$ is a union of balls. By the OBCT it follows ((ii) of IMMEDIATE 1 above) that

IMMEDIATE 2 (i) An open interval of $\mathbb{R}^n$ is a union of balls, and by OBCT(iii),
(ii) A ball is a union of open intervals. ///

Suppose $\emptyset \neq A \subseteq \mathbb{R}^n$ and $p \in \mathbb{R}^n$. p is called a *boundary point* of A if every ball $B(p, r)$ contains a point of A and a point of A's complement. Immediate from the OBCT is

IMMEDIATE 3 Let $\emptyset \neq A \subseteq \mathbb{R}^n$. $p \in \mathbb{R}^n$ is a boundary point of $A \Leftrightarrow$ every open interval containing p contains a point of A and a point of the complement of A. Compare the definition of a boundary point in [11, p.6/7].

Example 4 [3, 8.2 (d), (e), p.70] (i) The set $G = \{(\xi, \eta) \in \mathbb{R}^2 : 0 < \xi < 1, \eta = 0\}$ is *not* open in $\mathbb{R}^2$ (ii) The set $H = \{(\xi, \eta) \in \mathbb{R}^2 : 0 < \xi < 1\}$ is open in $\mathbb{R}^2$ *Proof* : (i) and (ii) are, clearly, immediate from IMMEDIATE 1(i). Compare a 'proof' of either using balls.///

(iii) The sets $G = \{(\xi, \eta, \zeta) \in \mathbb{R}^3 : \zeta > 0\}$ and $H = \{(\xi, \eta, \zeta) \in \mathbb{R}^3 : \xi > 0, \eta > 0, \zeta > 0\}$are open in $\mathbb{R}^3$ while the set $F = \{(\xi, \eta, \zeta) \in$

$\mathbb{R}^3 : \xi = \eta = \zeta\}$ is not open in $\mathbb{R}^3$. *Proof* Also immediate from IMMEDIATE 1(i): Compare a 'proof' using balls.///

Example 5 The points (1, 0) and (0, 1) in Figure 5 are boundary points of the cell diagrammatized there. *Proof* : Surround each of them with an open cell! ///.

Example 6 [1, last paragraph of page 244 spilling into p.245] Let $A_1, A_2, \ldots, A_n$ be non-empty open sets of $\mathbb{R}$. *CLAIM* : $A_1 \text{x} A_2 \text{x} \ldots \text{x} A_n$ is an open set of $\mathbb{R}^n$. *Proof of CLAIM* : Let $x = (x_1,$

$x_2, \ldots, x_n) \in A_1 \text{x} A_2 \text{x} \ldots \text{x} A_n$ and so by the topology of $\mathbb{R}$ there exist finite open intervals $I_1, I_2, \ldots, I_n$ in $\mathbb{R}$ such that $x_k \in I_k \subseteq A_k$, $k = 1, 2, ..., n$. Hence,

$x = (x_1, x_2, \ldots, x_n) \in I_1 \text{x} I_2 \text{x} \ldots \text{x} I_n \subseteq A_1 \text{x} A_2 \text{x} \ldots \text{x} A_n$. …(Δ)

The cell $I_1 \text{x} I_2 \text{x} \ldots \text{x} I_n$ is an open interval. By IMMEDIATE 1(i), therefore, x is interior to $A_1 \text{x} A_2 \text{x} \ldots \text{x} A_n$. Since x was arbitrary, $A_1 \text{x} A_2 \text{x} \ldots \text{x} A_n$ is an open set. Compare the 'proof' using balls of [1, p.244/245].

Example 7 Let $\varnothing \neq U \subseteq \mathbb{R}^n$ and suppose U is an open set, $p = (p_1, p_2, \ldots, p_n) \in U$ and $h = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^n$. *CLAIM* [12, last paragraph of p.288 spilling into p.289]: For some open interval of values of t, the vectors $p + th$ lie in U. /// Lang merely asserted, he didn't prove this claim. A proof here using the ball definition of "interior" and "openness" will certainly be complex. But, clearly, IMMEDIATE 1(i) offers a simple proof : $p \in I$ (a cell) $\subseteq U$. (See Figure 9 below)

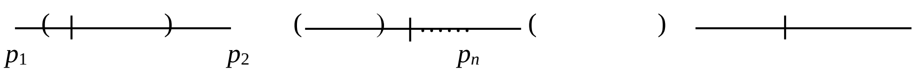

Fig. 9

And the existence of an open interval of values of t centered on 0 for which $p + th \in U$ is evident.

Example 8 Like in the preceding Example 7, the claim of openness of A_1 and B_1 in the proof of THEOREM 8.17 page 79 of [3] was only asserted but not proved. An approach through IMMEDIATE 1 as done in the preceding easily establishes this claim.

Example 9 ***The Partial Derivative*** The derivative $f'(a)$ of real-valued $f : I \to \mathbb{R}, \varnothing \neq I \subseteq \mathbb{R}$, is defined only at points a of a domain I **which is an interval**[13, Definition 6.1.1, p.158][4, Definition 5.1, p.104] just as solutions of ordinary differential equations are sought over intervals of $\mathbb{R}$ and *not* over arbitrary sets in $\mathbb{R}$. Precisely: Let I be an interval, $a \in I$ and $f : I \to \mathbb{R}$. Then, f is *differentiable at a* provided $\lim_{x \to a} f^{*a}(x)$ exists, where

$$f^{*a} : I - \{a\} \to \mathbb{R}.$$
$$x \mapsto \frac{f(x) - f(a)}{x - a}$$

And, we define

$$f'(a) = \lim_{x \to a} f^{*a}(x). \qquad \text{....(*)}$$

This informs why, for the definition of, say, the first partial derivative, $D_1 f(a)$, say, at a, of $f : A \to \mathbb{R}, \varnothing \neq A \subseteq \mathbb{R}^n$, $a = (a_1, a_2, ..., a_n) \in A$ **to make sense, there is the need for the existence of an interval I_1**, say, in $\mathbb{R}$ such that $a_1 \in I_1$ and $I_1 \text{x}\{a_2\}\text{x}\{a_3\}\text{x} \ldots \text{x}\{a_n\} \subseteq A$. Then, following (*), we define

$f^{p1} : I_1 \to \mathbb{R}, x \mapsto f(x, a_2, a_3, \ldots, a_n),\ x \in I_1$

and then, of course,

$$f^{p1*a_1} : I_1 - \{a_1\} \to \mathbb{R},$$
$$x \mapsto \frac{f^{p1}(x) - f^{p1}(a_1)}{x - a_1} = \frac{f(x, a_2, \ldots, a_n) - f(a_1, a_2, \ldots, a_n)}{x - a_1}.$$

And, then, define

$$D_1 f(a) \equiv \lim_{x \to a_1} f^{p1*a_1}(x)$$

Similarly, to define $D_2 f(a)$ there is the need to have an interval I_2 in $\mathbb{R}$ such that $a_2 \in I_2$ and $\{a_1\}\text{x}I_2\text{x}\{a_3\}\text{x} \ldots \text{x}\{a_n\} \subseteq A$. And then define

$f^{p2} : I_2 \to \mathbb{R},\ x \mapsto f(a_1, x, a_3, \ldots, a_n),\ x \in I_2$

and

$f^{p2*a_2} : I_2 - \{a_2\} \to \mathbb{R}$,

$$x \mapsto \frac{f^{p2}(x) - f^{p2}(a_2)}{x - a_2} = \frac{f(a_1, x, a_3, \ldots, a_n) - f(a_1, a_2, \ldots, a_n)}{x - a_2}.$$

And, then, define the second partial derivative. $D_2 f(a)$, of f at a, by

$$D_2 f(a) \equiv \lim_{x \to a_2} f^{p2*a_2}(x).$$

The requirements for, and the definitions of, $D_3 f(a), \ldots, D_n f(a)$ are now clear. One can refer to these definitions as the *puritanical* definitions of the partial derivatives. The reason for this is not far fetched; the reader should open several texts of Calculus on $\mathbb{R}^n$ and compare their definitions of $D_k f(a)$, k = 1, 2, …., n. The reader is *not* likely to find any of the texts mentioning

(i) the need for an interval I_k such that $a_k \in I_k$,
(ii) justification for the existence of I_k (E.g. an OBCT)
(iii) the functions f^{pk} and f^{pk*a_k}, and
(iv) the *puritanical* (i.e., the *very correct*) definition

$$D_k f(a) \equiv \lim_{x \to a_k} f^{pk*a_k}(x)$$

In another paper of the author, the author continues the discussion of the partial derivative, offering *two clarifications* on the definition of the partial derivative.

If $a \in A$ is interior, then IMMEDIATE 1(i) provides an open interval $I = I_1\text{x}I_2\text{x}I_3\text{x} \ldots \text{x}I_n$, say, indeed an open cell, such that

$a = (a_1, a_2, a_3, \ldots, a_n) \in I = I_1\text{x}I_2\text{x}I_3\text{x} \ldots \text{x}I_n \subseteq A$.

And then, $D_1 f(a), D_2 f(a), D_3 f(a), \ldots, D_n f(a)$ are all simultane- ously definable. And the OBCT has justified the existence of I_k, $k = 1, 2, \ldots, n$, and consequently the puritanical definition of the partial derivative at an interior point a of $A \subseteq \mathbb{R}^n$ for $f : A \to \mathbb{R}$.

Example 10 ***The Complex Derivative*** **[14, Definition 4.58, p.219][15, Definition 2.1,p.33].** Here we need to first note that the *topology of* $\mathbb{C}$ [| ≡ ball, interior point, open set, closed set etc, etc |] is the same as the *topology of the complex plane* [the *complex plane* ≡ $\mathbb{R}^2$ with the element $a = (a_1, a_2)$ of $\mathbb{R}^2$ identified with the complex number $a_1 + ia_2$|]. And so, IMMEDIATE 1 for the complex plane can easily be transferred to $\mathbb{C}$. Hence, if $\emptyset \neq D \subseteq \mathbb{C}$, $0 \neq h \in \mathbb{C}$, $z \in D$ *interior* to D and Re h and Im h small enough, of whatever sign, then $z + h$ *shall belong* to D, and so, for the complex valued function $f : D \to \mathfrak{C}$ $f(z + h)$ shall make sense and consequently $\lim\limits_{\substack{h \to 0 \\ z+h \in D}} \frac{f(z+h) - f(z)}{h}$ meaningful.

Example 11 ***Interval Characterization of Sequential Conver- gence in*** $\mathbb{R}^n$***,*** $n \geq 2$ **[3, Section 11, p.98-110][6, Section 5.61, p.218-221]** Let $a \in \mathbb{R}^n$ and I an interval in $\mathbb{R}^n$, $n \in \mathbb{N}$. *I is an interval about a* simply means $a \in I$. Recall the

Definition 1: Let $n \in \mathbb{N}$, $\emptyset \neq A \subseteq \mathbb{R}^n$, and $(x_k)_{k=1}^{\infty}$ a sequence in $\mathbb{R}^n$. The sequence $(x_k)_{k=1}^{\infty}$ is said to be *eventually in A* if there exists a positive integer, N, such that $x_k \in A$ for all $k \geq N$.

Definition 2: The sequence $(x_k)_{k=1}^{\infty}$ in $\mathbb{R}$ is said to *converge to a* $\in \mathbb{R}$ if for every $\varepsilon > 0$, the sequence $(x_k)_{k=1}^{\infty}$ is eventually in the open interval $(a - \varepsilon, a + \varepsilon)$.

THEOREM 3 The sequence $(x_k)_{k=1}^{\infty}$ in $\mathbb{R}$ converges to $a \in \mathbb{R} \Leftrightarrow$ for every open interval I about a, the sequen$(x_k)_{k=1}^{\infty}$ is eventually in I.

Proof Elementary Real Analysis (ERA) arguments. ///
Recall

***Definition* 3:** Let $n \in \mathbb{N}$, $n \geq 2$, $a \in \mathbb{R}^n$ and $(x_k)_{k=1}^{\infty}$ a sequence in $\mathbb{R}^n$. The sequence $(x_k)_{k=1}^{\infty}$ is said to *converge to a* if $(x_k)_{k=1}^{\infty}$ is eventually in every ball $B(a, r)$, $r > 0$.

THEOREM 5 Let $n \in \mathbb{N}$, $n \geq 2$, $a \in \mathbb{R}^n$. The sequence $(x_k)_{k=1}^{\infty}$ in $\mathbb{R}^n$ converges to $a \Leftrightarrow (x_k)_{k=1}^{\infty}$ is eventually in *every* open interval I about a.

Proof OBCT (i) for the forward implication $\Rightarrow$, and OBCT(ii) for the reverse implication $\Leftarrow$. ///

Let $n \in \mathbb{N}$, $n \geq 2$. In what follows, we want to use the functional notation in writing the ith coordinate of $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ [8, Remark 11, p.21], and so we write the ith coordinate of $x = (x_1, x_2, \ldots, x_n)$, x_i, as $x(i)$. Therefore, if $(x_k)_{k=1}^{\infty}$ is a sequence in $\mathbb{R}^n$, we may have a display of $(x_k)_{k=1}^{\infty}$ as follows.

$x_1 = (x_1(1), x_1(2), \ldots, x_1(n))$
$x_2 = (x_2(1), x_2(2), \ldots, x_2(n))$
$x_3 = (x_3(1), x_3(2), \ldots, x_3(n))$
.
.
.
$x_k = (x_k(1), x_k(2), \ldots, x_k(n))$
.
.
.

And so clearly $(x_k)_{k=1}^{\infty}$ has given rise to n real sequences

$(x_k(1))_{k=1}^{\infty} = (x_1(1), x_2(1), \ldots\ldots\ldots)$
$(x_k(2))_{k=1}^{\infty} = (x_1(2), x_2(2), \ldots\ldots\ldots)$
.
.
,
$(x_k(n))_{k=1}^{\infty} = (x_1(n), x_2(n), \ldots\ldots\ldots)$

the ith coordinates, $i = 1, 2, \ldots, n$, of the terms of $(x_k)_{k=1}^{\infty}$ giving rise to a sequence $(x_k(i))_{k=1}^{\infty}$.

THEOREM [3, Theorem 11.7, p.102][6, Theorem 61.3, p.220] 6 Let $n \in \mathbb{N}$, $n \geq 2$ and $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. The sequence $(x_k)_{k=1}^{\infty}$ in $\mathbb{R}^n$ converges to $a \Leftrightarrow$ the real sequence $(x_k(i))_{k=1}^{\infty}$ converges to $a_i = a(i)$, $i = 1, 2, \ldots, n$.

Proof $\Rightarrow$: Suppose $(x_k)_{k=1}^{\infty}$ converges to $a = (a_1, a_2, \ldots, a_n)$. Suppose I_i, $i = 1, 2, \ldots, n$, is an open interval in $\mathbb{R}$ about a_i. Then, $I = I_1 \mathrm{x} I_2 \mathrm{x} \ldots \mathrm{x} I_n$ is an open interval in $\mathbb{R}^n$ about $a = (a_1, a_2, \ldots, a_n) = (a(1), a(2), \ldots, a(n))$. By THEOREM 5 and the hypothesis, therefore, $(x_k)_{k=1}^{\infty}$ is eventually in I. And so, $(x_k(i))_{k=1}^{\infty}$ is eventually in I_i, $i = 1, 2, \ldots, n$. By THEOREM 3, therefore, $(x_k(i))_{k=1}^{\infty}$ converges to a_i. The proof of the implication $\Leftarrow$ is similar. ///

Example 12 ***Interval Characterization of Continuity*** Recall that if $\varnothing \neq A \subseteq \mathbb{R}$, $a \in A$ and $f : A \to \mathbb{R}$, then, f is said be *continuous at a* provided:

Whenever given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$\left.\begin{array}{l} x \in A \text{ and} \\ |x - a| < \delta(\varepsilon) \end{array}\right\} \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Clearly, this is equivalent to :

Whenever given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that
$x \in A \cap (a - \delta(\varepsilon), a + \delta(\varepsilon)) \Rightarrow f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$.
But this is also clearly equivalent to:

Whenever given an open interval I of $\mathbb{R}$ about $f(a)$, there exists an open interval J of $\mathbb{R}$ about a such that $f(A \cap J) \subseteq I$.
So, we have

THEOREM 1 Let $a \in A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Then, f is continuous at $a \Leftrightarrow$ for every open interval I of $\mathbb{R}$ about $f(a)$, there exists an open interval J of $\mathbb{R}$ about a such that $f(A \cap J) \subseteq I$. ///

Now let $p \in \mathbb{N}$ and $p \geq 2$. Denote by $\| \|_p$ the Euclidean norm of $\mathbb{R}^p$. Let $m, n \in \mathbb{N}$, $m \geq 2$, $n \geq 2$, $\varnothing \neq D \subseteq \mathbb{R}^n$, $a \in D$ and $f : D \to \mathbb{R}^m$. Recall that f is said to be *continuous at a* provided:
Whenever given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$\left.\begin{array}{l} x \in D \text{ and} \\ \| x - a \|_n < \delta(\varepsilon) \end{array}\right\} \Rightarrow \| f(x) - f(a) \|_m < \varepsilon.$$

Clearly, this is same as saying:

Whenever given a ball $B(f(a), \varepsilon)$, centered on $f(a)$ and of some radius $\varepsilon > 0$, in $\mathbb{R}^m$, there exists a ball $B(a, \delta(\varepsilon))$ in $\mathbb{R}^n$, centered on a and of some radius $\delta(\varepsilon) > 0$, such that
$f(D \cap B(a, \delta(\varepsilon)) \subseteq B(f(a), \varepsilon)$.
It is immediate from this and the OBCT that:

THEOREM 2 f is continuous at $a \Leftrightarrow$ for every open interval I of $\mathbb{R}^m$ about $f(a)$, there exists an open interval J of $\mathbb{R}^n$ about a such that
$f(D \cap J) \subseteq I$. ///

Let $n \in \mathbb{N}$, $n \geq 2$, $\varnothing \neq D \subseteq \mathbb{R}^n$, $a \in D$ and $f : D \to \mathbb{R}$. Recall here that f is said to be *continuous at a* provided:
Whenever given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$\left.\begin{array}{l} x \in D \text{ and} \\ \| x - a \|_n < \delta(\varepsilon) \end{array}\right\} \Rightarrow | f(x) - f(a) | < \varepsilon.$$

Clearly, this is same as saying that:
Whenever given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that
$f(D \cap (B(a, \delta(\varepsilon))) \subseteq (f(a) - \varepsilon, f(a) + \varepsilon)$.
From this and the OBCT, we clearly have

THEOREM 3 Let $n \in \mathbb{N}$, $n \geq 2$, $\varnothing \neq D \subseteq \mathbb{R}^n$, $a \in D$ and $f : D \to \mathbb{R}$. Then, f is continuous at $a \Leftrightarrow$ for every open interval I of $\mathbb{R}$ about $f(a)$ there exists an open interval J of $\mathbb{R}^n$ about a such that $f(D \cap J) \subseteq I$. ///

Let $\varnothing \neq D \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$. The function

$\text{coord}_k : D \to \mathbb{R}$, $x = (x_1, x_2, \ldots, x_n) \mapsto x_k$, $x \in D$,
is called the kth *coordinate function* on D, $k = 1, 2, \ldots, n$. Immedi- ate from THEOREM 3 is

THEOREM 4 Let $\varnothing \neq D \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, $a \in D$. The kth coordinate function, $k = 1, 2, \ldots, n$,

$\text{coord}_k : D \to \mathbb{R}$, $x = (x_1, x_2, \ldots, x_n) \mapsto x_k$, $x \in D$
is continuous, at a. ///

One formulates and proves easily the following.

THEOREM 5 Let $m \in \mathbb{N}$, $m \geq 2$, $\varnothing \neq A \subseteq \mathbb{R}$, $a \in A$ and $f : A \to \mathbb{R}^m$. Then, f is continuous at $a \Leftrightarrow$ for every open interval I of $\mathbb{R}^m$ about $f(a)$ there exists an open interval J of $\mathbb{R}$ about a such that $f(A \cap J) \subseteq I$. ///

Now, clearly, THEOREMS 1, 2, 3 and 5 together give the

Interval Characterization of Continuity 6 Let $m, n \in \mathbb{N}$, $\varnothing \neq D \subseteq \mathbb{R}^n$, $a \in D$ and $f : D \to \mathbb{R}^m$. Then, f is continuous at $a \Leftrightarrow$ for every open interval I about $f(a)$ there exists an open interval J about a such that $f(D \cap J) \subseteq I$. ///

We note, of course that by $\mathbb{R}^1$ is meant $\mathbb{R}$.

Let $m, n \in \mathbb{N}$, $m \geq 2$, $\varnothing \neq A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. If $a \in A$, then

$f(a) = (f(a)(1), f(a)(2), \ldots, f(a)(m)) \in \mathbb{R}^m$.

Then for $k \in \{1, 2, \ldots, m\}$ the function

$f_k : A \to \mathbb{R},\ a \mapsto f(a)(k),\ a \in A$

is called the *kth component function of* f [6, first line p.228]. And so we may write $f = (f_1, f_2, \ldots, f_m)$. We have

THEOREM [6, Lemma 65.1, p.231] 7 Let $m, n \in \mathbb{N}$, $m \geq 2$, $\varnothing \neq D \subseteq \mathbb{R}^n$, $a \in D$ and $f : D \to \mathbb{R}^m$. Then, f is continuous at $a \Leftrightarrow$ each of the component functions f_k, $k = 1, 2, \ldots, m$, is continuous at a.

Proof $\Rightarrow$: For definiteness suppose $k = 1$, and that f is continuous at a; we show that f_1 is continuous at a. And so let I_1 be an open interval in $\mathbb{R}$ about $f_1(a) = f(a)(1)$. Then, $I_1 \text{x} \mathbb{R} \text{x} \mathbb{R} \text{x} \ldots \text{x} \mathbb{R} \subseteq \mathbb{R}^m$ is an open interval about $f(a)$. By the *Interval Characterization of Continuity* 6 therefore there exists an open

interval $J \subseteq \mathbb{R}^n$ such that $a \in J$ and $f(D \cap J) \subseteq I_1 \text{x} \mathbb{R} \text{x} \mathbb{R} \text{x} \ldots \text{x} \mathbb{R}$. And so,

$f_1(D \cap J) \subseteq I_1$

from which follows again by the *Interval Characterization of Continuity 6*, that f_1 is continuous at a.

$\Leftarrow$: Suppose each f_k is continuous at a. Let $I = I_1 \text{x} I_2 \text{x} \ldots \text{x} I_m \subseteq \mathbb{R}^m$ be an open interval about $f(a)$. And so each $I_k \subseteq \mathbb{R}$ is an open interval about $f_k(a) = f(a)(k)$, $k = 1, 2, \ldots, m$. By hypothesis and the *Interval Characterization of Continuity 6*, therefore, there exists an open interval $J_k \subseteq \mathbb{R}^n$ such that $a \in J_k$ and

$f_k(D \cap J_k) \subseteq I_k$ …(Δ)

Clearly $J = \prod_{k=1}^{m} J_k$ is an open interval in $\mathbb{R}^n$ about a. From (Δ) follows that

$f(D \cap J) \subseteq I_1 \text{x} I_2 \text{x} \ldots \text{x} I_m = I$.

And so by the *Interval Characterization of Continuity* 6, f is continuous at a. ///

Recall

[3, THEOREM 15.2(c), p.147](Sequential Characteriz- ation of Continuity) 8 Let $m, n \in \mathbb{N}$, $\varnothing \neq D \subseteq \mathbb{R}^n$, $a \in D$ and $f : D \to \mathbb{R}^m$. Then, f is continuous at $a \Leftrightarrow$ for every sequence $(x_r)_{r=1}^{\infty}$ in D converging to a, $(f(x_r))_{r=1}^{\infty}$ converges to $f(a)$. ///

Immediate from this THEOREM 8, and THEOREM 7 is

THEOREM 9 Let $m, n \in \mathbb{N}$, $m \geq 2$, $\varnothing \neq D \subseteq \mathbb{R}^n$, $a \in D$ and $f = (f_1, f_2, \ldots, f_m) : D \to \mathbb{R}^m$. Then, f is continuous at $a \Leftrightarrow$ for every sequence $(x_r)_{r=1}^{n}$ in D converging to a, $(f_k(x_r))_{r=1}^{\infty}$ converges to $f_k(a)$. $k = 1, 2, \ldots, m$. ///

Now consider

CLAIM [3, Examples 15.5(i), p.151]: 10 Let $D = \mathbb{R}^2$, $a \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (2x + y, x - 3y), (x, y) \in \mathbb{R}^2$.

Then, f is continuous at a. *Proof of CLAIM* : Clearly,
$f = (f_1, f_2)$, where

$f_1 : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto 2x + y, (x, y) \in \mathbb{R}^2$
and

$f_2 : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x - 3y, (x, y) \in \mathbb{R}^2$.

Suppose $(x_r)_{r=1}^{\infty}$ is a sequence in $\mathbb{R}^2$ converging to $a = (a_1, a_2)$. Then,

$f_1(a) = 2a_1 + a_2$ and $f_2(a) = a_1 - 3a_2$,
and ,

$(x_r)_{r=1}^{\infty} = (x_r(1), x_r(2))_{r=1}^{\infty}$ converging to $a = (a_1, a_2)$ implies $(x_r(1))_{r=1}^{\infty}$ converges to a_1 and $(x_r(2))_{r=1}^{\infty}$ converges to a_2. And so, $(f_1(x_r))_{r=1}^{\infty}$ =

$(2x_r(1) + x_r(2))_{r=1}^{\infty}$ converges to $2a_1 + a_2 = f_1(a)$, and so, $(f_1(x_r))_{r=1}^{\infty}$ converges to $f_1(a)$.

Similarly, $(f_2(x_r))_{r=1}^{\infty} = (x_r(1) - 3x_r(2))_{r=1}^{\infty}$ converges to $a_1 - 3a_2 = f_2(a)$, and so $(f_2(x_r))_{r=2}^{\infty}$ converges to $f_2(a)$ By THEOREM 9 therefore, f is continuous at a. ///

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