THE DIFFUSIVE BEHAVIOURS ASSOCIATED WITH SOME OF THE OCEAN WAVE PROCESSES

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Abstract

This study describes the dynamic of the physical manifestations that characterise ocean wave events. The differential equation governing these processes and their respective solution techniques are in general quite familiar.

The similarity transformations techniques may also be familiar but the related parameters and variables derived here are quite new. They are indeed able to exhibit some of the physical characteristics associated with linear and nonlinear behaviour of ocean wave parameters such as the modulated frequency wave group velocity, the spectrum assumed narrow in this case.

Finally, the study appears to have brought to focus the remarkable effects of nonlinearity in the evolution of diffusive phenomena. These effect are indeed, drastic as shown in fig 1.

1.0   Introduction

This paper concerns the nonlinear diffusive phenomena and evolutional differential equations that describe their physical manifestations. A number of physical processes are diffusive in behaviour and hence, are related to these. The motions induced by water waves in ocean environment [3], energy related group velocity [2], heat wave, dispersive movement of atmospheric cloud, high frequency electromagnetic wave in plasma [3]; all these are highly diffusive in behaviour. Hence, these processes are governed by the identical evolution equations and in most cases, nonlinear. These equations are generally familiar. However, in this study, attempt will be made to integrate this equation not by the method of familiar integral transformation technique [6, 7].

Existing classical method (integral, Fourier) effectively solve linear evolution equation only in general. Similarity technique [1] on the other hand, provides an effective way out of complexities associated with both linear and nonlinear diffusive behaviour in a process and their inherent constraints [2, 7].

Consequently attempt will be made to determine suitable similarity parameters and valuables, that solve both linear and nonlinear evolution process with diffusive behaviour in the far field and subsequently applications will follow. We will appeal to ocean observed data [6, 4] eventually.

2.   A Unidirectional Propagating Process

A process propagating horizontally in x-direction is considered, y-axis is vertical and perpendicular x-axis. t > 0 denotes time duration. The evolution of the process denoted by V(x,t) is governed by the nonlinear diffusion equation [7]

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} = k\frac{\partial^2 V}{\partial x^2} \qquad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \qquad [1]$$

Here, the parameter k is the diffusion coefficient assumed to be constant in this study. The constraints on the V(x,t) will follow as this study progresses.

3.   Similarity Transformation Technique for Linear Process

We consider the linear problem stated as follows:

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2}, t > 0, x > 0 \quad - \qquad - \quad - \quad - \quad - \quad - \quad - \quad - \qquad [2]$$

These are associated with the following conditions

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$V(0,t) = 0, t > 0$ -   -   -   -   -   -   -   -   (2i)
$V(x,t)$ is bounded as $x \to \infty, t > 0$ -   -   -   -   -   -   (2ii)
$V(x,0) = V_0$ (a constant) -   -   -   -   -   -   -   (2iii)
Let $V(x, t) = f(\eta)$ and $\eta(x,t) = {}^x/_{h(t)}$ -   -   -   -   -   (3)

$$\eta_x = \frac{1}{h}(t), \eta_t(x,t) = -\left(\frac{x}{h(t)}\right)\frac{h'(t)}{h(t)} = -\eta(x,t)\frac{h'(t)}{h(t)}$$

$$V_x(x,t) = f'(\eta)\eta_x = \frac{1}{h(t)}f'(\eta), V_{xx}(x,t) = \frac{1}{h(t)}f''(\eta)\eta_x = \frac{1}{h^2(t)}f''(\eta)$$

$$V_t(x,t) = f'(\eta)\eta_t = -\eta(x,t)\frac{h'(t)}{h(t)}f'(\eta)$$

Substitute the above in (2)

$$\frac{k}{h^2}f''(\eta) = \frac{-h'(t)}{h(t)}\eta(x,t)f'(\eta), f''(\eta) = -\frac{h'h}{k}\eta f'(\eta)$$

Physical oceanography consideration gives

$hh' = 2k = \frac{1}{2}d/_{dt} h^2(t)$;thus, $\frac{h^2(t)}{2} = 2kt + B$. But $h(0) = 0$

Thus, $B = 0$; $h(t) = 2\sqrt{kt}$ -   -   -   -   -   -   -   (4)

Consequently
$f''(\eta) = -2\eta f'(\eta)$ -   -   -   -   -   -   -   (5)
The similarity transformation for this problem is now as follows from eqn. (3)
$V(x, t) = f(\eta)$ and $\eta(x,t) = {}^x/_{2\sqrt{kt}}$ -   -   -   -   -   (6)
and transformed equation (5) is a simple variable separable in form.
$V(0, t) = f(0) = 0$, $V(x, 0) = f(V(x, t) = f(\eta)$ and $f(\infty) = V_0$ and from eqn. (5)

$$\frac{f''(\eta)}{f'(\eta)} = -2\eta, \ln f'(\eta) = -\eta^2(x,t), f'(\eta) = k_0 e^{-\eta^2}$$

$f'(\eta) = k_0 \int_0^\eta e^{-\sigma^2}d\sigma$ -   -   -   -   -   -   -   (7)

$$f(\infty) = V_0 = k_0 \int_0^\eta e^{-\sigma^2}d\sigma = V(x,0) = k_0 \frac{\sqrt{\pi}}{2}$$

Using the definition of gamma function $\Gamma(\alpha) = \int_0^\eta R^{\alpha-1}e^{-R}dR$ and considering $\Gamma\left(\frac{1}{2}\right); k_0 = \frac{2V_0}{\sqrt{\pi}}$

Thus, $V(x, t) = \frac{2V_0}{\sqrt{\pi}}\int_0^\eta e^{-\sigma^2}d\sigma$ -   -   -   -   -   -   (8)

Where the standard error-function e.f is given by
$erf(\eta) = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\sigma^2}d\sigma$, thus
$V(x, t) = V_0 e. f(\eta)$ -   -   -   -   -   -   -   (9)

4.   The nonlinear diffusive processes [6, 2, 5]
This kind of diffusive phenomena are described by eqn. (1). The boundary condition is simply that $V(x,t) \to 0$ as $x \to \pm \infty$ but the initial data in same V_0. Sustained investigation suggests that the similarity transformation in eqn. (6) is not suitable in this case. However, further investigation suggests the following transformation:
$V(x, t) = V_0 t^q f(\eta), \eta(x,t) = xt^p$, -   -   -   -   -   -   (10)
q and p being similarity parameters and thus $\eta(x,t)$, the similarity variable are to be determined firstly.

From eqn. (10), $\eta_x(x,t) = t^p$, $\eta_{xx}(x,t) = 0$, $\eta_t(x,t) = {}^{p\eta(x,t)}/_t$
$V_t = V_0[qt^{q-1}f(\eta) + t^q f'(\eta)\eta_t] = V_0[qt^{q-1}f(\eta) + t^q f'(\eta)\eta_t]$
$= V_0 t^{q-1}[qf(\eta) + P\eta f'(\eta)], V_x = V_0 t^q f'(\eta)\eta_x = V_0 t^{p+q}f'(\eta)$
$V_{xx} = V_0 t^{p+q}f''(\eta)\eta_x = V_0 t^{p+2q}f''(\eta), VV_{xx} = V_0{}^2 t^{p+2q}f(\eta)f'(\eta)$

Substituting in eqn. (1), then
$t^{q-1}V_0[qf(\eta) + \eta pf'(\eta)] + V_0{}^2 t^{p+2q}f(\eta)f'(\eta) - kt^{q+2p}V_0 f''(\eta)] = 0$
$t^{q-1}[f(\eta) + \eta f'(\eta)] + V_0 t^{3q}f(\eta)f(\eta) - kt^{3q}f''(\eta)] = 0$
Provided p = q

$$= t^{-2q-1}[f(\eta) + \eta f'(\eta)] + \frac{V_0}{2}\frac{d}{d\eta}f^2(\eta) - kf''(\eta)] = 0$$

$1ft^{-2q-1} = 1$, then -2q-1=0 and hence, $q = \frac{-1}{2}$

The form of eqn. (1) which doesn't explicitly contain x and t is thus

$$-\frac{1}{2}[f(\eta) + \eta f'(\eta)] + \frac{V_0}{2}\frac{d}{d\eta}f^2(\eta) - kf''(\eta)] = 0 \qquad - \qquad - \qquad - \qquad - \qquad - \qquad (11)$$

The similarity transformation designed for (1) is now

$$V(x, t) = \frac{V_0}{\sqrt{t}}f(\eta), \eta(x, t) = \frac{x}{\sqrt{t}} \qquad - \qquad - \qquad - \qquad - \qquad - \qquad - \qquad - \qquad (12)$$

$$\int \eta f'(\eta)d\eta = \eta f(\eta) - \int f(\eta) + B, B = 0 \text{ since } f(\infty) = 0$$

Integrating (ii), we have

$$2kf'(\eta) + \eta f(\eta) = V_0 f^2(\eta) \qquad - \qquad - \qquad - \qquad - \qquad - \qquad - \qquad (13)$$

$$f'(\eta) + \frac{\eta f}{2k}(\eta) = \frac{V_0}{2k}f^2(\eta) - \qquad - \qquad - \qquad - \qquad - \qquad - \qquad - \qquad (14)$$

The constant of integration is zero since f(η) and the derivation are zero as $x \to \infty$

Eqn. (13) is essentially Bernoulli's ordinary differential equation. Its appearances in this study is interestingly an unexpected result. For the solution, however it may be put in the form,

$$f^{-2}(\eta)f'(\eta) + \frac{2}{2k}f^{-1}(\eta) = \frac{V_0}{2_1}k$$

Let $Z = f^{-1}(\eta) = Z(\eta), Z' = -f^{-2}(\eta)f'(\eta), f^{-2}(\eta)f'(\eta) = -Z$

and

$$Z' - \frac{\eta z}{2k} = -\frac{V_0}{2k} \qquad - \qquad - \qquad - \qquad - \qquad - \qquad - \qquad - \qquad - \qquad (15)$$

Eqn. (15) takes the form

$$\frac{d}{d\eta}\left[zexp\left[-\frac{\eta^2}{4k}\right]\right] = \frac{V_0}{2k}\exp\left[-\frac{\eta^2}{4k}\right], \qquad thus \; Z(\eta) = -exp\left[-\frac{\eta^2}{4k}\right]\frac{V_0}{2k}R(\eta)$$

$$R(\eta) = \int_{-\eta}^{\eta} \exp[-\sigma^2]\,d\sigma = \int_0^\eta \exp[-\sigma^2]\,d\sigma$$

for $\exp[-\sigma^2]$ is an even function of σ but $z(\eta) = f'(\eta)$, thus

Thus, $V_0 f(\eta) = -\frac{k}{R(\eta)}\exp(-\eta^2), t > 0 \qquad - \qquad - \qquad - \qquad - \qquad - \qquad - \qquad (16)$

$$V(x, t) = \frac{V_0}{\sqrt{t}}f(\eta) = \frac{-kexp(-\eta^2)}{\sqrt{t}R(\eta)} \qquad - \qquad - \qquad - \qquad - \qquad - \qquad - \qquad (17)$$

Eqn. (17) can be utilized to describe the far field behaviour of an ocean process as follows:

$\eta \longrightarrow \infty, t > 0, R(\eta) \longrightarrow \sqrt{\pi k}$ and V(x,t) $\longrightarrow$ 0 as expected.

In the subsequent calculations, the following ocean data are used [6, 2]. V_0 = 13m sec^{-1} and thus in the data for the modulation group velocity in the wave spectrum assumed narroew banded. Modulation wavelength L=10.2m; water depth from the free-surface h=10m; diffusion coefficient k= 31m^{2}sec^{-1}. The above are the characteristic value from an observed oceanographic data [2,6].

Conclusions

   Eqns. (16) and (17) illustrate some of the interesting diffuse behaviours in the concept of physical oceanography. In a slowly varying ocean medium, some of the important wave parameters involved are group velocity, wave number and frequency. These are often kinematically diffusive in their evolutions [4, 5]. Group velocity is denoted by V(x,t) in this consideration.It is the speed with which wave energy is transmitted. Adequate understanding of its evolution is essential in all coastal and in-shore ocean activities. Its main feature is this. It is kinematic in its evolution [2] and also diffusive even in the field [7]. This observation is justified analytically in this study.

Finally, this study suggests the unexpected catastrophic behaviour associated with the diffusive phenomenon at the origin of its profile (fig 1) in the nonlinear case. This event would be what a marine physicist or engineer would be happy to observe.
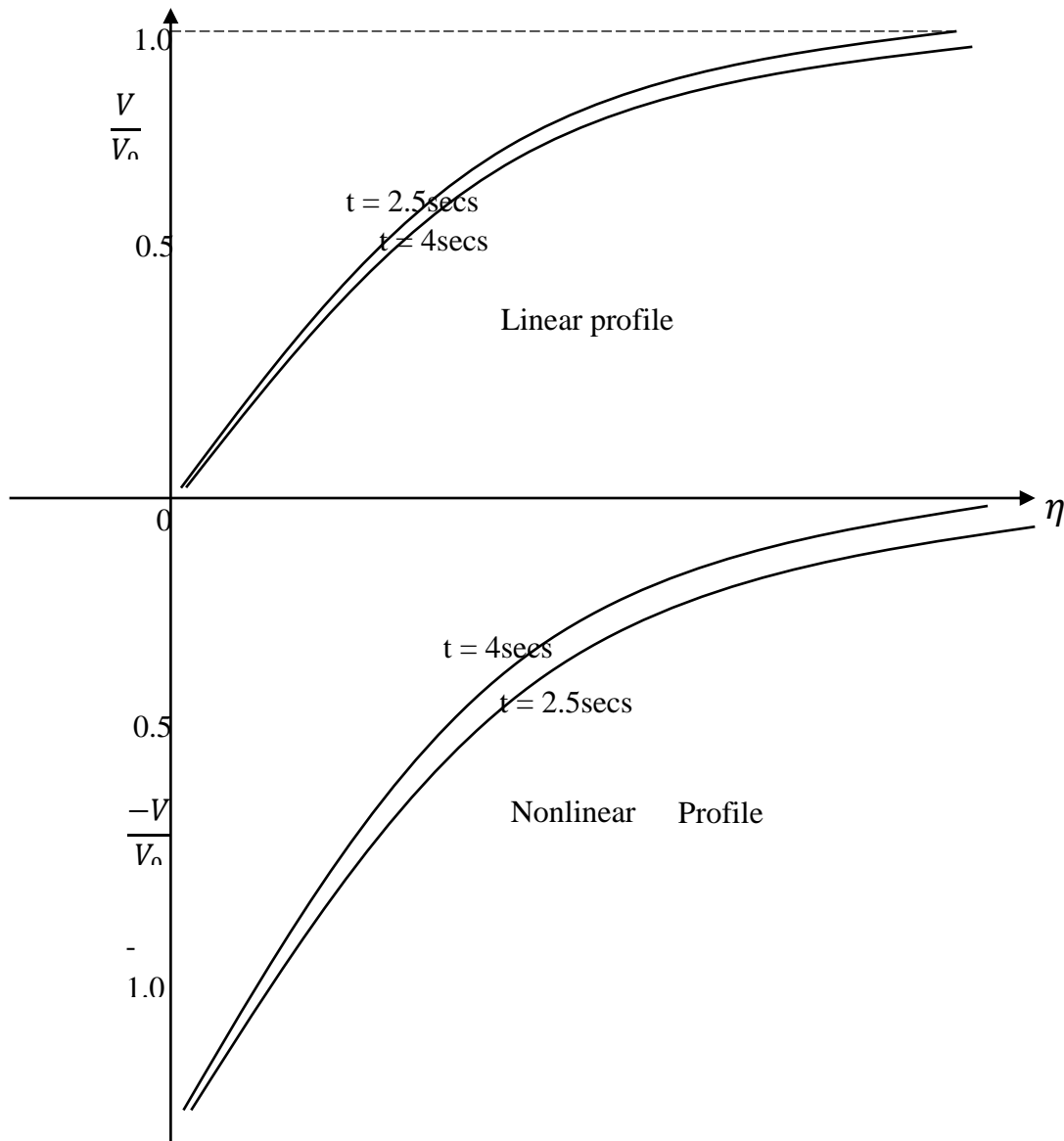
Fig I: Linear Profile (above), Non-linear profile (below)

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