Transactions of the Nigerian Association of Mathematical Physics
***Volume 9,* (*March and May, 2019*), pp137 – 148**
© Trans. of NAMP

MOTION AROUND TRIANGULAR EQUILIBRIUM POINTS OF PHOTOGRAVITATIONAL RESTRICTED PROBLEM OF THREE OBLATE BODIES HAVING CIRCULAR CLUSTER OF MATERIALS

Oni Leke[1] and Joel John Taura [2]

[1] Department of Mathematics, Statistics and Computer Science, College of Science, University of Agriculture, P.M.B. 2373, North-Bank, Makurdi, Benue-State, Nigeria
[2] Department of Mathematics and Computer Science, Faculty of Science, Federal University of Kashere, P.M.B 0182, Gombe, Gombe-State, Nigeria

Abstract

This paper studies motion of a test body around the triangular equilibrium points in the frame of the restricted three-body problem, when the three involved bodies have the shape of an oblate spheroid and are enclosed by a circular cluster of materials under effects of radiation pressure of the main bodies and small perturbations in the Coriolis and centrifugal forces. The equations of motion have been presented and the triangular equilibrium points computed. There exists a pair of triangular equilibrium points, defined by the oblateness of the three bodies, radiation pressures of the primaries, accumulation of materials and small perturbation in the centrifugal force. These equilibrium points are stable under certain conditions and the presence of accumulation of materials around the bodies draws the test body closer to the primaries. Hence, the circular cluster of materials increases the stability region and proves to be a stabilizing force.

Keywords: R3BP; equilibrium points; stability; forces

1. Introduction

The restricted problem of three bodies studies the motion of an infinitesimal mass moving under the gravitational effects of two finite masses, called primaries, which move in circular orbits around their center of mass on account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries. The study of the restricted three-body problem (R3BP) is of great theoretical, practical and educational relevance, and in its many variant, has been applied in several scientific fields, such as celestial mechanics, galactic dynamics, chaos theory and molecular physics. The R3BP is still a stimulating and active research field that has been receiving considerable attention of scientists and astronomers because of its applications in dynamics of the solar and stellar systems, lunar theory, and artificial satellites. There are so many examples of the restricted problem in space dynamics. One of them is the classical three-body problem viz; the Sun-Earth-Moon combination and describing the motion of the moon.

The solutions of the R3BP have been developed over the centuries; there exist five specific solutions called the equilibrium or Lagrangian points. If the third body is placed at any of these points with zero velocity in the coordinate system rotating with the primaries, it will remain at that point in the rotating system. Three of these points called the collinear equilibrium points L_1, L_2, L_3 are located on the line joining the primaries and were found by Euler while the other two equilibrium points L_4, L_5 called triangular equilibrium points were found by Lagrange [1]. The collinear equilibrium points are unstable points while the triangular points can be stable ([2], [3], [4] and [5]) in that a slight displacement of the test body away from the equilibrium points will not produce unbounded motion but rather an oscillation about the points. There are asteroids known as the Trojans librating about both triangular points in the Sun-Jupiter system.

Correspondence Author: Oni L., Email: lekkyonox4ree@yahoo.com, Tel: +2347053283453, +2347067917682 (JJT)

During the last century, several modifications of the classical problem have been introduced in order to make it more relevant and applicable to certain systems of Dynamical Astronomy. It is a well known fact that when one or both the primaries are a source of radiation, the classical R3BP fails to discuss the motion of the third body. This problem is called the photogravitational problem ([2], [6], [7]) and has been applied to the Sun-Planet-Particle and Galaxy Kernel-Sun-Particle. Radiation pressure act as an orbital perturbations and affects the orbits and trajectories of small bodies, all spacecrafts and all natural bodies (comets, asteroids, dust grains, gas molecules) and can cause dust grains to either leave the Solar system or spiral into the Sun. Also, if the effects of the Sun's radiation pressure on the spacecraft of the **Viking** program had been ignored, the spacecraft would have missed Mars orbit by 15,000 kilometers (Eugene Hecht).

In the formulation of the classical R3BP, the third body of infinitesimal mass is considered to move, only under the mutual gravitational force of the primaries, but in practice, Coriolis and centrifugal forces are effective and small perturbations affect these forces. An example is small deviation of disc stars on circular orbits. Therefore, it is important to include these forces in the study of the R3BP. Several interesting studies when Coriolis and centrifugal forces are perturbed have been carried out by [3], [8] and [5], among others. Also, the model of the classical R3BP considered all the bodies to be strictly spherical, but in actual sense, most celestial bodies are not perfect spheres, some have the shape of an oblate spheroid while some are triaxial in nature. For example, the Earth, Jupiter, Saturn and stars (Archerner, Antares and Altair) have the shape of an oblate spheroid while Haumea (a scalene dwarf planet is triaxial in shape. Also, neutron stars and black dwarfs which are a result of the cooling of white dwarfs are also oblate due to their rapid spinning after formation. Figure **1** shows the assignment of the semi-axes on a spheroid. It is an oblate spheroid when $c < a$ and prolate if $c > a$

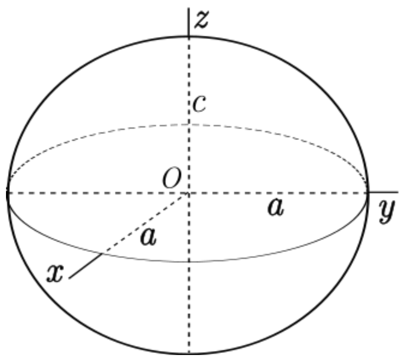

Fig 1: A spheroid showing the role of the semi-axes on the shape

The consideration of the lack of sphericity with respect to oblate shapes causes large perturbations in the study of the R3BP. It has been shown that additional equilibrium points referred to as the out-of-plane points [9] exist when oblateness of the primaries are included. Several other results have been recorded in the works [2], [3], [5], [7], [10], and [11].

In carrying out more investigations of the R3BP, some studies have examined the case when accumulation of debris disc or dust particles surrounds the bodies. {12] found a dust ring around a nearby star, e Eridani. [13] studied dust enshrouded AGB stars in the solar neighborhood. One notable example is in the HD 98800 system, which comprises two pairs of binary stars separated by around 34 AU (Fig **2**). The binary subsystem HD 98800 B, which consists of two stars of 0.70 and 0.58 solar masses in a highly eccentric orbit with semimajor axis 0.983 AU, is surrounded by a complex dust disc that is being warped by the gravitational effects of the mutually-inclined and eccentric stellar orbits [14] and [15]. The importance of the problem in astronomy and the R3BP has been addressed by [16], [17], [18], [19] and [20] where it was shown that the presence of accumulation of material around the bodies resulted in additional collinear equilibrium points of the system. Other works that took into account the case when the primaries are surrounded by circular cluster of materials under different assumptions include [4], [11], [21], [22], and [23].

Fig 2: An artist's impression of the binary star system **HD 98800 B** surrounded by a circular cluster of materials (**Credit**: NASA Spitzer Telescope)

In order to state what the present paper examines, it is important to reiterate that [5] extended the work in [3] by including the assumption that the third body also has the shape of an oblate spheroid. In our present study, we extend the work of [5] by looking at the case when the primaries are stars and are in addition enclosed by a circular cluster of material points. The equilibrium points are found and their stability investigated. The paper is organized such that section one gives the introduction. Section two captures the equations of the problem under effects of radiation, perturbations, oblateness of the three bodies and the gravitational potential from the cluster of material points. The investigation of the equilibrium points is done in section three while section four analyzes their linear stability. The discussion and conclusions are drawn in section five and six, respectively.

2. Equations of motion

Let m_1 and m_2 be the masses of the first and second primary which are massive radiating-oblate stars, respectively, and let m_3 be the mass of the third body having infinitesimal mass compared to the masses of the primaries. We consider the motion of a particle influenced by the gravitational force from the central binary and the accumulated cluster of materials. We use a model that best explains a flattened system given by [24] and expressed as

$$V(r,z) = -\frac{GM_d}{\sqrt{r^2 + \left(a + \sqrt{z^2 + b^2}\right)^2}}; \qquad (1)$$

where M_d is mass of the accumulated material point; a and b are parameters which determine the density profile of the material point. The ratio b/a is the measure of the flatness of the cluster and $r = \sqrt{x^2 + y^2}$. The behavior of the two parameters having dimension of distance determines the two limiting cases. When $a = b = 0$, equations (1) equal to the one when there is no flattening. If $a = 0$ the axial symmetry is reduced to its special case (the spherical one) while for $b = 0$ the system has a collapse. Therefore, by varying the ratio b/a of the parameters, one varies the flattening of the system: from zero (no flattening) towards infinity (collapse into the plane of symmetry). Now, as b/a is tending to zero the mass distribution of the cluster of materials becomes flatter and flatter.

Now, we assume that the three bodies have the shape of an oblate spheroid with both primaries radiating and surrounded by a cluster of materials. The modified potential in this case is

$$U^* = -Gm_3\left[m_1\left(\frac{q_1}{r_1} + \frac{\alpha_1 q_1}{2r_1^3} + \frac{\alpha_3}{2r_1^3}\right) + m_2\left(\frac{q_2}{r_2} + \frac{\alpha_2 q_2}{2r_2^3} + \frac{\alpha_3}{2r_2^3}\right) + \frac{M_d}{\left(r^2 + T^2\right)^{1/2}}\right] \qquad (2)$$

where $r_1^2 = (x - x_1)^2 + y^2$, $r_2^2 = (x - x_2)^2 + y^2$, $\alpha_i = \dfrac{AE_i^2 - AP_i^2}{5R^2} : \alpha_i << 1 : (i = 1,2,3)$.

G is the gravitational constant; q_1, q_2 are radiation factors of the first and second primary, respectively, while α_1, α_2 and α_3 are the parameters representing the oblateness of the first primary and second primary and the test body, respectively. r_1

and r_2 are the distances of the body from the first and second primary, respectively, and last term in the potential, is the gravitational potential due to the mass M_d of the disc: $z=0$; $T=a+b$ is the parameter which defines the density profile of the cluster of materials.

Now, the equations of motion of a passively gravitating third body in a barycentric rotating coordinate system, in the gravitational field of the primaries, based on the potential (2), have the form:

$$\ddot{x}-2n\dot{y}=n^2x-\frac{q_1(1-\mu)(x-\mu)}{r_1^3}-\frac{q_2\mu(x-\mu+1)}{r_2^3}-\frac{3\alpha_1q_1(1-\mu)(x-\mu)}{2r_1^5}-\frac{3\alpha_2q_2\mu(x-\mu+1)}{2r_2^5}$$
$$-\frac{3\alpha_3(1-\mu)(x-\mu)}{2r_1^5}-\frac{3\alpha_3\mu(x-\mu+1)}{2r_2^5}-\frac{M_dx}{(r^2+T^2)^{3/2}} \tag{3}$$
$$\ddot{y}+2n\dot{x}=n^2y-\frac{q_1(1-\mu)y}{r_1^3}-\frac{q_2\mu y}{r_2^3}-\frac{3\alpha_1q_1(1-\mu)y}{2r_1^5}-\frac{3\alpha_2q_2\mu y}{2r_2^5}-\frac{3\alpha_3(1-\mu)y}{2r_1^5}-\frac{3\alpha_3\mu y}{2r_2^5}$$
$$-\frac{M_dy}{(r^2+T^2)^{3/2}}$$

where

$$n^2=1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2+\frac{2M_dr_c}{(r_c^2+T^2)^{\frac{3}{2}}} \tag{4}$$

μ is the mass ratio of the stars and n is their mean motion defined by the oblateness of the primaries, the mass of the cluster points, the parameter which represents the density profile of the cluster of materials and r_c is the radial distance of the test body in the classical R3BP.

Now, due to small deviation of disc primaries on circular orbits, we assume that the Coriolis and centrifugal forces of the stars are slightly perturbed and so the system of equations of motion is recast to the form:

$$\ddot{x}-2\varphi n\dot{y}=U_x$$
$$\ddot{y}+2\varphi n\dot{x}=U_y \tag{5}$$

where

$$U=\frac{n^2\psi(x^2+y^2)}{2}+\frac{q_1(1-\mu)}{r_1}+\frac{q_2\mu}{r_2}+\frac{\alpha_1q_1(1-\mu)}{2r_1^3}+\frac{\alpha_2q_2\mu}{2r_2^3}+\frac{\alpha_3(1-\mu)}{2r_1^3}+\frac{\alpha_3\mu}{2r_2^3}+\frac{M_d}{(r^2+T^2)^{1/2}}$$

The parameters φ and ψ represent the small change in Coriolis and the centrifugal forces, respectively, and are such that $(\varphi-1)<<1$ and $(\psi-1)<<1$. Equations of motion (5) admits the Jacobi integral

$$C+(\dot{x}^2+\dot{y}^2)=2U \tag{6}$$

where C is the Jacobi constant

3. Triangular equilibrium points

Because no general solution in the R3BP is available, particular solutions are sought to obtain insight into the problem. These particular solutions which are referred to as the equilibrium points are found by setting the velocity and acceleration in equation (5) to zero:

$$n^2\psi x-\frac{q_1(1-\mu)(x-\mu)}{r_1^3}-\frac{q_2\mu(x-\mu+1)}{r_2^3}-\frac{3\alpha_1q_1(1-\mu)(x-\mu)}{2r_1^5}-\frac{3\alpha_2q_2\mu(x-\mu+1)}{2r_2^5}$$
$$-\frac{3\alpha_3(1-\mu)(x-\mu)}{2r_1^5}-\frac{3\alpha_3\mu(x-\mu+1)}{2r_2^5}-\frac{M_dx}{(r^2+T^2)^{3/2}}=0 \tag{7}$$

and
$$n^2\psi y-\frac{q_1(1-\mu)y}{r_1^3}-\frac{q_2\mu y}{r_2^3}-\frac{3\alpha_1q_1(1-\mu)y}{2r_1^5}-\frac{3\alpha_2q_2\mu y}{2r_2^5}-\frac{3\alpha_3(1-\mu)y}{2r_1^5}-\frac{3\alpha_3\mu y}{2r_2^5}-\frac{M_dy}{(r^2+T^2)^{3/2}}=0$$

Solving the pair of equations in (7) for x and y gives the coordinate of the triangular equilibrium points.

From first equation of (7), we get

$$(x-\mu)\left[n^2\psi-\frac{q_1(1-\mu)x}{r_1^3}-\frac{q_2\mu}{r_2^3}-\frac{3\alpha_1 q_1(1-\mu)}{2r_1^5}-\frac{3\alpha_2 q_2\mu}{2r_2^5}-\frac{3\alpha_3(1-\mu)}{2r_1^5}-\frac{3\alpha_3\mu}{2r_2^5}\right]-\frac{M_d x}{\left(r^2+T^2\right)^{3/2}}$$

$$+\mu\left[n^2\psi-\frac{q_2}{r_2^3}-\frac{3\alpha_2 q_2}{2r_2^5}-\frac{3\alpha_3}{2r_2^5}\right]=0$$

Now, substituting the second of (7) in the above equation (since $y\neq 0$), yields

$$\frac{M_d}{\left(r^2+T^2\right)^{3/2}}(x-\mu)=-\mu\left[n^2\psi-\frac{q_2}{r_2^3}-\frac{3\alpha_2 q_2}{2r_2^5}-\frac{3\alpha_3}{2r_2^5}\right]+\frac{M_d}{\left(r^2+T^2\right)^{3/2}}$$

Simplifying this equation, yields

$$n^2\psi-\frac{q_1}{r_1^3}-\frac{3\alpha_1 q_1}{2r_1^5}-\frac{3\alpha_3}{2r_1^5}-\frac{M_d}{\left(r^2+T^2\right)^{3/2}}=0$$

$$n^2\psi-\frac{q_2}{r_2^3}-\frac{3\alpha_2 q_2}{2r_2^5}-\frac{3\alpha_3}{2r_2^5}-\frac{M_d}{\left(r^2+T^2\right)^{3/2}}=0 \qquad (8)$$

When the primaries are not radiating and are spherical, equations (8) reduce respectively:

$$n^2\psi-\frac{1}{r_2^3}-\frac{3\alpha_3}{2r_2^5}-\frac{M_d}{\left(r^2+T^2\right)^{3/2}}=0$$

and

$$n^2\psi-\frac{1}{r_1^3}-\frac{3\alpha_3}{2r_1^5}-\frac{M_d}{\left(r^2+T^2\right)^{3/2}}=0$$

Further, when the cluster of materials is absent and the test body is spherical in shape, these equations reduce to

$$\psi-\frac{1}{r_1^3}=0\,, \quad \psi-\frac{1}{r_2^3}=0\,.$$

Therefore, the solution when these parameters are null but a slight deflection on circular orbit remains, is

$$r_1=r_2=\frac{1}{\psi^{1/3}} \qquad (9)$$

Hence, we use perturbation method to solve the system of equations (8), when all the parameters are present. To achieve this, we assume that the solutions are

$$r_1=\frac{1}{\psi^{1/3}}+\varepsilon_1, \quad r_2=\frac{1}{\psi^{1/3}}+\varepsilon_2 \qquad (10)$$

where $\varepsilon_i\,(i=1,2)$ are very small quantities.

Now, from equations (10), we get

$$r_1^{-3}=\psi\left(1-3\psi^{1/3}\varepsilon_1\right),\; r_1^{-5}=\psi^{5/3}\left(1-5\psi^{1/3}\varepsilon_1\right) \qquad (11)$$

$$r_2^{-3}=\psi\left(1-3\psi^{1/3}\varepsilon_2\right),\; r_2^{-5}=\psi^{5/3}\left(1-5\psi^{1/3}\varepsilon_2\right),$$

Substituting equations (4) and (11) in the equations of system (8) and solving, we get

$$\varepsilon_1=\frac{1}{3\psi^{1/3}}\left[-\frac{3}{2}(\alpha_1+\alpha_2)-(1-q_1)+\frac{3}{2}(\alpha_1+\alpha_3)\psi^{2/3}-\frac{M_d}{\left(r_c^2+T^2\right)^{3/2}}\left(2r_c-\frac{1}{\psi}\right)\right]$$

$$\varepsilon_2=\frac{1}{3\psi^{1/3}}\left[-\frac{3}{2}(\alpha_1+\alpha_2)-(1-q_2)+\frac{3}{2}(\alpha_1+\alpha_3)\psi^{2/3}-\frac{M_d}{\left(r_c^2+T^2\right)^{3/2}}\left(2r_c-\frac{1}{\psi}\right)\right] \qquad (12)$$

Now, substitute equations (12) in equations (10) and simplifying, to get

$$r_1 = \frac{1}{\psi^{1/3}}\left[1 - \frac{1}{2}(\alpha_1+\alpha_2) - \frac{1}{3}(1-q_1) + \frac{1}{2}(\alpha_1+\alpha_3)\psi^{2/3} - \frac{M_d(2\psi r_c - 1)}{3\psi\left(r_c^2+T^2\right)^{3/2}}\right]$$

$$r_2 = \frac{1}{\psi^{1/3}}\left[1 - \frac{1}{2}(\alpha_1+\alpha_2) - \frac{1}{3}(1-q_2) + \frac{1}{2}(\alpha_2+\alpha_3)\psi^{2/3} - \frac{M_d(2\psi r_c - 1)}{3\psi\left(r_c^2+T^2\right)^{3/2}}\right] \qquad (13)$$

The exact coordinate of the triangular equilibrium point is

$$x = \mu - \frac{1}{2} + \frac{r_2^2 - r_1^2}{2}, \qquad y = \pm\sqrt{\frac{r_1^2 + r_2^2}{2} - \frac{1}{4} - \frac{\left(r_2^2 - r_1^2\right)^2}{2}} \qquad (14)$$

Now, from equations (13), we find $r_2^2 - r_1^2$ and $r_2^2 + r_1^2$, and substitute the results in equations (14), to get

$$x = \mu - \frac{1}{2} + \left\{\frac{1}{\psi^{2/3}} - \frac{M_d(2\psi r_c - 1)}{3\psi^{5/3}\left(r_c^2+T^2\right)^{3/2}}\right\}\left[\frac{1}{3}(1-q_1) - \frac{1}{3}(1-q_2) - \frac{(\alpha_1-\alpha_2)\psi^{2/3}}{2}\right]$$

$$y = \pm\frac{\sqrt{4-\psi^{2/3}}}{2\psi^{1/3}}\left\{1 - \frac{2}{4-\psi^{2/3}}\left[\frac{1}{3}(1-q_1) + \frac{1}{3}(1-q_2) + (\alpha_1+\alpha_2) - \frac{1}{2}\psi^{2/3}(\alpha_1+\alpha_2+2\alpha_3) + \frac{2M_d(2\psi r_c - 1)}{3\psi\left(r_c^2+T^2\right)^{3/2}}\right]\right\} \qquad (15)$$

These are the coordinates of the triangular equilibrium points of the restricted problem of three oblate bodies under radiation pressure forces of the primaries, small change in the Coriolis and centrifugal forces when there is an enclosure of cluster of materials around the configuration. Using the software *Mathematica* {25] we plot the 3D-surface plots of the triangular points as shown in Figures 3, 4, 5, 6 and 7.

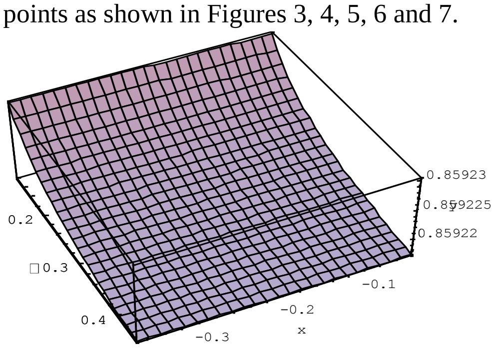

Fig 3: Surface plot of L_4 for $0.118166 \le \mu \le 0.450617$, $q_1 = 0.9988$, $q_2 = 0.9985$, $\psi = 1.002$, $\alpha_1 = 0.02$, $\alpha_2 = 0.018$, $\alpha_3 = 0.01$, $M_d = 0$.

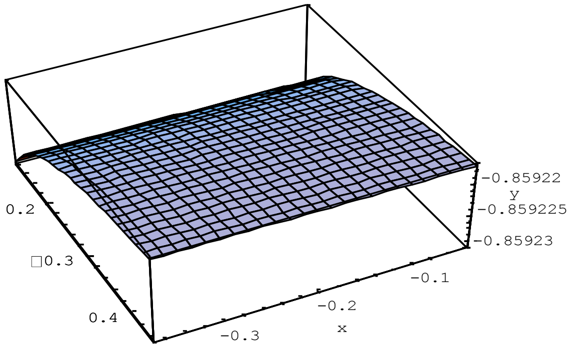

Fig 4: Surface plot of L_5 for $0.118166 \le \mu \le 0.450617$, $q_1 = 0.9988$, $q_2 = 0.9985$, $\psi = 1.002$, $\alpha_1 = 0.02$, $\alpha_2 = 0.018$, $\alpha_3 = 0.01$, $M_d = 0$.

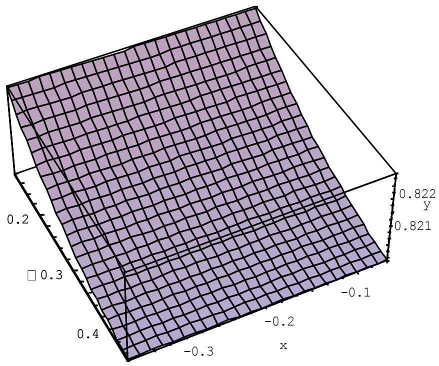

Fig 5: Surface plot of L_4 for $0.118166 \le \mu \le 0.450617$, $q_1 = 0.9988$, $q_2 = 0.9985$, $\psi = 1.002$, $\alpha_1 = 0.02$, $\alpha_2 = 0.018$, $\alpha_3 = 0.01$, $M_d = 0.09$

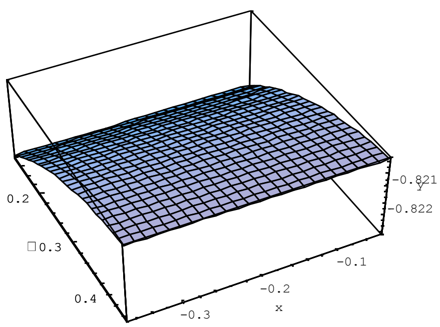

Fig 6: Surface plot of L_5 for $0.118166 \le \mu \le 0.450617$, $q_1 = 0.9988$, $q_2 = 0.9985$, $\psi = 1.002$, $\alpha_1 = 0.02$, $\alpha_2 = 0.018$, $\alpha_3 = 0.01$, $M_d = 0.09$

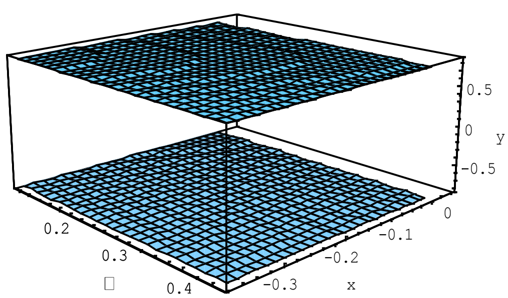

Fig 7: Surface plot of $L_{4,5}$ for $0.118166 \le \mu \le 0.450617$, $q_2 = 0.9985$, $\psi = 1.002$, $\alpha_1 = 0.02$ $\alpha_2 = 0.018$, $M_d = 0.09$

Figure **3** and **4** shows the positions L_4 and L_5 of the triangular equilibrium points when there are no cluster of materials $(M_d = 0)$ around the bodies. In this case, the test body is farther away from the line joining the stars. However, when materials begins to accumulate around the bodies, the mass M_d gradually increases and when it reaches up to 0.09, the test body is drawn nearer to the line on which the primaries lie and even goes closest, when the mass of accumulated materials reaches close to 0.1 (Fig. **5** and Fig **6**). Hence, an increase in the mass of cluster of materials produces a shift in the positions of the triangular equilibrium points and consequently makes the body to be positioned closer to the primaries; although, this is also influenced by the oblateness, radiation pressure and small deflections on circular orbit owning to the centrifugal force.

4. Stability of Triangular equilibrium points

In order to study the stability of any of the equilibrium points $L_i\left(i=1,2..5\right)$, we displace the test body a little from an equilibrium point, by applying a small velocity. Because of the additional imposed forces, the stability of the equilibrium points is expected to be affected by these forces. We introduce $x=x_0+\xi$ and $y=y_0+\eta$, where $\left(\xi,\eta\right)$ is a small displacement and substitute it in the equations of motion (5). We expand the equations of motion into first-order terms with respect to ξ and η to get the variational equations:

$$\ddot{\xi}-2n\varphi\dot{\eta}=U^0_{xx}\xi+U^0_{xy}\eta$$

$$\ddot{\eta}+2n\varphi\dot{\xi}=U^0_{xy}\xi+U^0_{yy}\eta$$

The superscript O indicates that the derivatives are to be calculated at the equilibrium points.
The associated characteristic equation is

$$\lambda^4-\left(U^0_{xx}+U^0_{yy}-4\varphi^2n^2\right)\lambda^2+U^0_{xx}U^0_{yy}-\left(U^0_{xy}\right)^2=0 \qquad (16)$$

To obtain the partial derivatives computed at the triangular point L_4 given in equation (15), we first get the expressions for r_i^{-3}, r_i^{-5} and $r_i^{-7}\ \left(i=1,2\right)$, from equations (13) and substitute them together with equations (4) and (15), in the derivatives U_{xx}, U_{yy} and U_{xy} to get

$$U^0_{xx}=\psi^{5/3}\left[\frac{3}{4}+\frac{27}{8}\alpha_1+\frac{3}{8}\alpha_2-3\mu(\alpha_1-\alpha_2)+\frac{5}{4\psi}\frac{M_d(2\psi r_c-1)}{\left(r_c^2+T^2\right)^{3/2}}\right]+\psi\left[\frac{1}{2}\left(\psi^{2/3}-2\right)(1-q_1)+(1-q_2)\right.$$
$$-\frac{\mu}{2}\left(\psi^{2/3}-4\right)(1-q_1)+\frac{\mu}{2}\left(\psi^{2/3}-4\right)(1-q_2)\Bigg]-\frac{3M_d\left(\mu-\frac{1}{2}\right)^2}{\left(r_c^2+T^2\right)^{3/2}}\left[1+\frac{1}{\psi^{2/3}\left(\mu-\frac{1}{2}\right)}\left\{\frac{1}{3}(1-q_1)+\frac{1}{3}(1-q_1)\right.\right.$$
$$\left.\left.-\frac{1}{2}\psi^{2/3}(\alpha_1-\alpha_2)\right\}\right] \qquad (17)$$

$$U^0_{xy}=\psi^{4/3}\sqrt{4-\psi^{2/3}}\left[-\frac{3}{4}+\frac{\left(4-4\psi^{2/3}+\psi^{4/3}\right)(1-q_1)}{2\psi^{2/3}\left(4-\psi^{2/3}\right)}+\frac{\left(\psi^{2/3}-2\right)(1-q_2)}{\psi^{2/3}\left(4-\psi^{2/3}\right)}+\frac{3}{8}\frac{\left(5\psi^{2/3}-24\right)}{\left(4-\psi^{2/3}\right)}\alpha_1\right.$$
$$+\frac{3}{8}\frac{\left(\psi^{2/3}-8\right)}{\left(4-\psi^{2/3}\right)}\alpha_2-\frac{3}{2}\frac{\psi^{2/3}}{\left(4-\psi^{2/3}\right)}\alpha_3-\frac{\left(16-5\psi^{2/3}\right)}{4\psi\left(4-\psi^{2/3}\right)}\frac{\left(2\psi r_c-1\right)M_d}{\left(r_c^2+T^2\right)^{3/2}}+\mu\left\{\frac{3}{2}-\frac{\left(\psi^{2/3}-2\right)}{2\left(4-\psi^{2/3}\right)}(1-q_1)\right.$$
$$\left.-\frac{\left(\psi^{2/3}-2\right)}{2\left(4-\psi^{2/3}\right)}(1-q_2)-\frac{3\left(3\psi^{2/3}-16\right)}{4\left(4-\psi^{2/3}\right)}(\alpha_1+\alpha_2)+\frac{3\psi^{2/3}}{\left(4-\psi^{2/3}\right)}\alpha_3+\frac{\left(16-5\psi^{2/3}\right)}{2\psi\left(4-\psi^{2/3}\right)}\frac{\left(2\psi r_c-1\right)M_d}{\left(r_c^2+T^2\right)^{3/2}}\right\}$$
$$\left.+\frac{3}{4\psi^{5/3}}\frac{\left(2\mu-1\right)M_d}{\left(r_c^2+T^2\right)^{5/2}}\right]$$

$$U^0_{yy}=\frac{3M_d\left(4-\psi^{2/3}\right)}{4\psi^{2/3}\left(r_c^2+T^2\right)^{5/2}}+\psi\left[\frac{3\left(4-\psi^{2/3}\right)}{4}+\frac{3\left(12-\psi^{2/3}\right)(\alpha_1+\alpha_2)}{8}+3\alpha_3\psi^{2/3}+\frac{(1-q_1)\left(2-\psi^{2/3}\right)}{2}\right.$$
$$\left.-(1-q_2)+\frac{\left(20-13\psi^{2/3}\right)}{4\psi}\frac{M_d\left(2\psi r_c-1\right)}{\left(r_c^2+T^2\right)^{3/2}}+\mu\left\{\frac{\left(4-\psi^{2/3}\right)}{2}(1-q_2)-\frac{\left(4-\psi^{2/3}\right)}{2}(1-q_1)\right\}\right]$$

Now, substituting these derivatives in the characteristic equation (16), we express $\varphi=1+\left(\varphi-1\right), \psi=1+\left(\psi-1\right)$ and simplify the result by retaining only linear terms in $\alpha_i\left(i=1,2,3\right), \left(1-q_1\right), \left(1-q_2\right), \left(\varphi-1\right), \left(\psi-1\right)$ and M_d, to get

$$\lambda^4-\left[3(\psi-1)-4-6(\alpha_1+\alpha_2)-8(\varphi-1)+3\left\{1+\frac{5}{2}\alpha_1+\frac{3}{2}\alpha_2+\alpha_3-\mu(\alpha_1-\alpha_2)+\frac{M_d(2\psi r_c-1)}{\psi(r_c^2+T^2)^{3/2}}\right.\right.$$
$$\left.\left.+\frac{\mu^2 M_d}{\psi(r_c^2+T^2)^{3/2}}-\frac{\mu M_d}{(r_c^2+T^2)^{3/2}}+\frac{M_d}{(r_c^2+T^2)^{3/2}}\right\}\right]\lambda^2+\frac{3}{4}\mu(1-\mu)\left[9+39(\alpha_1+\alpha_2)+12\alpha_3+22(\psi-1)\right.$$
$$\left.+2(1-q_1)+2(1-q_2)+\frac{M_d}{(r_c^2+T^2)^{3/2}}\left\{\frac{9}{(r_c^2+T^2)^{3/2}}+\frac{22(2\psi r_c-1)}{\psi}\right\}\right]=0 \qquad (18)$$

The roots are

$$\lambda_{1,2}=\pm s_1,\ \lambda_{3,4}=\pm s_2 \qquad (19)$$

where

$$s_{1,2}=\frac{1}{2}\left(-b\pm\sqrt{D}\right)$$

$$b=1-3(\psi-1)+8(\alpha-1)-\frac{3}{2}(\alpha_1-\alpha_2-2\alpha_3)+3\mu(\alpha_1-\alpha_2)+\frac{3M_d}{(r_c^2+T^2)^{3/2}}\times$$
$$\left\{\frac{8}{3}r_c-\frac{2\psi r_c-1}{\psi}-\frac{(\mu^2-\mu+1)}{(r_c^2+T^2)}\right\}$$

$$D=3\mu^2\left[9+39(\alpha_1+\alpha_2)+12\alpha_3+22(\psi-1)+2(1-q_1)+2(1-q_2)-\frac{22(1-2\psi r_c)M_d}{\psi(r_c^2+T^2)^{3/2}}+\frac{7M_d}{(r_c^2+T^2)^{5/2}}\right]$$
$$-3\mu\left[9+37\alpha_1+41\alpha_2+12\alpha_3+22(\psi-1)+2(1-q_1)+2(1-q_2)-M_d\left\{\frac{22(1-2\psi r_c)}{\psi(r_c^2+T^2)^{3/2}}-\frac{7}{(r_c^2+T^2)^{5/2}}\right\}\right]$$
$$+1+16(\alpha-1)-6(\psi-1)-3(\alpha_1-\alpha_2)-6\alpha_3+\frac{2M_d}{(r_c^2+T^2)^{3/2}}\left\{2r_c+\frac{3}{\psi}-\frac{3}{(r_c^2+T^2)}\right\}$$

D is the discriminant and the four roots in equation (19) determine whether the equilibrium point is stable. Now, since the value of D when $\mu=0$ and $\mu=\frac{1}{2}$ have opposite signs. Then, we must have a critical value of μ in the interval $0<\mu\le\frac{1}{2}$ at which $D=0$. This value is called the critical mass parameter and obtained in this case as

$$\mu_C=\frac{1}{2}\left(1-\sqrt{\frac{23}{27}}\right)-\frac{1}{9}\left(\frac{13}{\sqrt{69}}+1\right)\alpha_1-\frac{1}{9}\left(\frac{13}{\sqrt{69}}-1\right)\alpha_2-\frac{22}{9\sqrt{69}}\alpha_3+\frac{4(36\in-19\in')}{27\sqrt{69}}-\frac{2(1-q_1)}{27\sqrt{69}}$$
$$-\frac{2(1-q_2)}{27\sqrt{69}}+\frac{2M_d}{27\sqrt{69}(r_c^2+T^2)^{3/2}}\left[38-4r_c-\frac{61}{2(r_c^2+T^2)}\right] \qquad (20)$$

where $\in=\varphi-1$ and $\in'=\psi-1$ and the first term represents Routh's critical mass ratio (Szebehely 1967). The second, third and fourth expressions represent the effects arising from oblateness of the stars and the test body, respectively, while the fifth, sixth and seventh term are the value owning to small change in Coriolis and centrifugal forces, radiation pressure force of the first and second primary, respectively. The last expression stands for the effect of gravitational potential from the cluster of material around the bodies.

Now, we can state the stability criteria of the triangular equilibrium points taking into account the mass parameter μ and the critical mass function μ_C. It is seen that when $0<\mu<\mu_C$, $D>0$; the roots (19) are distinct pure imaginary numbers and the triangular point is stable. In this case, motion is bounded and composed of two harmonic motions. The general solution depends on roots (19) and given [1]:

$\xi = A_1 \cos s_1 t + A_2 \sin s_1 t + A_3 \cos s_2 t + A_4 \sin s_2 t$

$\eta = B_1 \cos s_1 t + B_2 \sin s_1 t + B_3 \cos s_2 t + B_4 \sin s_2 t$

where the coefficients A_i and $B_i \left(i = 1, 2..4\right)$ are the long and short periodic terms respectively.

When $\mu_C < \mu \leq \frac{1}{2}$ or $\mu = \mu_C$, we have $D < 0$ and $D = 0$, respectively. The triangular point is unstable in both cases due to a positive root which results in an unbounded motion and the test body will rapidly depart from the triangular point.

5. Discussion

The equations of motion of an infinitesimal body has been derived under the assumption that the three bodies involved in the model of the R3BP are surrounded by an accumulation of materials and have the shape of an oblate spheroid with further assumptions that both primaries are radiating and small perturbation in the Coriolis and centrifugal forces are considered to be effective. These equations are affected by radiation pressure, oblateness, potential from the cluster and the perturbations in the Coriolis and centrifugal forces of the primaries. These equations are similar but contain more parameters than other previous studies of [3], [4], [5], [11], [16]. [17], [18], [22] and [23]. The position of triangular equilibrium points L_4 and L_5 is given by equations (15). These points are defined by a small change in the centrifugal force, oblate shapes of the stars and the test body, radiation of the stars and the potential from the cluster of materials. When the cluster surrounding the radiating-oblate bodies are absent $M_d = 0$ and the coordinate reduces to those in [5]. When there are no perturbations in the centrifugal forces, the points fully coincide with those in [10]. The triangular points in [1], [2], [3], [4], [5], [11] can all be recovered from equation (15). Clearly, it is seen that when there are no cluster of materials $\left(M_d = 0\right)$ around the stars, the test body is farther away from the axis. However, when materials begins to cluster in circles around the bodies, the mass of the cluster increases and the test body draws nearer to the line on which the main bodies lie. So that more accumulation of cluster of materials produces a shift in the positions of the triangular equilibrium points and consequently makes the infinitesimal mass to be positioned closer to the primaries; although, this is also influenced by the oblateness, radiation pressure and small deflections on circular orbit owning to the centrifugal force.

The study of the stability of the equilibrium points does not yield anything different from the already established fact that the triangular pints are stable under the condition that the roots of the characteristic equation (18) are distinct pure imaginary roots when the mass parameter of the is less that the critical mass (20). Observe that the value of the critical mass is determined by Routh's critical mass value which is the first term of equation (20). The second, third and fourth expressions represent the effects arising from oblateness of the primaries and the test body, respectively, while the fifth, sixth and seventh term are the value owning to small change in Coriolis and centrifugal forces, radiation pressure force of the primary and second primary, respectively. The last expression stands for the effect of potential from the circular accumulation of materials around the stars. The critical mass calculated by [1], [2], [3], [4], [5], [10], [11], and [23] can all be verified from equation (20). We observe that μ_C is a decreasing function of the oblateness of the bodies, the radiation pressure force and small perturbation in centrifugal force but it is an increasing function of the small perturbation in the Coriolis force and cluster of materials. Therefore, the circular cluster of materials increases the stability region and so accumulation of materials around the stars proves to be a stabilizing force. The small perturbation in the Coriolis force is also a stabilizing force because its increases the stability region. The oblate shapes of the three bodies have destabilizing effects. Same hold for the radiation pressure of the primaries because their presence reduces the stability region.

6. Conclusion

A study of the R3BP when the motion of an oblate test body takes place under the gravitational influence of two massive oblate radiating bodies and enclosed by a circular cluster of material points coupled with small perturbations in the Coriolis and centrifugal forces, in the vicinity of the triangular equilibrium points has been studied. The equations of motion have been presented and a pair of triangular equilibrium points computed. These points are defined by the oblateness of the three bodies, radiation pressure of the primaries, small change in the centrifugal force and the presence of an enclosure of cluster of materials around the configuration. Motion around these points is points are stable.

The equilibrium points are very important in exploration and development of space. The Solar and Heliospheric Observatory (SOHO) lunched in 1995 and Microwave Anisotropy Probe (MAP) lunched in 2001 by NASA are currently in operation Sun-Earth L_1 and L_2, respectively. Solar TErrestrial RElations Observatory-Ahead (STEREO-A) made its closest pass to L_5 recently, on its orbit around the Sun. Asteroid 2010 SO16, is currently proximal to L_5 but at a high inclination.

References

[1] Szebehely, V.G.: (1967). *Theory of Orbits*. Academic Press, New York.

[2] Singh, J. and Ishwar, B.: (1999). Stability of triangular points in the generalized photogravitational restricted three-body problem. *Bulletin of the Astronomical Society of India*, *27*, 415

[3] AbdulRaheem, A. and Singh, J.: (2006). Combined effects of perturbations, radiation and oblateness on the stability of equilibrium points in the restricted three-body problem. *Astronomical Journal*, *131*, 1880.

[4] Kushvah, B.S.: (2008). Linear stability of equilibrium points in the generalized photogravitational Chermnykh's problem. *Astrophysics and Space Science*, *318*, 41.

[5] Singh, J. and Haruna, S.: (2014). Equilibrium points and stability under effects of radiation and perturbing forces in the restricted problem of three oblate bodies. *Astrophysics and Space Science*, *349*, 107

[6] Singh, J. and Leke, O.: (2010). Stability of the photogravitational restricted three-body problem with variable masses. *Astrophysics and Space Science, 326*, 305.

[7] Singh, J., Leke, O.: (2013). "Effects of oblateness, perturbations, radiation and varying masses on the stability of equilibrium points in the restricted three-body problem". *Astrophysics and Space Science, 344*: 51.

[8] Singh, J. and Begha, J.M.: (2011). Stability of equilibrium points in the generalized perturbed restricted three-body problem. *Astrophysics and Space Science, 331*, 511.

[9] Douskos, C.N. and Markellos, V.V.: (2006). Out-of-plane equilibrium points in the restricted three-body problem with oblateness. *Astronomy and Astrophysics, 446*, 357.

[10] Abouelmagd, E.I. and EL-Shaboury, S.M: (2012). Existence and stability of triangular points in the restricted three-body problem with axisymmetric bodies", *Astrophysics and Space Science*, *341*, 331.

[11] Singh, J. and Taura, J.J.: (2013). Motion in the generalized restricted three-body problem. *Astrophysics and Space Science*, *343*, 95.

[12] Greaves, J. S., Holland, W. S., Moriarty-Schieven, G., Jenness, T., Zuckerman, B., McCarthy, C., Dent, W. R. F., Webb, R.A., Butner, H .M., Gear, W.K. and Walker, H.J.:(1998). A dust ring around epsilon Eridani: analog to the young solar system. *Astrophysical Journal*, *506*,133.

[13] Olivier, E.A., Whitelock, P. and Marang, F.: (2001). Dust-enshrouded asymptotic giant branch stars in the solar neighborhood. *Monthly Notices of the Royal Astronomical Society 326*, 490.

[14] Akeson, R. L., Rice, W. K. M., Boden, A. F., Sargent, A. I., Carpenter, J. M. and Bryden, G. (2007). The Circumbinary Disc of HD 98800B: Evidence for Disc Warping. *The Astrophysical Journal, 670*, 1240.

[15] Verrier, P. E. and Evans, N. W.: (2008). HD 98800: a most unusual debris disc. *Monthly Notices of the Royal Astronomical Society, 390,* 1377.

[16] Jiang, I. G., and Yeh, L. C. 2003, *Int. J. Bifurcation Chaos*, **13**, 534.

[17] Jiang, I.G. and Yeh, L.C.: (2004). On the orbits of disk–star–planet systems. *Astronomical Journal, 128,* 923.

[18] Jiang, I.G. and Yeh, L.C.: (2006). On the Chermnykh-like problem: the equilibrium points. *Astrophysics and Space Science, 305,* 341.

[19] Jiang, I.G. and Yeh, L.C.: (2014). Galaxies with super massive binary black holes: (I) a possible model for the centers of core galaxies. *Astrophysics and Space Science, 349,* 881.

[20] Yeh, L.C., Jiang, I.G.: (2006). On the Chermnykh-Like Problems: II. The Equilibrium Points. *Astrophysics and Space Science, 306*, 189.

[21] Kishor, R. and Kushvah, B.S.: (2013) Linear stability and resonances in the generalized photogravitational Chermnykh-like problem with a disc. *Monthly Notices of the Royal Astronomical Society*, *436*, 1741.

[22] Singh, J., Leke, O., (2014a). Motion in a modified Chermnykh's restricted three-body problem with oblateness. *Astrophysics and Space Science*, *350*:143.

[23] Singh, J., Leke, O., (2014b). Analytic and numerical treatment of motion of dust grain particle around triangular equilibrium points with post-AGB binary star and disc. *Advances in Space Research 54*, 1659.

[24] Miyamoto, M., Nagai, R.: (1975). Three-dimensional models for the distribution of mass in galaxies *Astronomical Society of Japan Publications*, *27*, 533.

[25] Wolfram, S.: (2003). *The Mathematica Book.* 5th Edition. Wolfram Media, Champaign.