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PERIODIC ORBITS AROUND TRIANGULAR LIBRATION POINTS IN THE CR3BP WITH ZONAL HARMONICS AND POTENTIAL FROM A BELT

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Abstract

This paper studied Periodic orbits in an improved version of the circular restricted three-body problem where both primaries are considered as oblate spheroids; and surrounded by a belt of homogeneous circular planar cluster of material points centered at the mass center of the system. It examined the effects of the gravitational potential created by the belt and the oblateness up to zonal harmonic J_4 of the primaries on the periodic orbits around the stable triangular libration points. It has established that the periodic orbits in the vicinity of the triangular points for $0<\mu<\mu_c$, where $\mu_c\in(0,1/2)$ is the critical mass ratio influenced by the oblateness up to zonal harmonic J_4 of the primaries and potential from the belt, are ellipses. The elements of periodic motions: frequencies, eccentricities, and the directions of the principal axes of the ellipses are influenced by the resultant effects of the perturbations.

***Keywords*:** Restricted three-body problem, Triangular libration points, Zonal harmonic effect, Potential from a belt.

1. Introduction

The circular restricted three-body problem (CR3BP) presents the motion of a body possessing infinitesimal mass under the gravitational attraction of two massive bodies, which rotate around their barycenter on account of their mutual gravitational attraction only. It has attracted the interest of many researchers since it was at first considered by Euler in the 1772. The very obvious explanation for this persistent attention is that the model of the CR3BP can aid as an initial approximation in a number of real situations in astronomy. A deeper motivation originates probably from the reality that no general solution exists, despite the apparent easiness of the problem. Actually, it seems now most likely that such a general solution will not ever be found; for numerical studies show that the problem belongs to the general class of non-integrable dynamical systems. Regarding to its simplicity, the CR3BP can then serve as a good model problem for the study of non-integrable systems.

As acknowledged by [1], periodic orbits are the best opportunity for the understanding of the unresolved three-body problem. Also, periodic orbits shape the foundation around which orbits in general are organized. Accordingly, numerous studies have focused on periodic orbits, and this work is no exception.

The periodic orbits around the triangular points when both primaries are oblate spheroid as well as sources of radiation, under the effects of small perturbations in the Coriolis and centrifugal forces are examinedin [2].Presented in [3] are the periodic orbits generated by Lagrangian solutions of the restricted three–body problem when one of the primaries is an oblate spheroid. A systematic numerical exploration of the families of asymmetric periodic orbits of the restricted three-body problem when the primary bodies are equal and for the Earth-Moon mass ratio are outlined in[4].The existence of periodic orbits around the triangular points in the restricted three–body problem when the bigger primary is a triaxial, the smaller primary is considered as an oblate spheroid, working in the range of linear stability with the perturbed forces of Coriolis and centrifugal was studied in [5]. A Family of periodic orbits in the restricted four-body problem was established in [6]. Periodic orbits around the triangular points when the three participating bodies are oblate spheroids and the primaries are radiating was studied in [7].

Examined in [8] are orbits around the libration points when the more massive primary is radiating and the smaller is an oblate spheroid. Their study included the effects of oblateness up to 10^{-6} of the main term. Semi-analytic study of the periodic orbits

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around stable triangular libration points when the three participating bodies are modeled as oblate spheroids, under effect of, radiation of the main masses and small change in the Coriolis and centrifugal forces was performed in [9]. The motion around the triangular libration points, of a passively gravitating dust particle in the gravitational field of a low-mass post-AGB binary system, surrounded by circumbinary disc was investigated in [10]. In the framework of the elliptic restricted three-body problem, [11] studied the effects of oblateness, radiation and eccentricity of both primaries on the periodic orbits around the triangular libration points of oblate and luminous binary systems. The periodic structure of the restricted three–body problem considering the effect ofthe zonal harmonics J_2 and J_4 of the bigger primary was investigated in [12]. They showed that the triangular libration points in the restricted three–body problem havelong or short periodic orbits in the range $0<\mu<\mu_c$, where μ_c is the critical mass ratio and belongs to the open interval (0,1/2).

The effect of oblateness of both primaries up to zonal harmonic $J_4$and gravitational potential from a belt on the linear stability of the triangular libration points in the CR3B was looked at by [13]. It was found that, the triangular points are stable for $0<\mu<\mu_c$ and unstable for $\mu_c \le \mu \le \frac{1}{2}$, where μ_c is the critical mass ratio affected by the oblateness up to J_4 of the primaries and potential from the circular cluster of material points .

In our model, we study the periodic orbits around the triangular libration points within the stable region under the combined effects of oblateness up to zonal harmonic J_4 of the primaries and gravitational potential from a belt. The work is an extension of [13].

2. Equations of motion

We consider the motion of an infinitesimal body of mass m governed by the gravitational force from oblate primaries of masses m_1, m_2 $(m_1 > m_2)$ and a circumbinary belt of mass M_b centeredat the center of mass of the primaries. Let us use a coordinate system *oxyz* with origin at the centre of mass of the primaries and the *x*-axis is the line joining the primaries; while *y*-axis is perpendicular to it, the *z*-axis is perpendicular to the orbital plane of the primaries see Fig. 1. The distances of m from m_1 , m_2 are r_1 , r_2 respectively, and the distance between the primaries is *R*. We choose units for the mass and length such that the sum of the masses of the primaries and their separation distance are unity. The unit of time is chosen such that the gravitational constant is unity. Let $\mu = \frac{m_2}{m_1+m_2}$ be the mass parameter then, we have the masses $m_2 = \mu \; and \; m_1 = 1-\mu$; and let *oxyz* rotate about *z*-axis with constant angular velocity n, then the coordinates of m_1, m_2 and m are $(x_1,0,0)=(-\mu,0,0),(x_2,0,0)=(1-\mu,0,0) \; and \; (x,y,0)$ respectively.

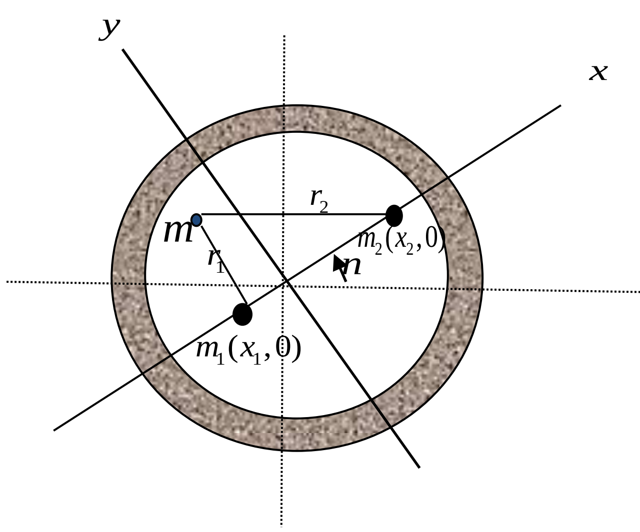

Fig.1: The planar configuration of the problem

Assuming that the primaries have their equatorial planes coinciding with the plane of motion, we denote the oblateness coefficients for the bigger primary as A*i,* $0<A_i = J_{2i}R_1^{2i} << 1$ and for the smaller primary as B*i,* $0<B_i = J_{2i}R_2^{2i} << 1$, $i=1,2$, where J_{2i} are zonal harmonic coefficients; R_1 , R_2 are the mean radii of m_1 and m_2 respectively. Then, in the dimensionless synodic coordinate system, the equations of motion of the infinitesimal bodyunder the influence of oblateness up to J_4 of the primaries and gravitational potential from the circumbinary belt are given as in [13].

$$\ddot{x}-2n\dot{y}=\Omega_x,\ \ddot{y}+2n\dot{x}=\Omega_y, \tag{1}$$

with

$$\Omega=\frac{n^2}{2}\left[(1-\mu)r_1^2+\mu r_2^2\right]+(1-\mu)\left(\frac{1}{r_1}+\frac{A_1}{2r_1^3}-\frac{3A_2}{8r_1^5}\right)+\mu\left(\frac{1}{r_2}+\frac{B_1}{2r_2^3}-\frac{3B_2}{8r_2^5}\right)+\frac{M_b}{(r^2+T^2)^{1/2}},$$

$$r_1^2=\left(x+\mu\right)^2+y^2, \qquad r_2^2=\left(x+\mu-1\right)^2+y^2,$$

$$n^2=1+\frac{3}{2}\left[A_1+B_1-\frac{5}{4}\left(A_2+B_2\right)\right]+\frac{2M_b r_c}{(r_c^2+T^2)^{3/2}}, \tag{2}$$

The over dot denotes differentiation with respect to time t, $\frac{M_b}{\left(r^2+T^2\right)^{1/2}}$ is the potential due to the circular cluster of material points [14, 15,16] where M_b is the total mass of the circular cluster of material points , r is the radial distance of the infinitesimal body and is given by $r^2=x^2+y^2$, $T=a+b$, a and b are parameters which determine the density profile of the circular cluster of material points . The parameter a controls the flatness of the profile and is known as the *flatness parameter*. The parameter b controls the size of the core of the density profile and is called the *core parameter*. When $a = b = 0$, the potential equals to the one by a point mass.

r_c is the radial distance of the infinitesimal body in the classical restricted three-body problem.

For simplicity, since $0.9\le r_c\le 1,\ \mu\in(0,1/2)$, we assume the variation in r_c to be negligible by regarding r_c as a constant; andwe set T=0.01 for all numerical investigations.

Equation (1) admits the energy integral of the form,

$$\dot{x}^2+\dot{y}^2=2\Omega\left(x,y\right)-C, \tag{3}$$

where C is the constant of integration, also known as Jacobi's constant.

3. Location of triangular libration points

The triangular points of the dynamical system denoted by $L_{4\,(5)}$ are the solutions of $\Omega_x=0,\ \Omega_y=0$ when $y\neq 0$; and are given by [13] as

$$x=\frac{1}{2}\left[1-2\mu+A_1-\frac{5A_2}{4}-B_1+\frac{5B_2}{4}\right],$$

$$y=\pm\frac{\sqrt{3}}{2}\left(1-\frac{A_1}{3}+\frac{5A_2}{12}-\frac{B_1}{3}+\frac{5B_2}{12}-\frac{4M_b(2r_c-1)}{9(r_c^2+T^2)^{3/2}}\right). \tag{4}$$

4Stability of the triangular libration points.

Denoting the positionof the triangular libration point as $L(x_0,y_0)$. Let a small displacement inthe coordinates (x_0,y_0) *)* be η,ξ . Then, following the same linear stability analysis used in [13], we have the variational equations of motion as

$$\ddot{\eta}-2n\dot{\xi}=\eta\Omega_{xx}\left(x_0,y_0\right)+\xi\Omega_{xy}\left(x_0,y_0\right),$$
$$\ddot{\xi}+2n\dot{\eta}=\eta\Omega_{yx}\left(x_0,y_0\right)+\xi\Omega_{yy}\left(x_0,y_0\right), \tag{5}$$

withits characteristic equation as

$$\lambda^4+(4n^2-\Omega_{xx}^0-\Omega_{yy}^0)\lambda^2+\Omega_{xx}^0\Omega_{yy}^0-\Omega_{xy}^{0\ 2}=0 \tag{6}$$

where

$$\Omega_{xx}^0=\frac{3}{4}+(\frac{27}{8}-3\mu)A_1-(\frac{165}{32}-\frac{75\mu}{16})A_2+(\frac{3}{8}+3\mu)B_1-(\frac{15}{32}+\frac{75\mu}{16})B_2+\frac{5M_b(2r_c-1)}{4(r_c^2+T^2)^{3/2}}+\frac{3M_b(\frac{1}{4}-\mu+\mu^2)}{(r_c^2+T^2)^{5/2}},$$

$$\Omega_{yy}^0=\frac{9}{4}+\frac{33A_1}{8}-(\frac{255}{32}-\frac{45\mu}{16})A_2+\frac{33B_1}{8}-(\frac{165}{32}+\frac{45\mu}{16})B_2+\frac{7M_b(2r_c-1)}{4(r_c^2+T^2)^{3/2}}+\frac{3M_b(\frac{3}{4})}{(r_c^2+T^2)^{5/2}},$$

$$\Omega_{xy}^0=\Omega_{yx}^0=\sqrt{3}\left\{\begin{array}{l}\frac{3}{4}-\frac{3\mu}{2}+(\frac{19}{8}-\frac{13\mu}{4})A_1+(5\mu-\frac{125}{32})A_2+(\frac{7}{8}-\frac{13\mu}{4})B_1\\ +(5\mu-\frac{35}{32})B_2+\left\langle\frac{11}{12}-\frac{11\mu}{6}\right\rangle\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{\frac{3}{2}M_b(\frac{1}{2}-\mu)}{(r_c^2+T^2)^{5/2}}\end{array}\right\}.$$

Simplifying Eq.6yields

$$\lambda^4 + Q\lambda^2 + N = 0, \qquad (7)$$

where

$$Q = 1 + 3(\mu - \frac{1}{2})A_1 + \frac{15}{2}(\frac{3}{4} - \mu)A_2 + 3(\frac{1}{2} - \mu)B_1 + \frac{15}{2}(\mu - \frac{1}{4})B_2 + \frac{M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}} - \frac{3M_b r_c^2}{(r_c^2+T^2)^{5/2}},$$

$$N = \left(\frac{27}{4} + \frac{117A_1}{4} - 45A_2 + \frac{117B_1}{4} - 45B_2 + \frac{33M_b(2r_c-1)}{2(r_c^2+T^2)^{3/2}} + \frac{27M_b}{4(r_c^2+T^2)^{5/2}}\right)\mu$$

$$-\left(\frac{27}{4} + \frac{117A_1}{4} - 45A_2 + \frac{117B_1}{4} - 45B_2 + \frac{33M_b(2r_c-1)}{2(r_c^2+T^2)^{3/2}} + \frac{27M_b}{4(r_c^2+T^2)^{5/2}}\right)\mu^2.$$

By Solving the characteristic equation Eq. (7)it was established in[13]that the triangularlibration points are stable when $0 < \mu < \mu_c$ and unstable for $\mu_c \le \mu \le \frac{1}{2}$, where μ_c is the critical mass parameter stated as

$$\mu_c = \frac{1}{2}\left(1 - \sqrt{\frac{23}{27}}\right) - \frac{1}{9}\left(1 + \frac{13}{\sqrt{69}}\right)A_1 + \frac{5}{18}\left(1 + \frac{25}{2\sqrt{69}}\right)A_2 + \frac{1}{9}\left(1 - \frac{13}{\sqrt{69}}\right)B_1$$

$$-\frac{5}{18}\left(1 - \frac{25}{2\sqrt{69}}\right)B_2 + \left[\frac{4(19-2r_c)\left(r_c^2+T^2\right)^{5/2} - 9(1+6r_c^2)\left(r_c^2+T^2\right)^{3/2}}{27\sqrt{69}\left(r_c^2+T^2\right)^4}\right]M_b.$$

5. Periodic Orbits around the Triangular Libration Points

The motions near the triangular libration points when the mass ratio $\mu \in (0, \mu_c)$ are bounded, while is unbounded in the interval $\mu_c \le \mu \le 1/2$, where $\mu_c \in (0, 1/2)$ is the critical mass ratio influenced by the oblateness up to the zonal harmonic J_4 and potential from the belt. We are to examine the nature of these orbits around the triangular libration points in the bounded region.

Now, in the stable region $0 < \mu < \mu_c$, the characteristic equation Eq.(7) has four distinct pure imaginary roots. The roots can be expressed as

$$\lambda_{1,2} = \pm i s_1,\ \lambda_{3,4} = \pm i s_2, \qquad (8)$$

with

$$s_1 = \sqrt{\frac{Q - \sqrt{\Delta}}{2}},\ s_2 = \sqrt{\frac{Q + \sqrt{\Delta}}{2}} \qquad (9)$$

and

$$\Delta = \left(27 + 117A_1 - 180A_2 + 117B_1 - 180B_2 + \frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}} + \frac{27M_b}{(r_c^2+T^2)^{5/2}}\right)\mu^2$$

$$-\left(27 + 111A_1 - 165A_2 + 123B_1 - 195B_2 + \frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}} + \frac{27M_b}{(r_c^2+T^2)^{5/2}}\right)\mu$$

$$+1 - 3A_1 + \frac{75A_2}{4} + 3B_1 + \frac{15B_2}{4} + \frac{2M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}} - \frac{6M_b r_c^2}{(r_c^2+T^2)^{5/2}}.$$

Hence, the motion around the triangular points is bounded and is composed of two harmonic motions specified by the variations η and ξ as:

$$\eta = D_1 \cos s_1 t + E_1 \sin s_1 t + D_2 \cos s_2 t + E_2 \sin s_2 t,$$

$$\xi = \bar{D}_1 \cos s_1 t + \bar{E}_1 \sin s_1 t + \bar{D}_2 \cos s_2 t + \bar{E}_2 \sin s_2 t, \qquad (10)$$

where

$$s_1 = \left[\mu(1-\mu)\left(\frac{27}{4} + \frac{117A_1}{4} - 45A_2 + \frac{117B_1}{4} - 45B_2 + \frac{33M_b(2r_c-1)}{2(r_c^2+T^2)^{3/2}} + \frac{27M_b}{4(r_c^2+T^2)^{5/2}}\right) - \frac{15A_2}{8} - \frac{15B_2}{8}\right]^{\frac{1}{2}} \quad (11)$$

and

$$s_2 = \Big[1 + \frac{27}{4}\mu(\mu-1) + (-6+117\mu^2-105\mu)\frac{A_1}{4} + (15+75\mu-90\mu^2)\frac{A_2}{2} + (6+117\mu^2-129\mu)\frac{B_1}{4} + (105\mu-90\mu^2)\frac{B_2}{2} + \left((2r_c+3) - \frac{33(2r_c-1)\mu}{2} + \frac{33(2r_c-1)\mu^2}{2}\right)\frac{M_b}{(r_c^2+T^2)^{3/2}} - \left(3r_c^2 + \frac{27\mu}{4} - \frac{27\mu^2}{4}\right)\frac{M_b}{(r_c^2+T^2)^{5/2}}\Big]^{\frac{1}{2}} \quad (12)$$

are the frequencies of the orbits, D_1, E_1, D_2, E_2 are arbitrary constants of integration and $\bar{D}_1, \bar{E}_1, \bar{D}_2, \bar{E}_2$ are related to the constants of integration. We obtain the relationship among the constants by substituting Eq. (10) in Eq. (5) and then equate coefficients of $\cos s_i, \sin s_i\ (i=1,2)$, we get

$$\bar{D}_i = \Gamma_i(2ns_i E_i - \Omega^0_{xy} D_i),$$
$$\bar{E}_i = -\Gamma_i(2ns_i D_i + \Omega^0_{xy} E_i), \quad (13)$$
$$where\ \ \Gamma_i = \frac{s_1^2 + \Omega^0_{xx}}{4n^2 s_1^2 + \Omega^{0\ 2}_{xy}} = \frac{1}{s_i^2 + \Omega^0_{yy}}\ \ (i=1,2).$$

It could be observed from Eq.11 and Eq.12 that $s_2 \geq s_1$, so s_1 is the frequency of long periodic orbit while s_2 is for short periodic orbit. The frequencies s_1 and s_2 are functions of the mass ratio and all the perturbations. Using Eq.11 and Eq.12,we analyze the effects of the perturbations on the frequencies which are presented in Table 1.

Table 1: Effects of the perturbations on the frequencies (μ=0.03 and T=0.01, r_c=0.99)

case	A_1	A_2	B_1	B_2	M_b	s_1	s_2
1	0	0	0	0	0	0.4431986010	0.8964234490
	0.01	0	0	0	0	0.4526994035	0.8837212513
	0.02	0	0	0	0	0.4620048701	0.8708337958
	0.03	0	0	0	0	0.4711265753	0.8577527324
2	0.01	0.005	0	0	0	0.4360340816	0.9076815245
	0.02	0.01	0	0	0	0.4288586460	0.9188016652
	0.03	0.015	0	0	0	0.4216724544	0.9297888201
	0	0	0.01	0	0	0.4526994035	0.8995350187
	0	0	0.02	0	0	0.4620048701	0.9026358623
3	0	0	0.03	0	0	0.4711265753	0.9057260899
	0	0	0.01	0.005	0	0.4360340816	0.9037896602
	0	0	0.02	0.01	0	0.4288586460	0.9110963176
	0	0	0.03	0.015	0	0.4216724544	0.9183448426
	0	0	0	0	0.01	0.4509309429	0.9039178042
4	0	0	0	0	0.02	0.4585329111	0.9113505328
	0	0	0	0	0.03	0.4660108860	0.9187231304
	0.01	0.005	0.01	0.005	0.01	0.43651609286	0.9223008711
5	0.02	0.01	0.02	0.01	0.02	0.42952187382	0.9474717904
	0.03	0.015	0.03	0.015	0.03	0.4222389193	0.9719910959

5.1 The nature of periodic orbits

The expansion of the potential Ω around the triangular libration points up to second order of η, ξ is expressed as

$$\Omega = \Omega^0 + \frac{\eta^2}{2}\Omega^0_{xx} + \eta\xi\Omega^0_{xy} + \frac{\xi^2}{2}\Omega^0_{yy}, \quad (14)$$

where

$$\Omega^0 = \frac{3}{2} + (\frac{5}{4} - \frac{\mu}{2})A_1 + (\frac{3\mu}{8} - \frac{21}{16})A_2 + (\frac{3}{4} + \frac{\mu}{2})B_1 - (\frac{3\mu}{8} + \frac{15}{16})B_2 + (\frac{r_c}{(r_c^2+T^2)^{3/2}} + \frac{1}{(r_c^2+T^2)^{1/2}})M_b,$$

$$\Omega^0_x = \Omega^0_y = 0.$$

$$\Omega_{xx}^0 = \frac{3}{4} + (\frac{27}{8} - 3\mu)A_1 - (\frac{165}{32} - \frac{75\mu}{16})A_2 + (\frac{3}{8} + 3\mu)B_1 - (\frac{15}{32} + \frac{75\mu}{16})B_2 + \frac{5M_b(2r_c - 1)}{4(r_c^2 + T^2)^{3/2}} + \frac{3M_b(\frac{1}{4} - \mu + \mu^2)}{(r_c^2 + T^2)^{5/2}},$$

$$\Omega_{yy}^0 = \frac{9}{4} + \frac{33A_1}{8} - (\frac{255}{32} - \frac{45\mu}{16})A_2 + \frac{33B_1}{8} - (\frac{165}{32} + \frac{45\mu}{16})B_2 + \frac{7M_b(2r_c - 1)}{4(r_c^2 + T^2)^{3/2}} + \frac{3M_b(\frac{3}{4})}{(r_c^2 + T^2)^{5/2}},$$

$$\Omega_{xy}^0 = \Omega_{yx}^0 = \sqrt{3}\left\{ \begin{array}{l} \frac{3}{4} - \frac{3\mu}{2} + (\frac{19}{8} - \frac{13\mu}{4})A_1 + (5\mu - \frac{125}{32})A_2 + (\frac{7}{8} - \frac{13\mu}{4})B_1 \\ +(5\mu - \frac{35}{32})B_2 + \left\langle \frac{11}{12} - \frac{11\mu}{6} \right\rangle \frac{M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{\frac{3}{2}M_b(\frac{1}{2} - \mu)}{(r_c^2 + T^2)^{5/2}} \end{array} \right\}.$$

These derivatives are given in view of Eq.6 the superscript 0 indicates that, the potential Ω and its derivatives have been evaluated at any of the libration points Eq. 4.

Now, the potential Ω satisfies the Jacobi integral $2\Omega = C$ of Eq. 3, that is

$$\eta^2\Omega_{xx}^0 + 2\eta\xi\Omega_{xy}^0 + \xi^2\Omega_{yy}^0 = C - 2\Omega^0. \qquad (15)$$

The discriminant of the quadratic part ofEq. 15 is

$$\begin{vmatrix} \Omega_{xx}^0 & \Omega_{xy}^0 \\ \Omega_{xy}^0 & \Omega_{yy}^0 \end{vmatrix} = \Omega_{xx}^0\Omega_{yy}^0 - \left(\Omega_{xy}^0\right)^2$$

$$= \left(\mu - \mu^2\right)\left(\frac{27}{4} + \frac{117A_1}{4} - 45A_2 + \frac{117B_1}{4} - 45B_2 + \frac{33M_b(2r_c - 1)}{2(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{4(r_c^2 + T^2)^{5/2}}\right)$$

$$> 0,$$

which indicates that the periodic orbits around the triangular libration pointsare elliptical. Thus, eccentricities, inclination and semi major axes of the orbits will be used to determine the shapes, orientation and sizes of the orbits.

5.2 Orientation of the elliptical orbit

Eq. 14 contains a bilinear term $\eta\xi$ which causes the rotation of the principal axes of the ellipse with respect to the coordinates system $0\eta\xi$ by an angle β Fig. 2. This advices the introduction of a normal reference frame of coordinates system $0\bar{\eta}\bar{\xi}$ such that the bilinear term is zero.

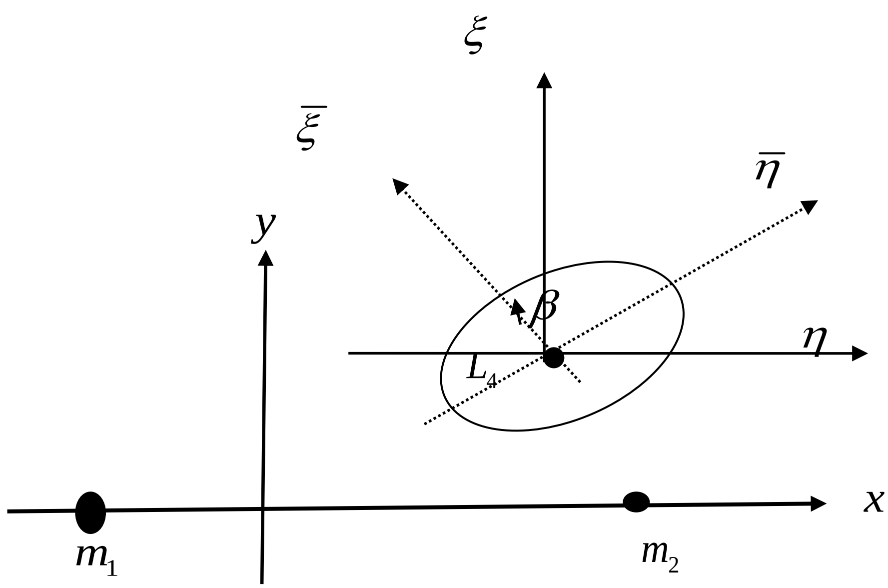

Fig.2: Orientation of the principal axes

L_5

The relations between the normal coordinates $(\bar{\eta}, \bar{\xi})$ and the previous coordinates system (η, ξ) are given by

$$\eta = \bar{\eta}\cos\beta - \bar{\xi}\sin\beta,$$
$$\xi = \bar{\eta}\sin\beta + \bar{\xi}\cos\beta. \qquad (16)$$

Using Eq.16 in Eq.14, we obtain the potential function under this transformation as

$$\bar{\Omega} = \frac{\Omega^0_{xx}}{2}\left(\bar{\eta}^2\cos^2\beta + \bar{\xi}^2\sin^2\beta - 2\bar{\eta}\bar{\xi}\cos\beta\sin\beta\right) + \frac{\Omega^0_{yy}}{2}\left(\bar{\eta}^2\sin^2\beta + \bar{\xi}^2\cos^2\beta + 2\bar{\eta}\bar{\xi}\cos\beta\sin\beta\right)$$
$$+\Omega^0_{xy}\left(\bar{\eta}^2\cos\beta\sin\beta - \bar{\xi}^2\cos\beta\sin\beta + \bar{\eta}\bar{\xi}(\cos^2\beta - \sin^2\beta)\right) + \Omega^0. \qquad (17)$$

Now, setting the bilinear terms in Eq. 17 to zero we have,

$$\tan 2\beta = \frac{2\Omega^0_{xy}}{\Omega^0_{xx} - \Omega^0_{yy}}. \qquad (18)$$

Substituting the values of the partial derivatives in Eq. 18,applying Binomial expansion and ignoring higher powers in small quantities, we get

$$\tan 2\beta = \pm\frac{4}{3}\sqrt{3}\left[\frac{3}{4} - \frac{3\mu}{2} + \left(2 - 4\mu + 3\mu^2\right)A_1 + \left(-\frac{5}{2} + \frac{25\mu}{8} - \frac{15\mu^2}{8}\right)A_2 + \left(-1 + 2\mu - 3\mu^2\right)B_1 \right. \qquad (19)$$
$$\left. + \left(\frac{5}{4} - \frac{5\mu}{8} + \frac{15\mu^2}{8}\right)B_2 + \left(\frac{2}{3} - \frac{4\mu}{3}\right)\frac{M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{3M_b(-\mu + 3\mu^2 - 2\mu^3)}{2(r_c^2 + T^2)^{5/2}}\right].$$

Thus, Eq.19 gives the orientation of the elliptical orbits. Note that the positive sign refers to the periodic motion around L_4 while negative one gives the motion around L_5. The effects of the perturbations on this angle of rotation when μ=0.03, T=0.01 and r_c=0.99 are outlined in Table 2.

Table 2: Effects of the perturbations on the angle of rotation of the principal axis

Case	A_1	A_2	B_1	B_2	M_b	$\tan 2\beta$	β
1	0	0	0	0	0	1.6281	29.2208
	0.01	0	0	0	0	1.6716	29.5555
	0.02	0	0	0	0	1.7151	29.8776
	0.03	0	0	0	0	1.7586	30.1877
2	0.01	0.005	0	0	0	1.6438	29.3430
	0.02	0.01	0	0	0	1.6595	29.4634
	0.03	0.015	0	0	0	1.6752	29.5822
	0	0	0.01	0	0	1.6064	29.0483
	0	0	0.02	0	0	1.5846	28.8724
3	0	0	0.03	0	0	1.5628	28.6930
	0	0	0.01	0.005	0	1.6206	29.1615
	0	0	0.02	0.01	0	1.6131	29.1018
	0	0	0.03	0.015	0	1.6055	29.0417
	0	0	0	0	0.01	1.6417	29.3271
4	0	0	0	0	0.02	1.6554	29.4320
	0	0	0	0	0.03	1.6690	29.5357
	0.01	0.005	0.01	0.005	0.01	1.6499	29.3899
5	0.02	0.01	0.02	0.01	0.02	1.6717	29.5558
	0.03	0.015	0.03	0.015	0.03	1.6934	29.7186

5.3 Equations of the ellipses

The quadratic part of Eq. 15admits the characteristic equation

$$\bar{\lambda}^2 - \left(\Omega^0_{xx} + \Omega^0_{yy}\right)\bar{\lambda} + \Omega^0_{xx}\Omega^0_{yy} - \left(\Omega^0_{xy}\right)^2 = 0. \qquad (20)$$

Utilizing the values of the partial derivatives of Eq. 14 in Eq. 20, yields

$$\bar{\lambda}^2 - \left(3 + \left(\frac{15}{2} - 3\mu\right)A_1 - \left(\frac{105}{8} - \frac{15\mu}{2}\right)A_2 + \left(\frac{9}{2} + 3\mu\right)B_1 - \left(\frac{45}{8} + \frac{15\mu}{2}\right)B_2 + \frac{3M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{3M_b r_c^2}{(r_c^2 + T^2)^{5/2}}\right)\bar{\lambda} \qquad (21)$$
$$+\left(\mu - \mu^2\right)\left(\frac{27}{4} + \frac{117A_1}{4} - 45A_2 + \frac{117B_1}{4} - 45B_2 + \frac{33M_b(2r_c - 1)}{2(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{4(r_c^2 + T^2)^{5/2}}\right) = 0.$$

The roots of Eq.21 are

$$\bar{\lambda}_1=\frac{9}{4}\left(\mu-\mu^2\right)+\left(\frac{39}{4}\mu-\frac{39}{4}\mu^2\right)A_1-\left(15\mu-15\mu^2\right)A_2+\left(\frac{39}{4}\mu-\frac{39}{4}\mu^2\right)B_1-\left(15\mu-15\mu^2\right)B_2 \qquad (22)$$
$$+\frac{11}{2}(\mu-\mu^2)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{9}{4}(\mu-\mu^2)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}.$$

and

$$\bar{\lambda}_2=3-\frac{9}{4}\left(\mu-\mu^2\right)+\left(\frac{15}{2}-\frac{51}{4}\mu+\frac{39}{4}\mu^2\right)A_1-\left(\frac{105}{8}-\frac{45}{2}\mu+15\mu^2\right)A_2+\left(\frac{9}{2}-\frac{27}{4}\mu+\frac{39}{4}\mu^2\right)B_1 \qquad (23)$$
$$-\left(\frac{45}{8}-\frac{15}{2}\mu+15\mu^2\right)B_2+\left(3-\frac{11}{2}(\mu-\mu^2)\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\left(3-\frac{9}{4}(\mu-\mu^2)\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}.$$

Now, the equations of motion of the infinitesimal mass can be expressed as

$$\ddot{\bar{\eta}}-2\dot{\bar{\xi}}=\bar{\lambda}_1\bar{\eta}, \qquad (24)$$
$$\ddot{\bar{\xi}}+2\dot{\bar{\eta}}=\bar{\lambda}_2\bar{\xi};$$

and the new potential function takes the form

$$\bar{\Omega}=\Omega^0+\frac{1}{2}\bar{\lambda}_1\bar{\eta}^2+\frac{1}{2}\bar{\lambda}_2\bar{\xi}^2.$$

So that the periodic solutions Eq. 10 can be written in the new coordinates as

$$\bar{\eta}=G_1\cos s_1t+H_1\sin s_1t+G_2\cos s_2t+H_2\sin s_2t, \qquad (25)$$
$$\bar{\xi}=\bar{G}_1\cos s_1t+\bar{H}_1\sin s_1t+\bar{G}_2\cos s_2t+\bar{H}_2\sin s_2t.$$

The relationships between the constants are given by

$$\bar{G}_i=\sigma_iH_i,$$
$$\bar{H}_i=-\sigma_iG_i, \qquad (26)$$
$$where \quad \sigma_i=\frac{(s_i^2+\bar{\lambda}_1)}{2s_i}=\frac{2s_i}{s_i^2+\bar{\lambda}_2} \quad (i=1,2).$$

From Eq. 25 we observe that by choosing suitable initial conditions, the short or the long periodic terms can be eliminated from the solution. Now, let us assume that the constants associated to the short periodic terms are zeros, so that Eq. 25 reduces to

$$\bar{\eta}=G_1\cos s_1t+H_1\sin s_1t,$$
$$\bar{\xi}=\bar{G}_1\cos s_1t+\bar{H}_1\sin s_1t. \qquad (27)$$

Let $(\bar{\eta}_0,\bar{\xi}_0)$ be the coordinates of the triangular libration point taken as the initial conditions at the initial time *t*, then Eq. 27 becomes

$$\bar{\eta}=\bar{\eta}_0\cos s_1t+\frac{\bar{\xi}_0}{\sigma_1}\sin s_1t, \qquad (28)$$
$$\bar{\xi}=\bar{\xi}_0\cos s_1t-\sigma_1\bar{\eta}_0\sin s_1t.$$

Eq.28 signifies a particular solution with only two arbitrary constants, which cannot be freely selected. Thus, eliminating sine and cosine from the equations of Eq.28 produces the equation of the elliptical orbit of the long periodic orbit as

$$\frac{\bar{\eta}^2}{a_1^2}+\frac{\bar{\xi}^2}{b_1^2}=1, \qquad (29)$$
$$where \quad a_1^2=\bar{\eta}_0^{\,2}+\frac{\bar{\xi}_0^{\,2}}{\sigma_1^2} \quad and \quad b_1^2=\sigma_1^2\bar{\eta}_0^{\,2}+\bar{\xi}_0^{\,2}.$$

Eq. 29 is the equation of the elliptical orbits of the long periodic orbit, a_1 its length of the semi-major axis and $b_1$the semi-minor axis. Therefore, its eccentricity $e_1$isgiven by

$$e_1^2=1-\frac{b_1^2}{a_1^2}=1-\sigma_1^2.$$

In view of Eq. 26,

$$e_1 = 1-\left(\frac{3}{2}\mu-6\mu^2\right)+\left(\mu+35\mu^2\right)A_1-\left(-\frac{5}{12}+\frac{25}{4}\mu+\frac{185}{4}\mu^2\right)A_2-\left(2\mu-44\mu^2\right)B_1$$
$$-\left(-\frac{5}{12}-\frac{5}{4}\mu+\frac{275}{4}\mu^2\right)B_2+\left(\frac{95}{6}\mu-\frac{31}{3}\mu^2\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}} \quad (30)$$
$$+\left(\frac{(21+12r_c^2)\mu}{4}+\frac{(51-21r_c^2)\mu^2}{4}\right)\frac{M_b}{(r_c^2+T^2)^{5/2}}.$$

Now, let the coordinates of the triangular libration points be taken as the initial conditions, then from Eq. 29, we obtained the lengths of the semi-major axis a_1 and the semi-minor axis b_1 of the long period orbits as

$$a_1=\frac{\sqrt{5}}{2}\left[1+\frac{1}{10\mu}-\frac{2}{5}\mu+\frac{2}{5}\mu^2+\left(\frac{34}{15}-\frac{2}{5}\mu\right)A_1+\left(-3+\frac{\mu}{2}-\frac{1}{3\mu}+\frac{1}{36\mu^2}\right)A_2+\left(\frac{37}{15}+\frac{1}{5\mu}+\frac{2}{5}\mu\right)B_1\right. \quad (31)$$
$$\left.+\left(-4-\frac{\mu}{2}+\frac{1}{6\mu}+\frac{1}{36\mu^2}\right)B_2+\left(\frac{29}{30\mu}-\frac{47}{45}\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\left(\frac{7+8r_c^2}{20\mu}+\frac{204-259r_c^2}{740}\right)\frac{M_b}{(r_c^2+T^2)^{5/2}}\right].$$

$$b_1=\frac{\sqrt{3}}{2}\left[1+\frac{\mu}{2}-4\mu^2-\left(\frac{1}{3}-\frac{2\mu}{3}+\frac{49\mu^2}{3}\right)A_1+\left(\frac{5}{18}+\frac{25\mu}{18}+\frac{1085\mu^2}{18}\right)A_2\right.$$

and

$$-\left(\frac{1}{3}+\frac{\mu}{3}+\frac{34\mu^2}{3}\right)B_1+\left(\frac{5}{18}+\frac{25\mu}{18}+\frac{595\mu^2}{36}\right)B_2-\left(\frac{4}{9}+\frac{95\mu}{27}-\frac{221\mu^2}{9}\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}} \quad (32)$$
$$\left.-\left(\frac{(21+r_c)\mu}{12}+\frac{(11-10r_c)\mu^2}{4}\right)\frac{M_b}{(r_c^2+T^2)^{5/2}}\right].$$

In the same manner, the constants associated with the long periodic terms can be set to zero in Eq. 25, to obtain equation of the elliptical orbit of the short periodic orbit as

$$\frac{\bar{\eta}^2}{a_2^2}+\frac{\bar{\xi}^2}{b_2^2}=1,$$

$$\text{where } a_2^2=\bar{\eta}_0^2+\frac{\bar{\xi}_0^2}{\sigma_2^2} \quad \text{and} \quad b_2^2=\sigma_2^2\bar{\eta}_0^2+\bar{\xi}_0^2.$$

So that the eccentricity e_2, the semi-major axis a_2 and the semi-minor axis b_2 of the short period orbits are:

$$e_2=\frac{\sqrt{3}}{2}\left\{1+\frac{3}{8}\mu+\frac{75}{16}\mu^2+\left(\frac{3}{4}-\frac{9}{8}\mu+\frac{169}{4}\mu^2\right)A_1-\left(\frac{55}{33}+\frac{1195}{128}\mu+\frac{935}{8}\mu^2\right)A_2\right.$$
$$+\left(\frac{1}{4}-\frac{19}{8}\mu+49\mu^2\right)B_1-\left(\frac{15}{32}+\frac{75}{128}\mu+\frac{9445}{128}\mu^2\right)B_2-\left(\frac{(2r_c+3)}{6}-\frac{(98r_c-73)\mu}{24}\right. \quad (33)$$
$$\left.\left.-\frac{(1285r_c-521)\mu^2}{24}\right)\frac{M_b}{(r_c^2+T^2)^{3/2}}+\left(\frac{r_c^2}{2}+\frac{33r_c^2\mu}{16}+\frac{(531-51r_c^2)\mu^2}{64}\right)\frac{M_b}{(r_c^2+T^2)^{5/2}}\right\},$$

$$a_2=\frac{\sqrt{13}}{2}\left\{1+\frac{23\mu}{26}+\frac{683\mu^2}{52}+\left(\frac{24}{13}-\frac{103\mu}{26}+\frac{2817\mu^2}{26}\right)A_1-\left(\frac{465}{104}+\frac{10315}{416}\mu-\frac{32535}{104}\mu^2\right)A_2\right. \quad (34)$$
$$+\left(\frac{4}{13}-\frac{185\mu}{26}+\frac{3303\mu^2}{26}\right)B_1-\left(\frac{85}{104}+\frac{395}{416}\mu+\frac{80505}{104}\mu^2\right)B_2+\left(\frac{4r_c+38}{39}-\frac{(174r_c+102)\mu}{13}\right.$$
$$\left.\left.+\frac{(2422r_c-12867)\mu^2}{52}\right)\frac{M_b}{(r_c^2+T^2)^{3/2}}+\left(\frac{18r_c^2}{13}+\frac{297r_c^2\mu}{52}+\frac{(4779-459r_c^2)\mu^2}{208}\right)\frac{M_b}{(r_c^2+T^2)^{5/2}}\right\}$$

and

$$b_2=\frac{\sqrt{13}}{4}\left\{1-\frac{25\mu}{104}-\frac{121\mu^2}{208}+\left(-\frac{21}{52}+\frac{5\mu}{8}-\frac{1383\mu^2}{104}\right)A_1+\left(\frac{285}{416}+\frac{125\mu}{128}+\frac{1155\mu^2}{52}\right)A_2\right.$$
$$+\left(-\frac{23}{52}+\frac{115\mu}{104}-\frac{1239\mu^2}{104}\right)B_1+\left(\frac{245}{416}-\frac{1175\mu}{1664}+\frac{1875\mu^2}{128}\right)B_2+\left(-\frac{73}{156}+\frac{85\mu}{208}-\frac{1421\mu^2}{208}\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}} \quad (35)$$
$$\left.+\left(-\frac{3r_c^2}{26}-\frac{3r_c^2\mu}{208}+\frac{(1353r_c^2-1593)\mu^2}{832}\right)\frac{M_b}{(r_c^2+T^2)^{5/2}}\right\}$$

respectively. Using Eq. 30 to Eq. 35 we investigate the effects of the various perturbations on the eccentricities and the semi axes of the long and short periodic orbits and are presented in Tables 3, 4 and 5 correspondingly.

Table 3: Effects of the perturbations on the eccentricities (μ=0.03,T=0.01, r_c=0.99)

case	A_1	A_2	B_1	B_2	M_b	e_1	e_2
1	0	0	0	0	0	0.9604000000	0.8794217342
2	0.01	0	0	0	0	0.9610150000	0.8859539473
	0.02	0	0	0	0	0.9616300000	0.8924861604
	0.03	0	0	0	0	0.9622450000	0.8990183735
	0.01	0.005	0	0	0	0.9619527083	0.8770688199
	0.02	0.01	0	0	0	0.9635054166	0.8747159055
	0.03	0.015	0	0	0	0.9650581250	0.8723629912
3	0	0	0.01	0	0	0.9601960000	0.8813516718
	0	0	0.02	0	0	0.9599920000	0.8832816094
	0	0	0.03	0	0	0.9597880000	0.8852115470
	0	0	0.01	0.005	0	0.9621574583	0.8789582448
	0	0	0.02	0.01	0	0.9639149166	0.8784947555
	0	0	0.03	0.015	0	0.9656723750	0.8780312661
4	0	0	0	0	0.01	0.9677578486	0.8776093784
	0	0	0	0	0.02	0.9751156973	0.8757970226
	0	0	0	0	0.03	0.9824735460	0.8739846668
5	0.01	0.005	0.01	0.005	0.01	0.9710680153	0.8747929747
	0.02	0.01	0.02	0.01	0.02	0.9817360306	0.8701642152
	0.03	0.015	0.03	0.015	0.03	0.9924040460	0.8655354557

Table 4: Effects of the perturbations on the semi axes of the long periodic

case	A_1	A_2	B_1	B_2	M_b	a_1	b_1
1	0	0	0	0	0	4.8318000356	0.8758980933
2	0.01	0	0	0	0	4.8570079752	0.8730572413
	0.02	0	0	0	0	4.8822159149	0.8702163893
	0.03	0	0	0	0	4.9074238546	0.8673755373
	0.01	0.005	0	0	0	4.9507444280	0.8746753857
	0.02	0.01	0	0	0	5.0696888203	0.8734526782
	0.03	0.015	0	0	0	5.1886332127	0.8722299706
3	0	0	0.01	0	0	5.2789860253	0.8728364049
	0	0	0.02	0	0	5.7261720150	0.8697747164
	0	0	0.03	0	0	6.1733580047	0.8667130279
	0	0	0.01	0.005	0	5.4601341020	0.8742840505
	0	0	0.02	0.01	0	6.0884681685	0.8726700077
	0	0	0.03	0.015	0	6.7168022350	0.8710559649
4	0	0	0	0	0.01	5.3793997766	0.8707785846
	0	0	0	0	0.02	5.9269995176	0.8656590759
	0	0	0	0	0.03	6.4745992586	0.8605395671
5	0.01	0.005	0.01	0.005	0.01	6.1266782354	0.8679418342
	0.02	0.01	0.02	0.01	0.02	7.4215564353	0.8599855751
	0.03	0.015	0.03	0.015	0.03	8.7164346351	0.8520293159

Table 5: Effects of the perturbations on the semi axes of the short periodic orbit.

case	A_1	A_2	B_1	B_2	M_b	a_2	b_2
1	0	0	0	0	0	1.8719294178	0.8944154974
2	0.01	0	0	0	0	1.9048268143	0.8908364070
	0.02	0	0	0	0	1.9377242108	0.8872573167
	0.03	0	0	0	0	1.9706216073	0.8836782263
	0.01	0.005	0	0	0	1.8603571063	0.8941462283
	0.02	0.01	0	0	0	1.8487847948	0.8938769593
	0.03	0.015	0	0	0	1.8372124833	0.8936076903
3	0	0	0.01	0	0	1.8756893837	0.8906309599
	0	0	0.02	0	0	1.8794493497	0.8868464225
	0	0	0.03	0	0	1.8832093156	0.8830618851
	0	0	0.01	0.005	0	1.8617857366	0.8934401792
	0	0	0.02	0.01	0	1.8516420555	0.8924648611
	0	0	0.03	0.015	0	1.8414983743	0.8914895429
4	0	0	0	0	0.01	1.9060494140	0.8891331907
	0	0	0	0	0.02	1.9401694102	0.8838508841
	0	0	0	0	0.03	1.9742894065	0.8785685775
5	0.01	0.005	0.01	0.005	0.01	1.8843334214	0.8878886036
	0.02	0.01	0.02	0.01	0.02	1.8967374249	0.8813617098
	0.03	0.015	0.03	0.015	0.03	1.9091414285	0.8748348160

6. Discussions

As indicated in Eq.10the motion in the neighborhood of the stable triangular libration points of the circular restricted three-body problem under the combined infleunce of oblateness up to the coefficients J_4 of the primaries and the gravitational potential from a belt, is confined and comprises of two harmonic motions. Eq.11 and Eq. 12identify the occurrences for the short and long periodic orbits, respectively. Section 5.1 has certified that these orbits are ellipses with their origin at the libration points while Eq. 19indicates the directions of the principal axes of the ellipses. The eccentricity, lengths of semi–major and semi–minor axes of the long periodic orbits are itemized in Eq.30, Eq.31 and Eq.32 respectively; whileEq.33,Eq. 34 andEq.35 provide the eccentricity, lengths of semi–major and semi–minor axes of the short periodic orbits correspondingly.

The elements of periodic motions (i.e. frequencies, eccentricity, and the directions of the principal axes of the ellipses) reduce to those of classical problem [17] when the oblateness of the primaries and the gravitational potential from the belt are neglected.

The numerical explorations Tables 1-5underline the effects of each disturbing force on the periodic elements: frequency, angle of rotation of the principal axis, eccentricity and lengths of the semi-axes of the elliptic orbits of the infinitesimal mass. Examining Table 1exposes that the frequency of the long periodic orbit s_1 decreases in the presence of oblateness up to the coefficient J_4 of the primaries (Cases 2 and 3) and due to the combined effects of oblatenes up J_4 of the primaries with the potential from belt (Case 5).

Studying Table 2 indicates that oblateness up to J_2 and Up to J_4 of the smaller primary (Case 3) diminishes the angle of the rotation of the principal axes of the ellipses, whereas oblateness up to J_4 of the bigger primary, the potential from the belt (Case 4) and the combined effects of all the perturbations (Case 5), amplify the angle of rotation of the principal axes.

Table 3portrays that the eccentricity $e_1$of the long periodic orbit lessens in the presence of oblateness of the smaller primary up to J_2 (Case3), while it raises owing to oblateness up to J_4 of the primaries (cases2 and 3) or potential from the belt (Case 4) or due to combined effects of the perturbations (Case 5). The Table advances that the eccentricity e_2 of the short periodic orbit boosts due to the oblateness up to J_2 of the primaries (Cases 2 or 3), whereas its value declinesas a results of oblateness up to J_4 of the primaries (cases2 or 3) or the potential from the belt (Case 4) or the combined effects of the perturbations (Case 5).

Table 4 shows that the length of the semi-major axis a_1 of the long periodic orbit increases in the presence of any or all of the perturbations whereas the length of the semi-minor axis $b_1$of the long periodic orbit decreases in the presence of any or all of the perturbations.

Table 5 indicates that the length of the semi-major axis a_2 of the short periodic orbit, increases only on account ofthe oblateness up to J_2 of the bigger primary (Case2) or the oblateness up to J_2 of the smaller primary (Case3) or the potential from belt (Case 4) or the combined effects of all the perturbations (Case5Furthermore, the table discloses that the length of the semi-minor axis b_2 of the short periodic orbit diminishes in the presence of any one or all of the perturbations.

7. Conclusion

The periodic orbits in the vicinity of the triangular points for $0 < \mu < \mu_c$, where $\mu_c \in (0, 1/2)$ is the critical mass ratio influenced by the oblateness and radiation of the primaries and potential from the belt are ellipses. The values of the angular frequency of the long elliptic orbit s_1, the eccentricity of the short elliptic orbit e_2, the semi-minor axes of the elliptic orbits b_1, $b_2$decrease, whereasthe frequency of the short periodic orbits s_2, the eccentricity of the long periodic orbit e_1, the semi-major axes of the orbits a_1, a_2 and the angle of rotation of the principal axes β increase owing to the combined effects of the oblateness up to the coefficients J_4 of the primaries and the gravitational potential from the belt.

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