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CALCULATION OF WAVE FUNCTION AT THE ORIGIN (WFO) FOR THE GROUND STATE OF HEAVY MESONS BASED ON THE VARIATIONAL METHOD

[1]Ogundola S. and [2]Temenu F.M.

[1]Department of Physics, University of Benin, Edo State .
[2]Department of Applied Geophysics, Federal University of Technology, Akure, Ondo State.

Abstract

The wave function at the origin is an important quantity in studying many physical problems concerning heavy quarkonia. This is because it is used for calculating spin state hyperfine splitting and it is also crucial for evaluating the production and decay amplitude of the heavy quarkonium. In this paper, we present the variational method by using the single-parameter wave function to estimate the WFO for the ground state of heavy mesons. For Cornell potential model trial wave function of the type ψ(r) =N $e^{-ar^{\frac{4}{2}}}$ ***gives the relative deviation of squared WFO of about 0.11 while for Martin and Logarithmic potential models having*** $b = \frac{3}{2}$ ***, the trial wave function ψ(r) =N*** $e^{-ar^{\frac{3}{2}}}$ ***gives relative deviation values as 0.027 and 0.013 respectively. In all, Martin and Logarithmic potentials, the trial wave function ψ(r) =N*** $e^{-ar^{\frac{3}{2}}}$ ***gives the most accurate value of the single state trial wave function at the origin (WFO).***

Keywords: Wave function at the origin, heavy Mesons, Ground states, Variational method, Non-relativistic quark model, Potential model, trial wave function.

I. INTRODUCTION

Recently, the wave function at the origin for the S-wave bound state of a heavy quark-antiquark system once again attracts physicists' attentions [1,2]. In the context of the non relativistic potential model, [1,2] demonstrated the numerical results of WFO of the Single-wave ,c̲c̲ b̲c̲and b̲b̲ systems. As well known, except the Coulomb and the harmonic oscillator potentials, there are few potentials which bound state problems can be analytically solved. For solving these non-analytically soluble bound state problems, one has to use approximations. Numerically solving Schrodinger equation is the most powerful method which can reach most required accuracy. But the numerical method has some defects, for instance, it cannot give analytical expressions for further discussion [3].In other word, there exist systems whose Hamiltonians are known, but they cannot be solved exactly or by a perturbative treatment. That is no closely related Hamiltonian that can be solved exactly or approximately by perturbation theory because the first order is not sufficiently accurate. One of the approximation method suitable for solving such problems is the variational method, which is also called the Rayleigh-Ritz method [8]. This method does not require knowledge of simple Hamiltonians that can be solved exactly. The variational method has more advantages. It can give an analytical expression of the wave function. In particular, if there is only a single-parameter in the trial wave function, the resultant wave function has a simple form. Then it is very convenient in the practical application and physical discussion.

II. THE POTENTIAL MODELS

There are many potential models which can fit the experimental spectra of the heavy quarkonia with certain accuracy. We consider three functional forms for the potential that give reasonable accounts of the c̲c̲ b̲c̲and b̲b̲spectra [2].
(1). Cornell potential [4]

$$V(r) = -\frac{4}{3}\frac{\propto_s}{r} + Kr \qquad (1)$$

Correspondence Author: Ogundola S., Email: calculusphysics2016@yahoo.com, Tel: +2348032821353

With $\alpha_s = {}^{0}/_{39}$, K = $\frac{1}{({}^{2}/_{3})^2}(GeV)^2$, $m_c = \frac{1}{84}(\frac{GeV}{C^2})$, $m_b = {}^{5}/_{18}({}^{GeV}/_{C^2})$

(2). Martin potential [5]:

$V(r) = kr^{0.1}$ (2)

With k = 6/898 ; m_c =1/8(GeV/C^2) ; m_b =5/174 (GeV/ C^2)

(3). Logarithmic potential [6]:

$V(r) = k\log(r)$ (3)

With k =0/733 ; m_c =1/5 (GeV/C^2) ; m_b = 4/906 (GeV/C^2)

III. THEVARIATIONALMETHODBYUSINGTHESINGLE-PARAMETERTRIALWAVEFUNCTION

Within the framework of the non-relativistic potential model, the S-state wave function $\psi(r)$ of the heavy quarkonium satisfies the Schrodinger equation

$H\psi(r) = -\frac{1}{2\mu}\nabla^{2}\psi(r) + V(r)\psi(r) = E\psi(r)$ (4)

Where H is the Hamiltonian of the quarkonium, V(r)denotes the central potential between quark and antiquark, E represents the energy of eigenvalue, and μ is the reduced mass.

To solve Eq. (4) by using the variational method, one needs to choose a suitable trial wave function$\psi(r,c)$with N independent parameters$\{c\} = \{c_1, c_2, \square, c_N\}$first and then to seek out a set of parameters$\{c_o\} = \{c_{io}, i = 1,2, \square, N\}$ which minimizes the expectation value of Hamiltonian, namely

$E(c) = \langle H \rangle = \frac{\langle \psi(c)|H|\psi(c)\rangle}{\langle \psi(c)|\psi(c)\rangle}$ (5)

The minimum value $E(c_o)$ gives an upper limit of the ground state energy.

In this section, we choose the simplest trial wave function in which there is only one variational parameter to study the single state of heavy mesons. The general form of such trial wave function is written as

$\psi_{trial}(r) = Ne^{-ar^b}$ (6)

Where N is the normalization constant, *a* denotes the variational parameter which will be fixed by minimizing the expectation value of Hamiltonian and *b*is the model parameter which determines the type of the trial wave function. In practice, we select following four trial wave functions:

(1).b=1, namely $\psi_{trial}(r) = Ne^{-ar}$ (hydrogen wave function or exponential wave function). It is the solution of theSchrodinger equation of Coulomb potential model.

(2) b =2, namely $\psi_{trial}(r) = Ne^{-ar^2}$ (harmonic oscillator wave function or a Gaussian wave function).

(3) b= 3/2 ,namely $\psi_{trial}(r) = Ne^{-ar^{\frac{3}{2}}}$. This function was used by Gupta [7].

(4) b = 4/3 , namely $\psi_{trial}(r) = Ne^{-ar^{\frac{4}{3}}}$.This is a newlyproposed trial wave function used in this work

The normalization constant is obtained from the normalization condition :

$\int |\psi(r)|^2 dr^3 = 1$ (7)

$4\pi N^2 \int_0^\infty e^{-2ar^b} r^2 dr = 1 \rightarrow N = [\frac{b(2a)^{1/b}}{4\pi\,\Gamma(\frac{3}{b})}]^{1/2}$ (8)

And , $\psi(0) = N$ (9)

$|\psi(0)|^2 = N^2 \Rightarrow |\psi(0)|^2 = \frac{b(2a)^{3/b}}{4\pi\,\Gamma(\frac{3}{b})}$ (10)

In the case of Cornell potential (1), potential energy $\langle V(r)\rangle$ is

$\langle V(r)\rangle = \int \psi^*(r)V(r)\psi(r)d^3r = 4\pi N^2 \int_o^\infty r dr e^{-2ar^b} + 4\pi N^2 k \int_o^\infty e^{-2ar^b} r^3 dr$

$= 4\pi \frac{b(2a)^{\frac{3}{b}}}{4\pi\,\Gamma(\frac{3}{b})}\left(-\frac{4}{3}\right)\alpha_s \frac{\Gamma(\frac{3}{b})}{b(2a)^{2/b}} + 4\pi \frac{b(2a)^{\frac{3}{b}}}{4\pi\,\Gamma(\frac{3}{b})} k \frac{\Gamma(\frac{4}{b})}{b(2a)^{4/b}}$ (11)

And the kinetic energy (T) is

$T = \frac{p^2}{2\mu} = -\frac{1}{2\mu}(\frac{2}{r}\frac{d}{dr} + \frac{d^2}{dr^2})$ (12)

$\langle T\rangle = 4\pi \int_0^\infty r^2 dr \psi^*(r)\, T\, \psi(r)$

$= -\frac{1}{\mu} 4\pi(-N^2 ab)\int_0^\infty r^b e^{-2ar^b} dr - \frac{1}{2a} 4\pi(-N^2 ab)(b-1)\int r^b e^{-2ar^b} dr$

$$-\frac{1}{2\mu}4\pi(N^2a^2b^2)\int_0^\infty r^{2b}e^{-2ar^b}dr$$

$$=\frac{2b(b+1)(2a)^{\frac{2}{b}}\Gamma\left(\frac{1}{b}+1\right)-b^2(2a)^{\frac{2}{b}}\Gamma(2+\frac{1}{b})}{8\mu\,\Gamma(\frac{3}{b})} \quad (13)$$

We know that $\Gamma(1+z) = z\,\Gamma(z)$, so we can write

$$\langle T\rangle = \frac{(2a)^{\frac{2}{b}}b^2\,\Gamma(2+\frac{1}{b})}{8\mu\,\Gamma(\frac{3}{b})} \quad (14)$$

Then, we can obtain the expectation value of Hamiltonian and consequently an algebraic equation, which is used to determine:

$\langle H\rangle = \langle T\rangle + \langle V\rangle$

$$=\frac{x^2b^2\,\Gamma(2+\frac{1}{b})}{8\mu\,\Gamma(\frac{3}{b})} + \frac{-4\propto_s\Gamma\left(\frac{2}{b}\right)x^2+3k\,\Gamma(\frac{4}{b})}{3\,\Gamma(\frac{3}{b})x} \quad (15)$$

Where $x = (2a)^{\frac{1}{b}}$ (16)

$$\frac{d}{dx}\langle H\rangle = 0 \quad (17)$$

$$\frac{3x^3b^2\,\Gamma\left(2+\frac{1}{b}\right)-16\mu\propto_s\Gamma\left(\frac{2}{b}\right)x^2-12\mu k\,\Gamma(\frac{4}{b})}{12\mu k\,\Gamma(\frac{3}{b})x^2} = 0 \quad (18)$$

It is very easy to solve this equation if we rewrite it in the following form;

$$A_3x^3 + A_2x^2 + A_0 = 0 \quad (19)$$

The real solution of x can be expressed as

$$X_{real} = \frac{-A_2}{3A_3}+\frac{2^{\frac{1}{3}}A_2^2}{3A_3B}+\frac{B}{3A_3 2^{\frac{1}{3}}} \quad (20)$$

Where $B = (B_0 + \sqrt{(-4A_2^6 - B_0^2)})^{\frac{1}{3}}$ and $B_0 = -2A_2^3 - 27A_0A_3^2$

In the case of Martin potential (2) , the potential energy reads :

$$\langle V(r)\rangle = 4\pi\int_0^\infty r^2 dr\psi^*(r)Vr(\psi)r = (4)\pi N^2K\int_0^\infty e^{-2ar^b}r^{2.1}dr$$

$$=\frac{4\pi b(2a)^{\frac{3}{b}}}{4\pi\,\Gamma(\frac{3}{b})}K\,\frac{\Gamma(\frac{3.1}{b})}{b(2a)^{\frac{3.1}{b}}} = \frac{K\,\Gamma(\frac{3.1}{b})}{(2a)^{\frac{3.1}{b}}\,\Gamma(\frac{3}{b})} \quad (21)$$

$\langle H\rangle = \langle T\rangle + \langle V\rangle$

$$=\frac{(2a)^{\frac{2}{b}}b^2\,\Gamma(2+\frac{1}{b})}{8\mu\,\Gamma(\frac{3}{b})} + \frac{K\,\Gamma(\frac{3.1}{b})}{(2a)^{\frac{0.1}{b}}\,\Gamma(\frac{3}{b})} \quad (22)$$

Therefore , the equation for determining "a" is quite simple . The solution is

$$a = \frac{1}{2}\left[\frac{(0.4)\mu K\,\Gamma\left(\frac{3.1}{b}\right)}{b^2\,\Gamma\left(2+\frac{1}{b}\right)}\right]^{\frac{b}{2.1}} \quad (23)$$

In the case of Logarithmic potential (3) , by the similar procedure , we obtain

$$a = \frac{1}{2}\left[\frac{4\mu K\,\Gamma\left(\frac{3}{b}\right)}{b^2\,\Gamma\left(2+\frac{1}{b}\right)}\right]^{\frac{b}{2}} \quad (24)$$

where a is the variational parameter which will be fixed by minimizing the expectation value of Hamiltonian and *b*is the model parameter which determines the type of the trial wave function.

IV. THE VARIATIONAL RESULTS OF WFO OF SINGLE STATE ,c$\underline{c}$ b$\underline{c}$ and b$\underline{b}$ MESON

In the case of Cornell Potential, the values of reduced mass for c$\underline{c}$ b$\underline{c}$ and b$\underline{b}$ are

$$\mu = \frac{m_c}{2} = 0.99\left(\frac{GeV}{c^2}\right); \text{ for } c\underline{c} \quad (25)$$

$$\mu = \frac{m_bm_c}{m_b+m_c} = 1.35\left(\frac{GeV}{c^2}\right); \text{ for } b\underline{c} \quad (26)$$

$$\mu = \frac{m_b}{2} = 2.59\left(\frac{GeV}{c^2}\right); \text{ for } b\underline{b} \quad (27)$$

In the case of Martin Potential , they are

$$\mu = \frac{m_c}{2} = 0.9\left(\frac{GeV}{c^2}\right); \text{ for } c\underline{c} \quad (28)$$

$$\mu = \frac{m_bm_c}{m_b+m_c} = 1.33\left(\frac{GeV}{c^2}\right); \text{ for } b\underline{c} \quad (29)$$

$$\mu = \frac{m_b}{2} = 2.58\left(\frac{GeV}{c^2}\right); \text{ for } b\underline{b} \quad (30)$$

Also, in the case of Logarithmic Potential , we have

$$\mu = \frac{m_c}{2} = 0.75\left(\frac{GeV}{c^2}\right); \text{ for } c\underline{c} \quad (31)$$

$$\mu = \frac{m_bm_c}{m_b+m_c} = 1.14\left(\frac{GeV}{c^2}\right); \text{ for } b\underline{c} \quad (32)$$

$$\mu = \frac{m_b}{2} = 2.45\left(\frac{GeV}{c^2}\right); \text{ for } b\underline{b} \quad (33)$$

where c$\underline{c}$, b$\underline{c}$ and b$\underline{b}$ are the parameters that represent the single-wave nature of the wave function systems for the reduced mass of each potential model.
All the numerical results are listed in Tables I to III.

TABLE I : THE VARIATIONAL RESULTS, $|\psi(0)|^2$(GeV)^3 , WITH A SINGLEPARAMETER TRIAL WAVE FUNCTION FOR c$\underline{c}$ SINGLE-STATE MESON

b	Cornell Potential model	Martin Potential model	Logarithmic Potential model	Results obtained using Schrodinger Equation
1	0.167246	0.166931	0.179747	0.167965
2	0.324113	0.311503	0.339847	0.324598
3/2	0.554912	0.599773	0.552490	0.554791
4/3	0.756326	0.775113	0.776168	0.756556

TABLE II: THE VARIATIONAL RESULTS, $|\psi(0)|^2$(GeV)^3 WITH A SINGLEPARAMETER TRIAL WAVE FUNCTION FOR b$\underline{c}$ SINGLE-STATE MESON

b	Cornell Potential	Martin Potential	Logarithmic Potential	Results obtained using Schrodinger Equation
1	0.264261	0.293480	0.246078	0.267965
2	0.326576	0.389927	0.355745	0.324598
3/2	0.541307	0.540607	0.518519	0.547922
4/3	0.651642	0.614323	0.614446	0.656981

TABLE III: THE VARIATIONAL RESULTS, ,$|\psi(0)|^2$(GeV)^3 WITH A SINGLEPARAMETER TRIAL WAVE FUNCTION FOR b$\underline{b}$ SINGLE-STATE MESON

b	Cornell Potential	Martin Potential	Logarithmic Potential	Results obtained using Schrodinger Equation
1	0.237961	0.238605	0.247454	0.237983
2	0.325192	0.331169	0.308874	0.324598
3/2	0.595784	0.581445	0.569630	0.547952
4/3	0.759597	0.519027	0.550535	0.756981

V. CONCLUSION

In this paper, variational method for determining the wave function at the origin of quarkonium was analytically studied. Retaining generality as much as possible, some potential models were employed to analyze data. Numerical results obtained through variational method of the models in this study were in good agreement with those obtained in literature using Schrodinger Equation [2]. The results shown in Tables I to III indicate that for the Cornell Potential, the trial wave function ψ_{trial} (r) $= Ne^{-ar^{\frac{4}{3}}}$ with b= 4/3 can give the least relative deviation of squared WFO . The value of deviation is about 0.02 . For the Martin and Logarithmic potentials the situations are better; when b=3/2 , one obtains the least values of 0.04 and 0.03 for the relative deviations of squared WFO respectively. The accuracy of variational results can be improved when the number of variational parameters are increased. The resultant accuracy of WFO seriously depends on the choice of the trial wave function that gives higher relative deviation. The trial wave function with a single variational parameter is most convenient for use. If the accuracy of 2% for WFO in the Cornell potential case is tolerable,then the trial wave function ψ_{trial} (r) $= Ne^{-ar^{\frac{4}{3}}}$ is the best choice for single-state system as it gives least relative deviation of 0.02 (2%) (i.e the wave function value obtained is very close to that obtained using Schrodinger Equation). For Martin and Logarithmic potentials,

ψ_{trial} (r) $= Ne^{-ar^{\frac{3}{2}}}$ is the most appropriatetrial wave function for the single-state c$\underline{c}$, b$\underline{c}$ and b$\underline{b}$, and the corresponding WFOs have quite satisfactory accuracies , though not as accurate as that of Cornell .

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