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OPTIMAL INVESTMENT PROBLEM IN A FINANCIAL INSTITUTION UNDER STOCHASTIC INTEREST RATE AND STOCHASTIC VOLATILITY

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Abstract

This paper study the problem of how a financial institution can optimally allocate its wealth in three assets namely; treasury, security and loan. Derived the Hamilton – Jacobi – Bellman equation associated with the optimization problem through the application of dynamic programming principle and solve the resulted partial differential equation equivalent of the Hamilton – Jacobi – Bellman equation explicitly in the case of constant relative risk aversion (CRRA) utility function. We also presented numerical examples to illustrate the dynamics of the optimal investment policy (strategy).

Keywords: Financial institution, investment strategy, stochastic optimization theory, stochastic interest rate, stochastic volatility, Affine model

1. Introduction

The problem of finding an optimal portfolio in a continuous time market setting has been a subject of research in recent time. Portfolio can disperse risk and increase revenues. In recent years, some scholars are concerned with the optimal investment problems in financial institutions under stochastic framework. Current literature has been largely focusing on financial institutions optimal asset allocation in stochastic interest framework. However, in a more realistic world, one should account for both stochastic interest rates and stochastic volatility [1]. Also, in attempt to manage their assets, financial institutions try to lower risk by diversifying their investment portfolio through investment in different types of assets [2].

For instance, [3] studied an optimal assets allocation problem with stochastic interest rates which takes into account specific features of bank. Their goal is to present a numerical aspect of the derived Hamilton – Jacobi – Bellman (HJB) equation and to focus on the optimal assets allocation model results from a practical viewpoint. Similarly, [4] also considered assets allocation problem. In their work, they illustrated that it is possible to use an analytic approach to optimize assets allocation strategies for banks. They formulated an optimal bank valuation problem through optimal choices of loan rate and demand which leads to maximal deposits, provisions for deposits withdrawals and bank profitability subject to cash flow, loan demand, financing and balance sheet constraints.

Several studies have also investigated the assets allocation problems using stochastic control theory developed by [5] and [6]in discrete and continuous time setting [7-9]. The approach solved nonlinear partial differential HJB equation to find the closed form solution for the value function.

Also in a work by [2] determined an optimal rate at which additional debt and equity should be raised and strategy for the allocation of bank equity. They employed dynamic programming algorithm for stochastic optimization to verify their results. In another work by [10], they obtained an analytical solution for the associated HJB in a case where the utility functions are either of power, exponential or logarithmic type. Here, the control variates are the depository consumption, value of the depository financial institutions invested in loans, and provisions for loan losses.

Furthermore, Martingale approach in analyzing the bank behavior has been used in recent papers. The research work by [11] considered a theoritical quantitative approach for bank liquidity provisioning, solved a nonlinear stochastic optimal

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liquidity risk management problem for subprime originators with deposit inflow rates and marketable securities allocation as controls using the Martingale approach. Here, they provided an explicit expression for the aggregate liquidity risk when a locally risk minimizing strategy is utilized. The Martingale method also frequently appears in areas such as optimal design and assets allocation of a pension fund or life insurance policy [12-13].The partial differential equation derived via martingale method is much simpler to solve than the highly non linear HJB equation associated with the dynamic programming method.

Also, the work of [15] investigated an optimal investment strategy for banks funds in treasuries and securities in a risk and regret theoretical framework. Evidence of portfolio shifting are found in [16] and [17], where they suggested that banks may change their balance sheets in ways that can cause procyclicality. The research paper by [4] also modeled non – risk – based and risk – based capital adequacy. Specifically, they constructed a continuous time stochastic models for the dynamics of the leverage, equity and Tier 1 ratios and derived the CAR. They also show the relevant of their result to the banking sector by studying an optimal control problem in which an optimal assets allocation strategy is derived for the leverage ratio on a given time interval. Precisely, they determined the optimal expected terminal utility of the leverage ratio and derived the optimal assets allocation strategy that make it possible to maximize the expected terminal utility of the leverage ratio on a given time interval.

Therefore, many mathematical models have been formulated over the past years to explore the dynamics of asset allocation problem in financial institutions under stochastic interest rate setting. In our contribution, we explore dynamics of a financial institution asset allocation problem in a stochastic interest rate and stochastic volatility framework. Our goal is to maximize an expected utility of the assets at a future time.

2. The mathematical models formulation

We consider a financial institution that dynamically allocates its wealth among three assets namely: treasury, loan and security. The assets prices satisfy the geometric Brownian motion, assets can be bought and sold without incurring any transaction costs or restriction on short sales and the interest rate is described by Affine model. The risk preference of the investor satisfies CRRA utility function.

2.1 The Financial Market

We consider a complete and frictionless financial market which is continuously open over a fixed time interval $[0, T]$ and Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is the filtration generated by the Brownian motions, $\mathbb{P}$ is the real world probability.The first asset in the financial market is a riskless treasury and its price at time t can be denoted as $S_0(t)$. It evolves according to the following stochastic differential equation

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt, \qquad S(0) = S_0 \tag{1}$$

The dynamics of the short rate process, $r(t)$, is given by the stochastic differential equation (Affine model)

$$dr(t) = \big(a - br(t)\big)dt - \sigma_r\sqrt{r(t)}dw_r(t), \quad r(t) = r_0 \tag{2}$$

Where a, b and$\sigma_r = \sqrt{k_1}$ are constants.

The second asset is a loan to be amortized over a period $[0, T]$ whose price at time $t \geq 0$ is denoted by $L(t)$. Its dynamics can be described by the stochastic differential equation:

$$\frac{dL(t)}{L(t)} = \big(r(t) + b_1\lambda_r k_1 r(t)\big)dt + b_1\sigma_r\sqrt{r(t)}dw_r(t) \tag{3}$$

whereb_1, λ_randk_1 are constants. The loan return has a risk premium $b_1\lambda_r r(t)$ that changes with t both implicitly through the dependence on $r(t)$ and explicitly through the dependence on b_1.

The third asset in the financial market is a risky security whose price is denoted by $S(t), t \geq 0$. Its dynamics can be described by the equation:

$$\frac{dS(t)}{S(t)} = \big(r(t) + \upsilon\eta(t) + \sigma_s\lambda_r k_1 r(t)\big)dt + \sigma_s\sigma_r\sqrt{r(t)}dw_r(t) + \sqrt{\eta(t)}dw_s(t) \tag{4}$$

The volatility $\eta(t)$ is assumed to satisfy the Heston model:

$$d\eta(t) = \alpha\big(\delta - \eta(t)\big)dt + \sigma_\eta\sqrt{\eta(t)}dw_r(t) \tag{5}$$

whereα, δ and $\sigma_\eta$are positive constant and satisfied the condition $2\alpha\delta > \sigma_\eta^2$ and it ensures $\eta(t) > 0$ $\forall\ t \in [0, T]$.

Here we assume that there is no correlation between $w_s(t)$and $w_r(t)$, and between $w_\eta(t)$ and $w_r(t)$. The correlation between $w_s(t)$and $w_\eta(t)$is ρ.

2.2 The Portfolio of the Financial Institution

Let $X(t)$ denotes the value of the financial institution assets portfolio at time $t \in [0, T], \pi_s(t)$ and $\pi_l(t)$ denote the amount invested in the security and loan respectively. Therefore, $\pi_B(t) = X(t) - \pi_s(t) - \pi_l(t)$ denotes the amount invested in the riskless asset. The dynamics of the assets portfolio is given by

$$dX(t) = \big(X(t) - \pi_s(t) - \pi_l(t)\big)\frac{dS_0(t)}{S_0(t)} + \pi_s(t)\frac{dS(t)}{S(t)} + \pi_l(t)\frac{dL(t)}{L(t)}$$
$$= [X(t)r(t) + \pi_s(t)\upsilon\eta(t) + \pi_s(t)\sigma_s\lambda_r k_1 r(t) + \pi_l(t)b_1\lambda_r k_1 r(t)]dt$$
$$+ \left[\pi_s(t)\sigma_s\sigma_r\sqrt{r(t)} + \pi_l(t)b_1\sigma_r\sqrt{r(t)}\right]dw_r(t) + \pi_s(t)\sqrt{\eta(t)}dw_s(t) \quad (6)$$

2.3 Admissible Strategy

An investment strategy $\Pi(t) = (\pi_s(t), \pi_l(t))$ is said to be admissible if the following conditions are satisfied.

1. $\pi_s(t)$ and $\pi_l(t)$ are all $f_t -$ measurable.
2. $E\left(\int_0^T (\pi_s^2(t)\eta(t) + [\pi_s(t)\sigma_s\sigma_r + \pi_l(t)b_1\sigma_r]^2 r(t))dt\right) < \infty$
3. The stochastic differential equation (3.6) has a unique solution $\forall\, \pi(t) = (\pi_s(t), \pi_l(t))$.

2.4 The Portfolio Optimization Problem

Let the set of all admissible strategy be denoted by Π. Under the portfolio (6), the financial institution looks for an optimal strategy $\pi_s{}^*(t)$ and $\pi_l{}^*(t)$ which maximizes the expected utility of the terminal wealth. i.e.:

$$\max_{\pi(t)\in\Pi} E[U(X(T))] \quad (7)$$

Based on the classical tools of stochastic optimal control, we state the problem as follows:

Maximize $E[U(X(T))]$

Subject to:

$$dr(t) = \big(a - br(t)\big)dt - \sigma_r\sqrt{r(t)}dw_r(t)$$
$$d\eta(t) = \alpha\big(\delta - \eta(t)\big)dt + \sigma_\eta\sqrt{\eta(t)}dw_r(t)$$
$$dX(t) = [X(t)r(t) + \pi_s(t)\upsilon\eta(t) + \pi_s(t)\sigma_s\lambda_r k_1 r(t) + \pi_l(t)b_1\lambda_r k_1 r(t)]dt$$
$$+ \left[\pi_s(t)\sigma_s\sigma_r\sqrt{r(t)} + \pi_l(t)b_1\sigma_r\sqrt{r(t)}\right] dw_r(t) + \pi_s(t)\sqrt{\eta(t)}dw_s(t)$$
$$X(0) = x_0, r(0) = r_0, \eta(0) = \eta_0$$

where $0 \le t \le T$ and $X(0) = x_0, r(0) = r_0, \eta(0) = \eta_0$ are the initial conditions of the optimal control problem.

The objective is to maximize the expected utility of the financial institution's portfolio at future date $T > 0$. That is, find the optimal value function

$$H(t, r, \eta, x) = \max_{\pi(t)\in\Pi} E\big[U\big(X(T)\big)\big| r(t) = r, \eta(t) = \eta, X(t) = x\,\big] \quad (8)$$

and the optimal strategy is $\pi^*(t) = (\pi_s{}^*(t), \pi_l{}^*(t))$ such that

$$\underset{\pi^*(t)}{\mathrm{H}}(t, r, \eta, x) = H(t, r, \eta, x) \quad (9)$$

2.5 The Derivation of the Hamilton – Jacobi – Bellman Equation Associated with the Portfolio Optimization Problem

The Hamilton – Jacobi – Bellman equation associated with the portfolio optimization problemis:

$$\max_{\pi(t)\in\Pi}\{H_t + [X(t)r(t) + \pi_s(t)\upsilon\eta(t) + \pi_s(t)\sigma_s\lambda_r k_1 r(t) + \pi_l(t)b_1\lambda_r k_1 r(t)]H_x$$
$$+\frac{1}{2}(\pi_s^2(t)\eta(t) + \left[\pi_s(t)\sigma_s\sigma_r\sqrt{r(t)} + \pi_l(t)b_1\sigma_r\sqrt{r(t)}\right]^2)H_{xx} - [\pi_s(t)\sigma_s\sigma_r^2 r(t)$$
$$+\pi_l(t)b_1\sigma_r^2 r(t)]H_{xr} + [\rho\pi_s(t)\sigma_\eta\eta(t)]H_{x\eta} + [a - br(t)]H_r + \frac{1}{2}\sigma_r^2 r(t)H_{rr}$$
$$+\alpha[\delta - \eta(t)]H_\eta + \frac{1}{2}\sigma_\eta^2\eta(t)H_{\eta\eta}\} = 0 \quad (11)$$
$$H(T, r, \eta, x) = U(x) \quad (12)$$

where $H_t, H_\eta, H_x, H_r, H_{xx}, H_{rr}, H_{\eta\eta}, H_{x\eta}$ and H_{xr} denote partial derivatives of first and second orders with respect to t, r, η and x respectively.

Differentiating (11) with respect to $\pi_s(t)$ *and* $\pi_l(t)$, we obtain

$$(\upsilon\eta + \sigma_s\lambda_r k_1 r)H_x + (\pi_s(t)\eta + (\pi_s(t)\sigma_s^2\sigma_r^2 r + \pi_l(t)b_1\sigma_s\sigma_r^2 r)H_{xx}$$
$$-\sigma_s\sigma_r^2 rH_{xr} + \rho\sigma_\eta\eta H_{x\eta} = 0 \quad (13)$$

and

$$b_1\lambda_r k_1 rH_x + (\pi_s(t)b_1\sigma_s\sigma_r^2 r + \pi_l(t)b_1^2\sigma_r^2 r)H_{xx} - b_1\sigma_r^2 rH_{xr} = 0 \quad (14)$$

Solving (13) and (14) for $\pi_s(t)$ and $\pi_l(t)$ give the first order maximizing conditions for the optimal strategy $\big(\pi_s^*(t), \pi_s^*(t)\big)$.From equation (14),

$$\pi_l(t) = \frac{H_{xr}}{b_1 H_{xx}} - \frac{\lambda_r k_1 H_x}{b_1\sigma_r^2 H_{xx}} - \frac{\pi_s(t)\sigma_s}{b_1} \quad (15)$$

Substituting for $\pi_l(t)$ in equation (13) and simplifying, we obtain

$$\pi_s^*(t) = -\upsilon\frac{H_x}{H_{xx}} - \rho\sigma_\eta\frac{H_{x\eta}}{H_{xx}} \quad (16)$$

Substituting (16) in (15) and simplifying gives

$$\pi_l^*(t) = \frac{H_{xr}}{b_1 H_{xx}} + \frac{(\upsilon\sigma_s\sigma_r^2 - \lambda_r k_1)H_x}{b_1\sigma_r^2 H_{xx}} + \frac{\rho\sigma_\eta\sigma_s H_{x\eta}}{b_1 H_{xx}} \quad (17)$$

Substituting (16) and (17) in (11) gives the partial differential equation (PDE) for the value function.

$$H_t + xrH_x - \left(\frac{\upsilon^2\eta}{2} + \frac{\lambda_r^2 k_1^2 r}{2\sigma_r^2}\right)\frac{H_x^2}{H_{xx}} - \rho^2\sigma_\eta^2\eta\frac{H_{x\eta}^2}{2H_{xx}} - \sigma_r^2 r\frac{H_{xr}^2}{2H_{xx}} - \rho\sigma_\eta\eta\upsilon\frac{H_x H_{x\eta}}{H_{xx}}$$

$$+\lambda_r k_1 r\frac{H_x H_{xr}}{H_{xx}} + (a - br)H_r + \frac{1}{2}\sigma_r^2 rH_{rr} + \alpha(\delta - \eta)H_\eta + \frac{1}{2}\sigma_\eta^2\eta H_{\eta\eta} = 0 \quad (18)$$

The problem now is solving (18) for the value function and replace it in (16) and (17).

3 The Solution of the Optimization Problem

For CRRA utility function, we conjecture a solution to the equation (18) in the following form:

$$H(t,r,\eta,x) = \frac{x^\beta}{\beta}f(t,r,\eta), \quad \beta < 1, \beta \neq 0 \quad (19)$$

With the boundary condition:

$$f(T,r,\eta) = 1 \quad (20)$$

From (19), we have

$$\left.\begin{array}{l} H_t = \frac{x^\beta}{\beta}f_t, H_x = x^{\beta-1}f, \quad H_r = \frac{x^\beta}{\beta}f_r, \quad H_\eta = \frac{x^\beta}{\beta}f_\eta, H_{xx} = (\beta-1)x^{\beta-2}f \\ H_{xr} = x^{\beta-1}f_r, H_{x\eta} = x^{\beta-1}f_\eta, \quad H_{rr} = \frac{x^\beta}{\beta}f_{rr}, \quad H_\eta = \frac{x^\beta}{\beta}f_{\eta\eta} \end{array}\right\} \quad (21)$$

Where $H_t, H_x, H_r, H_\eta, H_{xx}, H_{xr}, H_{x\eta}, H_{rr}$ and $H_{\eta\eta}$ represent the first order and second order partial derivatives of H with respect t, x, r and η.

Introducing these derivatives in (21) into (18) and dividing through by $\frac{x^\beta}{\beta}$ yields

$$f_t + \left[r\beta - \left(\frac{\beta\upsilon^2\eta}{2(\beta-1)} + \frac{\beta\lambda_r^2 k_1^2 r}{2\sigma_r^2(\beta-1)}\right)\right]f - \frac{\beta\rho^2\sigma_\eta^2\eta f_\eta^2}{2(\beta-1)f} - \frac{\beta\sigma_r^2 r f_r^2}{2(\beta-1)f}$$

$$+\left[\alpha(\delta-\eta) - \frac{\beta\rho\sigma_\eta\eta\upsilon}{\beta-1}\right]f_\eta + \left[\frac{\beta\lambda_r k_1 r}{\beta-1} + (a-br)\right]f_r + \frac{1}{2}\sigma_r^2 r f_{rr} + \frac{1}{2}\sigma_\eta^2\eta f_{\eta\eta} = 0 \quad (22)$$

We conjecture $f(t,r,\eta)$ as the following:

$$\left.\begin{array}{l} f(t,r,\eta) = e^{D_1(t)+D_2(t)r+D_3(t)\eta} \\ D_1(t) = D_2(t) = D_3(t) = 0 \end{array}\right\} \quad (23)$$

From (23), we have

$$\left.\begin{array}{l} f_t = (D_1'(t) + D_2'(t)r + D_3'(t)\eta)f \\ f_r = D_2(t)f, \quad f_\eta = D_3(t)f \\ f_{rr} = D_2^2(t)f, \quad f_{\eta\eta} = D_3^2(t)f \end{array}\right\} \quad (24)$$

Where f_t, f_r, f_η, f_{rr} and $f_{\eta\eta}$ represent the first order and second order partial derivatives of f with respect t, r and η respectively. Hence substituting for f_t, f_r, f_η, f_{rr} and $f_{\eta\eta}$ in (22) gives

$$[D_1'(t) + aD_2(t) + \alpha\delta D_3(t)]f + rf\left[D_2'(t) + \left(\beta - \frac{\beta\lambda_r^2 k_1^2}{2\sigma_r^2(\beta-1)}\right)\right.$$

$$\left. + \left(\frac{1}{2}\sigma_r^2 - \frac{\beta\sigma_r^2}{2(\beta-1)}\right)D_2^2(t) + \left(\frac{\beta\lambda_r k_1}{\beta-1} - b\right)D_2(t)\right] +$$

$$\eta f\left[D_3'(t) - \left(\frac{\beta\upsilon^2}{2(\beta-1)}\right) + \left(\frac{1}{2}\sigma_\eta^2 - \frac{\beta\rho^2\sigma_\eta^2}{2(\beta-1)}\right)D_3^2(t) - \left(\alpha + \frac{\beta\rho\sigma_\eta\upsilon}{\beta-1}\right)D_3(t)\right] = 0 \quad (25)$$

Eliminating the dependency on r *and* η, we decompose (25) into

$$D_1'(t) + aD_2(t) + \alpha\delta D_3(t) = 0 \quad (26)$$

$$D_2'(t) + \left(\frac{1}{2}\sigma_r^2 - \frac{\beta\sigma_r^2}{2(\beta-1)}\right)D_2^2(t) + \left(\frac{\beta\lambda_r k_1}{\beta-1} - b\right)D_2(t) + \left(\beta - \frac{\beta\lambda_r^2 k_1^2}{2\sigma_r^2(\beta-1)}\right) = 0 \quad (27)$$

$$D_3'(t)+\left(\frac{1}{2}\sigma_\eta^2-\frac{\beta\rho^2\sigma_\eta^2}{2(\beta-1)}\right)D_3^2(t)-\left(\alpha+\frac{\beta\rho\sigma_\eta\upsilon}{\beta-1}\right)D_3(t)-\left(\frac{\beta\upsilon^2}{2(\beta-1)}\right)=0 \qquad (28)$$

Observe that (27) and (28) are the general Riccati equations.
Now, we turn to solving the above three equations. From (26),

$$D_1'(t)=-aD_2(t)-\alpha\delta D_3(t)$$

$$D_1(t)=-\left(a\int_t^T D_2(t)\,dt+\alpha\delta\int_t^T D_3(t)\,dt\right) \qquad (29)$$

From (27), we have that

$$\frac{dD_2(t)}{dt}=\left(\frac{\beta\sigma_r^2}{2(\beta-1)}-\frac{1}{2}\sigma_r^2\right)D_2^2(t)+\left(b-\frac{\beta\lambda_r k_1}{\beta-1}\right)D_2(t)+\left(\frac{\beta\lambda_r^2k_1^2}{2\sigma_r^2(\beta-1)}-\beta\right) \qquad (30)$$

The discriminant $=B^2-4AC=b^2+\dfrac{\beta(2b\lambda_r k_1-2\sigma_r^2-\lambda_r^2k_1^2)}{1-\beta}$ since $\beta<1$

where $A=\left(\dfrac{\beta\sigma_r^2}{2(\beta-1)}-\dfrac{1}{2}\sigma_r^2\right), B=\left(b-\dfrac{\beta\lambda_r k_1}{\beta-1}\right), C=\left(\dfrac{\beta\lambda_r^2k_1^2}{2\sigma_r^2(\beta-1)}-\beta\right)$

let$\Delta_0=b^2+\dfrac{\beta(2b\lambda_r k_1-2\sigma_r^2-\lambda_r^2k_1^2)}{1-\beta}$, then

$$M_{1,2}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}=\frac{\left(b+\frac{\beta\lambda_r k_1}{1-\beta}\right)\pm\sqrt{\Delta_0}}{\left(\frac{\sigma_r^2}{1-\beta}\right)}, \qquad \beta<1$$

Equation (27) has different solutions depending on whether $\Delta_0>0, \Delta_0=0$ and$\Delta_0<0$. Now, let $\Delta_0>0$ then the quadratic function has two different roots denoted by M_1 and M_1 respectively such that

$$\frac{dD_2(t)}{dt}=A[(D_2(t)-M_1)(D_2(t)-M_2)] \qquad (31)$$

Therefore, equation (31) becomes

$$\frac{dD_2(t)}{(D_2(t)-M_1)(D_2(t)-M_2)}=Adt \qquad (32)$$

$$\frac{1}{M_1-M_2}\left(\frac{1}{D_2(t)-M_1}-\frac{1}{D_2(t)-M_2}\right)dD_2(t)=Adt \qquad (33)$$

The integral of (33) with respect to t, from t to T is:

$$\frac{1}{M_1-M_2}\int_t^T\left(\frac{dD_2(s)}{D_2(s)-M_1}-\frac{dD_2(s)}{D_2(s)-M_2}\right)=A\int_t^T ds$$

$$D_2(t)=\frac{M_1M_2-M_1M_2e^{A(M_1-M_2)(T-t)}}{M_1-M_2e^{A(M_1-M_2)(T-t)}}$$

Note that

$$A=\left(\frac{\beta\sigma_r^2}{2(\beta-1)}-\frac{1}{2}\sigma_r^2\right)=-\left(\frac{\beta\sigma_r^2}{2(1-\beta)}+\frac{1}{2}\sigma_r^2\right)\text{for}\beta<1$$

Therefore,

$$D_2(t)=\frac{M_1M_2-M_1M_2e^{-\left(\frac{1}{2}\sigma_r^2+\frac{\beta\sigma_r^2}{2(1-\beta)}\right)(M_1-M_2)(T-t)}}{M_1-M_2e^{-\left(\frac{1}{2}\sigma_r^2+\frac{\beta\sigma_r^2}{2(1-\beta)}\right)(M_1-M_2)(T-t)}} \qquad (34)$$

Next we solve for $D_3(t)$ in (28)

$$D_3'(t)=\left(\frac{\beta\rho^2\sigma_\eta^2}{2(\beta-1)}-\frac{1}{2}\sigma_\eta^2\right)D_3^2(t)+\left(\alpha+\frac{\beta\rho\sigma_\eta\upsilon}{\beta-1}\right)D_3(t)+\left(\frac{\beta\upsilon^2}{2(\beta-1)}\right) \qquad (35)$$

From (35), we have

$$A_1=\left(\frac{\beta\rho^2\sigma_\eta^2}{2(\beta-1)}-\frac{1}{2}\sigma_\eta^2\right), B_1=\left(\alpha+\frac{\beta\rho\sigma_\eta\upsilon}{\beta-1}\right), C_1=\left(\frac{\beta\upsilon^2}{2(\beta-1)}\right)$$

The discriminant $=B_1^2-4A_1C_1=\alpha^2-\dfrac{2\beta\rho\sigma_\eta\upsilon\alpha}{1-\beta}-\dfrac{\beta\upsilon^2\sigma_\eta^2}{1-\beta}, \qquad \beta<1$

Again, let$\Delta_1=\alpha^2-\dfrac{2\beta\rho\sigma_\eta\upsilon\alpha}{1-\beta}-\dfrac{\beta\upsilon^2\sigma_\eta^2}{1-\beta}$

Then,

$$M_{3,4} = \frac{\left(\alpha - \frac{\beta\rho\sigma_\eta \upsilon}{1-\beta}\right) \pm \sqrt{\Delta_1}}{\left(\sigma_\eta^2 + \frac{\beta\rho^2\sigma_\eta^2}{1-\beta}\right)}, \qquad \beta < 1$$

Equation (28) has different solution depending on whether $\Delta_1 > 0, \Delta_1 = 0$ and $\Delta_1 < 0$. Let $\Delta_1 > 0$, then the quadratic function has two distinct roots denoted by M_3 and M_4 respectively such that

$$\frac{dD_3(t)}{dt} = A_1[(D_3(t) - M_3)(D_3(t) - M_4)] \qquad (36)$$

$$\frac{dD_3(t)}{(D_3(t) - M_3)(D_3(t) - M_4)} = A_1 dt \qquad (37)$$

$$\frac{1}{M_3 - M_4}\left(\frac{1}{D_3(t) - M_3} - \frac{1}{D_3(t) - M_4}\right) dD_3(t) = A_1 dt \qquad (38)$$

The integral of (38) from t to T with respect to t is:

$$\frac{1}{M_3 - M_4}\int_t^T \left(\frac{1}{D_3(s) - M_3} - \frac{1}{D_3(s) - M_4}\right) dD_3(s) = A_1 \int_t^T ds$$

$$D_3(t) = \frac{M_3M_4 - M_3M_4 e^{A_1(M_3 - M_4)(T-t)}}{M_3 - M_4 e^{A_1(M_3-M_4)(T-t)}}$$

Observe that

$$A_1 = \left(\frac{\beta\rho^2\sigma_\eta^2}{2(\beta - 1)} - \frac{1}{2}\sigma_\eta^2\right) = -\left(\frac{\beta\rho^2\sigma_\eta^2}{2(1-\beta)} + \frac{1}{2}\sigma_\eta^2\right), \ \ \beta < 1$$

Therefore,

$$D_3(t) = \frac{M_3M_4 - M_3M_4 e^{-\left(\frac{1}{2}\sigma_\eta^2 + \frac{\beta\rho^2\sigma_\eta^2}{2(1-\beta)}\right)(M_3 - M_4)(T-t)}}{M_3 - M_4 e^{-\left(\frac{1}{2}\sigma_\eta^2 + \frac{\beta\rho^2\sigma_\eta^2}{2(1-\beta)}\right)(M_3 - M_4)(T-t)}} \qquad (39)$$

Theorem 1

From (16), (17), (21) and (24), the optimal proportion of wealth invested in security, loan and treasury under stochastic interest rates and stochastic volatility framework, and in the case of CRRA utility function is given by:

$$\pi_{sp}^*(t) = \frac{\nu}{1-\beta} + \frac{\rho\sigma_\eta D_3(t)}{1-\beta}$$

$$\pi_{lp}^*(t) = \frac{(\lambda_r k_1 - \upsilon\sigma_s\sigma_r^2)}{b_1\sigma_r^2(1-\beta)} - \frac{D_2(t)}{b_1(1-\beta)} - \frac{\rho\sigma_s\sigma_\eta D_3(t)}{b_1(1-\beta)}$$

$$\pi_{0p}^*(t) = 1 - \pi_{sp}^*(t) - \pi_{lp}^*(t)$$

$$= 1 + \frac{\upsilon\sigma_s\sigma_r^2 - \upsilon b_1\sigma_r^2 - \lambda_r k_1}{b_1\sigma_r^2(1-\beta)} + \frac{1}{b_1(1-\beta)}D_2(t) + \frac{\rho\sigma_\eta(\sigma_s - b_1)}{b_1(1-\beta)}D_3(t)$$

4. Numerical Examples

Here, we present the numerical simulation for the evolution of the optimal investment strategy and the effects of some of the market parameters on optimal investment strategy. We assume that the investment period $T = 10$ years, $k = 0$. The remaining parameters:$a = 0.0187, b = 0.2339, r_0 = 0.05, \eta_0 = 1$,
$\beta = -2, \lambda_r = 1, k_1 = 0.0073, \sigma_r = 0.0854, \alpha = 2, \delta = 0.3, \rho = 0.5, \sigma_\eta = 1, \upsilon = 1.5, b_1 = 0.7$,
$\sigma_s = 0.02$ are gotten from ([13], [18])

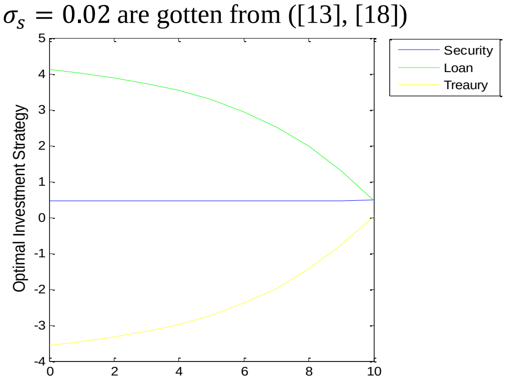


Fig. 1 The effect of time on the optimal investment strategy

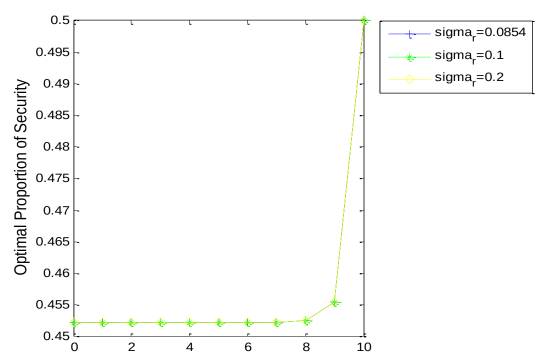


Fig. 2 The effect of the parameter $\sigma_r$onSecurity

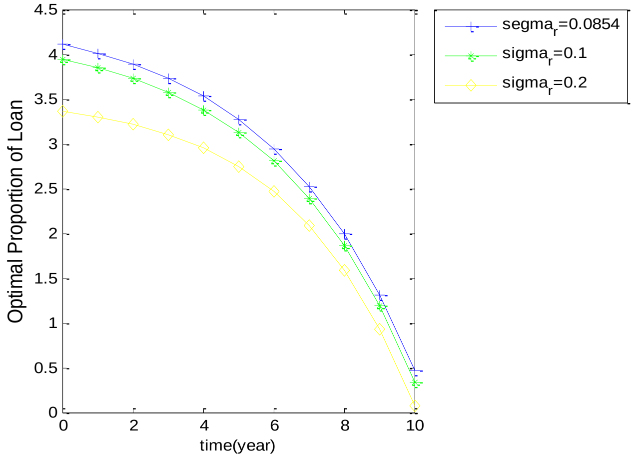


Fig. 3 The effect of the parameter $\sigma_r$on Loan

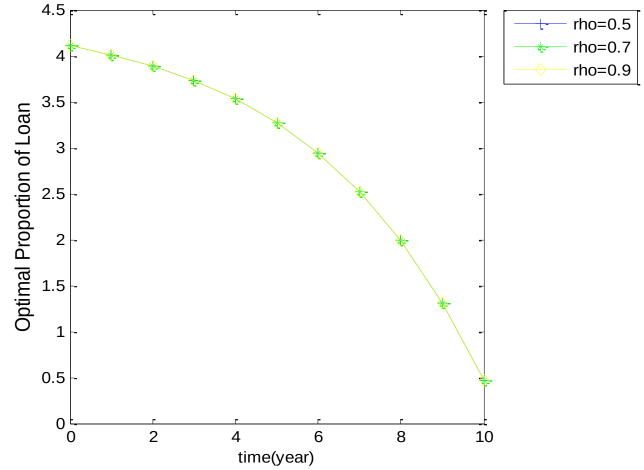


Fig. 6 The effect of the positive parameter ρon Loan

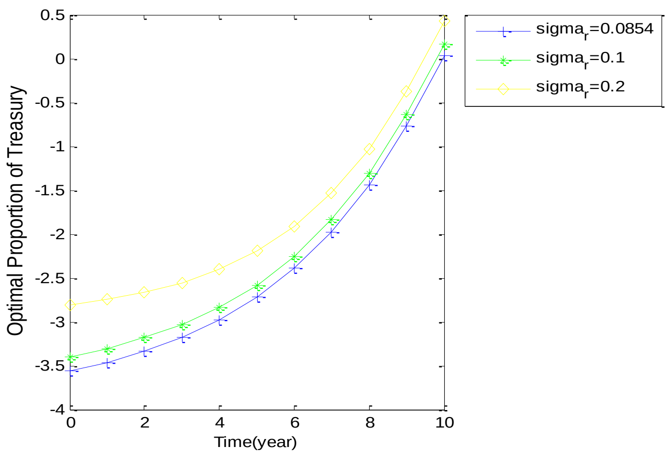


Fig. 4 The effect of the parameter $\sigma_r$on Treasury

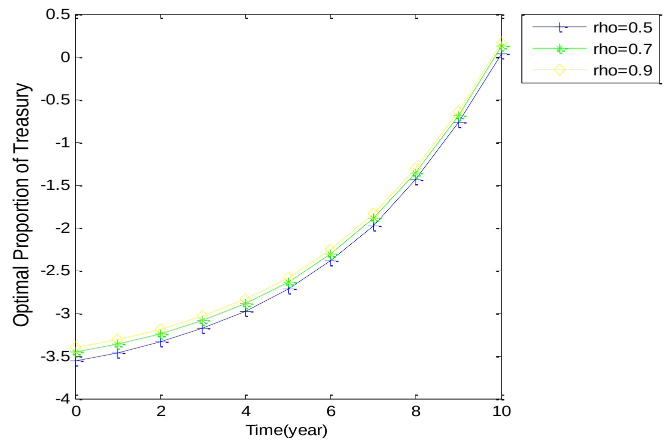


Fig.7 The effect of the positive parameter ρonTreasury

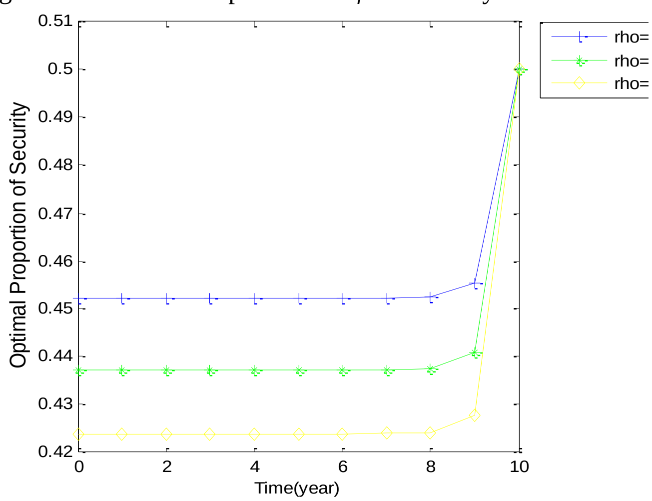


Fig. 5 The effect of the positive parameter ρ on Security

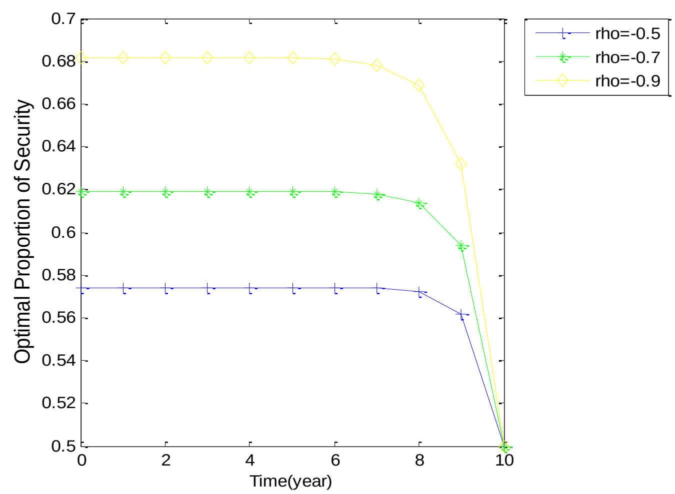


Fig. 8 The effect of the negative parameter ρon Security

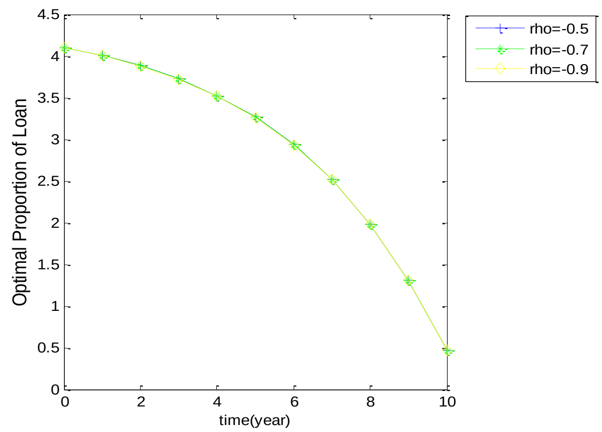
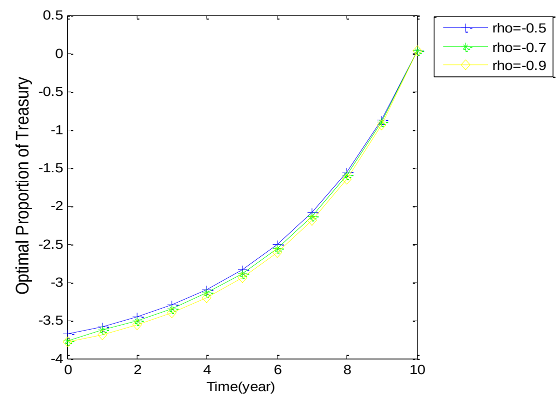


Fig. 9 The effect of the negative parameter ρon Loan



F ig. 10 The effect of the negative parameter ρ on Treasury

Figure 1 illustrates the trends of how the optimal proportion of the wealth invested in the three assets change with time. From figure 1, there is a positive relationship between optimal investment in the treasury and time. That is, as time increases so also the optimal investment in the treasury. However, the optimal proportion invested in the security almost remains unchanged and the optimal proportion invested loan decreases. Figure 1 also shows that the optimal proportion invested in the treasury is negative at the beginning of the investment horizon which indicates that the investor takes a short position in the treasury. The treasury short position enables the investor to invest more in the risky instruments within the period but toward the end of the investment period, the investor invests more in the treasury to reach the optimal investment strategy.

Figure 2 to figure 4 gives the relationship between optimal investment strategy and the parameter σ_r. From figure 2, we found that the optimal proportion invested in security remains unchanged but the optimal proportion invested in the loan decreases as σ_r increases as shown in figure 3. This illustrates that the interest rate has little influence on the optimal investment in the security. While the optimal proportion invested in the treasury increases as σ_r increases as shown in figure 4. This illustrates the intuitive observation that if the optimal investment in security remains almost unchanged and the optimal investment in the loan decreases then the optimal investment in the treasury increases.

The Heston model, $\eta(t)$ reflects the volatility of the risky asset's price therefore optimal investment strategy depends on the parameters of $\eta(t)$. Thus we plot the effects of the parameters of $\eta(t)$ on the optimal investment strategy. We observed that the optimal proportion invested in the security decreases for $\rho > 0$ and increases for $\rho < 0$. The parameter ρ, reflects the correlation between the security's price and its volatility. The uncertainties of the two processes change in the same way when ρ is positive and change in different ways when ρ is negative. Therefore, the investor invests less money in the security when $\rho > 0$ as ρ increases as illustrated by figure 5while the investor's investment remains unchanged in loan as shown in figure 6 and increases in treasury as shown in figure 7. The investor invests more in the security when $\rho < 0$ as ρ decreases as shown in figure 8 to hedge the risk which is consistent with intuition and this numerical example is illustrated by figure 8. However, the investor's investment remains unchanged in loan as shown in figure 9 and decreases in treasury as shown in figure 10.

5. Conclusion

Allocating optimally the financial institution's resources among competing investments is very important. In this research work, we considered optimization problem of a financial institution assets where the interest rate is driven by stochastic Affine interest rate model and the volatility of the security is described by the Heston stochastic volatility model. Here, the investor objective is to maximize the terminal wealth. The interest rate model is stochastic and follows the CIR Affine model. The volatility of the security is also stochastic and obeys the Heston's stochastic volatility model. Therefore, the investor has to deal with the risk of both interest rate and volatility. Under the asset portfolio optimization problem, the financial market consists of three assets namely; security, loan and treasury. We derived the optimal investment strategy under the CRRA utility function, obtained the explicit solution of the (resulted Hamilton – Jacobi – Bellman equation) for the optimal asset allocation problem and analyze the behavior of the optimal portfolio via some numerical examples with interpretation of its economic meanings in the real market. Some of the results we got are:

i. The optimal investment strategy is to diversify the financial institution portfolio away from the risky assets and toward the riskless treasury.

ii. Increasing the volatility of the interest rate causes shift of wealth from security and loan into treasury. This is as the result of the fact that investment in security and loan become more risky as the interest rate becomes more volatile

iii. The parameter ρ reflects the correlation between the security price and its volatility. The uncertainties of the two processes change in same sense when ρ is positive and change in different ways when ρ is negative. Therefore, the security price and its volatility are negatively correlated.

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