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CAPITAL OPTIMIZATION PROBLEM IN A FINANCIAL INSTITUTION

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Abstract

In this paper, we considered capital optimization problem of a financial institution where we assumed that the interest rate and volatility are constant. We derived the Hamilton – Jacobi – Bellman (HJB) equation associated with the capital optimization problem through the application of stochastic optimization theory. We also solved explicitly for optimal investment strategy for the case of exponential utility function. Lastly, we applied it to a financial institution in Nigeria and found that the optimal investment strategy is to diversify the financial institution investment away from the riskless asset (treasury) and toward the risky assets namely; security and loan when $\theta_1$is small (for investor with less risk averse policy), as income of security and loan increases the investor increases investment in the security and loan, and decreases investment in treasury to reach the optimal investment strategy and as the volatilities of the security and loan increases, the investor invests less in the risky assets (security and loan) and invests more in the riskless asset (treasury) to arrive at the optimal investment strategy.

Keywords: Financial institution, investment strategy, stochastic optimization theory, capital, portfolio

1. Introduction

A dynamic portfolio or optimal asset allocation is very crucial in a financial institution management. Also, optimal assets allocation and capital management play a very important role in financial institutions. Interest in this topic in stochastic framework has grown commensurately [1 – 3].

In [3], an optimal assets allocation problem with stochastic interest rates which takes into account specific features of bank was considered. Their goal was to present a numerical aspect of the derived Hamilton – Jacobi – Bellman (HJB) equation and to focus on the optimal assets allocation model results from a practical viewpoint. Similarly, [4] also considered assets allocation problem. In their work, they illustrated that it is possible to use an analytic approach to optimize assets allocation strategies for banks. They formulated an optimal bank valuation problem through optimal choices of loan rate and demand which leads to maximal deposits, provisions for deposits withdrawals and bank profitability subject to cash flow, loan demand, financing and balance sheet constraints.

The work by [5] considered a bank that invests in both liquid and non - liquid assets in their work. The goal of the investor is to maximize its shareholders' profit while satisfying some regulatory constraints. They studied the sensitivity of the shareholders' gain and optimal portfolio allocations, and the associated bondholders' payoff to the minimal capital requirement and liquidity ratio. In their research, they found that tightening the liquidity constraint adversely affects the rates of return on investment while preventing some large losses that occur when the portfolio is very illiquid and stiffening the minimal capital requirement penalizes the shareholders but seems to have little influence on the bondholders.

An optimal investment strategy for banks funds in treasuries and securities in a risk and regret theoretical framework has also been considered by [6]. Evidence of portfolio shifting is found in [7 – 8], where they suggested that banks may change their balance sheets in ways that can cause procyclicality. In[4], the authors modeled non – risk – based and risk – based capital

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adequacy. Specifically, they constructed a continuous time stochastic models for the dynamics of the leverage, equity and Tier 1 ratios and derived the capital adequacy ratio (CAR). They also show the relevant of their result to the banking sector by studying an optimal control problem in which an optimal assets allocation strategy is derived for the leverage ratio on a given time interval. Precisely, they determined the optimal expected terminal utility of the leverage ratio and derived the optimal assets allocation strategy that make it possible to maximize the expected terminal utility of the leverage ratio on a given time interval.

The motivation for the current study lies in the work of [3], where the authors looked at how a financial institution can optimally allocate its wealth among its assets namely; security, loan and treasury, and also manage its capital under Stochastic Interest Rates. The current study modified the existing model and study the optimal investment strategies of the financial institution's capital under constant interest rate and constant volatility, estimate some of the parameters of the models using data obtained from [9] by the method of maximum likelihood [10] and applied it to a financial institution in Nigeria.

2 Formulation of optimization problem and its transformation to partial differential equation

2.1 The assets and liability models in the financial market for the financial institution

Here, we assume that the financial institution can invest in a financial market that consists of three assets. The first asset in the financial market is a riskless treasury and its price at time t can be denoted by $S_0(t)$. It evolves according to the following stochastic differential equation:

$$\frac{dS_0(t)}{S_0(t)} = r_0 dt, \qquad S_0(0) = s_0 \tag{1}$$

The second asset in the financial market is a risky security whose price is denoted by $S(t), t \geq 0$. Its dynamics can be described by the equation (Grant and Peter, 2014):

$$\frac{dS(t)}{S(t)} = \left(r(t) + \upsilon\sigma_1 + \sigma_s\lambda_r k_1 r(t)\right)dt + \sigma_s\sigma_r\sqrt{r(t)}dw_r(t) + \sigma_1 dw_s(t) \tag{2}$$

From equation (2), if we assume that the volatility scale factor σ_s which measures how the risk sources of interest rate affect the price of the security is equal to zero (the risk sources of the interest rate have no effect on the price of the security) and the interest rate is constant, then the modified security model is given by:

$$\frac{dS(t)}{S(t)} = (r_0 + \lambda_1\sigma_s)dt + \sigma_s dw_s(t), \qquad S(0) = s_0 \tag{3}$$

where $r_0,\ \lambda_1$ and σ_s are constants. Let $\lambda_1\sigma_s = \lambda_s$, then (3) becomes

$$\frac{dS(t)}{S(t)} = (r_0 + \lambda_s)dt + \sigma_s dw_s(t), \qquad S(0) = s_0 \tag{4}$$

where r_0, λ_s and σ_s are constants.

The third asset is a loan to be amortized over a period $[0, T]$ whose price at time $t \geq 0$ is denoted by $L(t)$. Let also assume that the price of the asset can be describe by a stochastic differential equation similar to (4):

$$\frac{dL(t)}{L(t)} = (r_0 + \lambda_l)dt + \sigma_l dw_l(t), \qquad L(0) = l_0 \tag{5}$$

where r_0, λ_l and σ_l are constants.

Furthermore, Let us assume that the financial institution liabilities results only from deposits made by customers. Let also assume that the dynamics of the deposits satisfies the following SDE:

$$dD(t) = \lambda_d dt + \sigma_d dw_d(t) \qquad D(0) = d_0 \tag{6}$$

2.2 The derivation of the financial institution's capital model equation

By definition, the financial institution capital can be defined as the difference between the value of the financial institution's assets and liabilities. In this case, the assets are; treasury, security and loan, and the liability is the deposit. Let $V(t)$ denotes the financial institution asset at time $t \in [0.T]$, $m_1(t)$ and $m_2(t)$ denote the amounts invested in the security and loan respectively. Therefore,

$$m_3(t) = V(t) - m_1(t) - m_2(t)$$

denotes the amount invested in the riskless asset (treasury). Therefore, the model equation for the financial institution capital is given by:

$$A(t) = V(t) - D(t)$$

Therefore, the stochastic differential equation describing the financial institution capital is:

$$dA(t) = dV(t) - dD(t)$$

Note that:

$$dV(t) = \left(V(t) - m_1(t) - m_2(t)\right)\frac{dS_0(t)}{S_0(t)} + m_1(t)\frac{dS(t)}{S(t)} + m_2(t)\frac{dL(t)}{L(t)} \tag{7}$$
$$= (V(t)r_0 + m_1(t)\lambda_s + m_2(t)\lambda_l)dt + m_1(t)\sigma_s dw_s(t) + m_2(t)\sigma_l dw_s(t)$$

Hence, the financial institution capital model is:

$$dA(t) = (V(t)r_0 + m_1(t)\lambda_s + m_2(t)\lambda_l - \lambda_d)dt + m_1(t)\sigma_s dw_s(t) + m_2(t)\sigma_l dw_l(t) - \sigma_d dw_d(t) \tag{8}$$

Definition 1: Admissible strategy for the capital model equation

An investment strategy $\gamma(t) = (m_1(t), m_2(t))$ is said to be admissible if the following conditions are satisfied.

i. $m_1(t)$ and $m_2(t)$ are all $f_t -$ measurable.

ii. $E\left(\int_0^T ((m_1(t)\sigma_1)^2 + (m_2(t)\sigma_2)^2)dt\right) < \infty$

iii. The stochastic differential equation (8) has a unique solution $\forall\, \theta(t) = (m_1(t), m_2(t))$.

Definition 2: Constant absolute risk aversion (CARA) utility function

Here, we describe the financial institution's objective with an exponential utility function under capital optimization problem as:

$$U(x) = -\frac{1}{\theta_1}e^{-\theta_1 x}, \quad (\theta_1 > 0 \text{ is a positive constant}) \tag{9}$$

The absolute risk aversion of the decision maker with the utility described in equation (9) is constant i.e.

$$R(x) = A(x) = -\frac{U''(x)}{U'(x)} = -\frac{(-\theta_1 e^{-\theta_1 x})}{e^{-\theta_1 x}} = \theta_1$$

and $U'(x)$ and $U''(x)$ denote the first and second derivatives of $U(x)$ with respect to x. Therefore, equation (9) describes a CARA utility function.

2.3 The formulation of the financial institution's capital optimization problem

Let the set of all admissible strategy be denoted by $\gamma_1(t)$. Under the capital model (8), the financial institution objective is to maximize the expected utility of its capital at future time $T > 0$. i.e.:

$$\max_{\gamma(t)\in\gamma_1} E[U(V(T))] \tag{10}$$

Based on the classical tools of stochastic optimal control, we state the problem as follows:

Maximize $E[U(V(T))]$

Subject to the following constraints

$$dA(t) = (V(t)r_0 + m_1(t)\lambda_l + m_2(t)\lambda_l - \lambda_d)dt + m_1(t)\sigma_l dw_s(t) + m_2(t)\sigma_l dw_l(t) - \sigma_d dw_d(t)$$

where $V(0) = V_0, r_0 = r_1$ are the initial conditions of the capital optimization problem and $0 \le t \le T$.

Applying the classical tools of stochastic optimal control theory, the value function can be define for $0 \le t \le T$ as:

$$H(t, A) = \sup_{\gamma(t)} E[U(A(T)|A(t) = A)] \tag{11}$$

Here we assumed that there is no correlation between $w_s(t)$ and $w_l(t)$, between $w_s(t)$ and $w_d(t)$, between $w_l(t)$ and $w_d(t)$. The correlation between $w_d(t)$ and $w_d(t)$ is 1, the correlation between $w_s(t)$ and $w_s(t)$ is 1 and the correlation between $w_l(t)$ and $w_l(t)$ is 1.

2.4 The transformation of the optimization problem into partial differential equation

From maximum principle we derive the Hamilton – Jacobi – Bellman equation associated with the capital optimization problem as

$$H_t + \sup_{\gamma_1}\left\{[V(t)r_0 + m_1(t)\lambda_s + m_2(t)\lambda_l - \lambda_d]H_A + \frac{1}{2}[m_1^2(t)\sigma_s^2 + m_2^2(t)\sigma_l^2 + \sigma_d^2]H_{AA}\right\} = 0 \tag{12}$$

whereH_t, H_A and H_{AA} denote partial derivatives of first and second orders with respect to t and A respectively.

Now, the first order maximizing conditions for the optimal investment strategy $\gamma(t)$ (i.e. differentiating (12) with respect to $m_1(t)$ and $m_2(t)$) gives

$$\lambda_s H_v + \frac{1}{2}(2m_1(t)\sigma_s^2)H_{AA} = 0 \tag{13}$$

$$\lambda_l H_v + \frac{1}{2}(2m_2(t)\sigma_l^2)H_{AA} = 0 \tag{14}$$

Solving the equations (13) and (14) gives

$$m_1^*(t) = -\frac{\lambda_s H_A}{\sigma_s^2 H_{AA}} \quad (15)$$

$$m_2^*(t) = -\frac{\lambda_l H_A}{\sigma_l^2 H_{AA}} \quad (16)$$

Substituting for $m_1^*(t)$ and $m_1^*(t)$ in equation (12), and simplifying we obtain

$$H_t + (V(t)r_0 - \lambda_d)H_A - \left(\frac{\lambda_s^2}{2\sigma_s^2} + \frac{\lambda_l^2}{2\sigma_l^2}\right)\frac{H_A^2}{H_{AA}} + \frac{1}{2}\sigma_3^2 H_{AA} = 0 \quad (17)$$

where equation (17) is the partial differential equation equivalent to the Hamilton – Jacobi – Bellman equation (12).

3. The analytical Solution of the formulated capital optimization problem under the exponential utility function

In the case of exponential utility, we conjecture the solution of (17) as:

$$H(t, A) = -\frac{1}{\theta_1} exp\{-\theta_1 A + g(t)\} \quad (18)$$

$$g(T) = 0 \quad (19)$$

Then from (18) we have that:

$$\left.\begin{aligned} H_t &= g_t H \\ H_A &= \theta_1 H \\ H_{AA} &= -\theta_1^2 H \end{aligned}\right\} \quad (20)$$

where $H_t, H_A, H_{AA}$are partial derivatives. Hence substituting for H_t, H_A and H_{AA} into (17) and simplifying gives

$$H\left(g_t + \theta_1(Vr_0 - \lambda_d) + \left(\frac{\lambda_s^2}{2\sigma_s^2} + \frac{\lambda_l^2}{2\sigma_l^2}\right) - \frac{1}{2}\sigma_d^2\theta_1^2\right) = 0 \quad (21)$$

Eliminating H in (21) gives

$$g_t + \theta_1(Vr_0 - \lambda_d) + \left(\frac{\lambda_s^2}{2\sigma_s^2} + \frac{\lambda_l^2}{2\sigma_l^2}\right) - \frac{1}{2}\sigma_d^2\theta_1^2 = 0 \quad (22)$$

Let

$$n = \theta_1(Vr_0 - \lambda_d) + \left(\frac{\lambda_s^2}{2\sigma_s^2} + \frac{\lambda_l^2}{2\sigma_l^2}\right) - \frac{1}{2}\sigma_d^2\theta_1^2$$

Then from equation (22), we that

$$g_t + n = 0 \quad (23)$$

$$g(t) = -\int_t^T n ds$$

$$g(t) = -n(T - t) \quad (24)$$

$$= -\left(\theta_1(Vr_0 - \lambda_d) + \left(\frac{\lambda_s^2}{2\sigma_s^2} + \frac{\lambda_l^2}{2\sigma_l^2}\right) - \frac{1}{2}\sigma_d^2\theta_1^2\right)(T - t)$$

$$g(t) = \left(\frac{1}{2}\sigma_d^2\theta_1^2 - \theta_1(Vr_0 - \lambda_d) - \left(\frac{\lambda_s^2}{2\sigma_s^2} + \frac{\lambda_l^2}{2\sigma_l^2}\right)\right)(T - t) \quad (25)$$

Theorem 1: Capital optimization problem

From equations (15), (16) and (20), the optimal investment strategy for the case of exponential utility function is given by:

$$m_1^*(t) = \frac{\lambda_s}{\sigma_s^2\theta_1}$$

$$m_2^*(t) = \frac{\lambda_l}{\sigma_l^2\theta_1}$$

$$m_3^*(t) = V - \frac{\lambda_s}{\sigma_s^2\theta_1} - \frac{\lambda_l}{\sigma_l^2\theta_1}$$

4. Numerical examples

Here, we present the numerical simulation for the evolution of the optimal investment strategy derived in the previous section. We take the investment period= 10 years, $\theta_1 = 1, v_0 = 1$ and assumed that$\lambda_l = 0.0031$, $\sigma_l = 0.0874$. The remaining parameters $\lambda_s = 0.0022, \sigma_s = 0.0748$, are estimated fromdata obtained from Nigeria Stock Exchange fact book.

Table 1: Table of optimal investment strategies for capital optimization problem

S/n	Risk Aversion Parameter	Optimal Investment Strategy
1	$\theta_1 = 1$	$m_1 = 0.3932$ $m_2 = 0.4059$ $m_3 = 0.2010$
2	$\theta_1 = 5$	$m_1 = 0.0786$ $m_2 = 0.0812$ $m_3 = 0.8402$

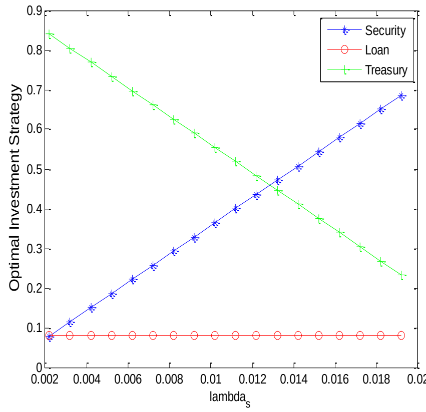

Fig. 1: The effect of the parameter λ_s on theoptimal investment strategy

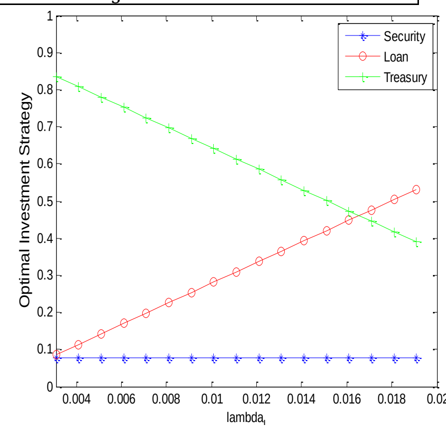

Fig. 2: The effect of the parameter $\lambda_l$on theoptimal investment strategy

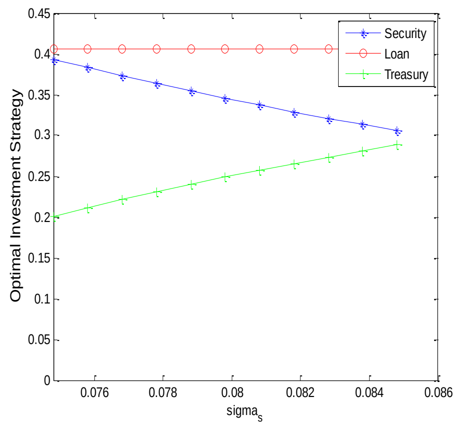

Fig. 3: The effect of the parameter $\sigma_s$on the optimal investment strategy

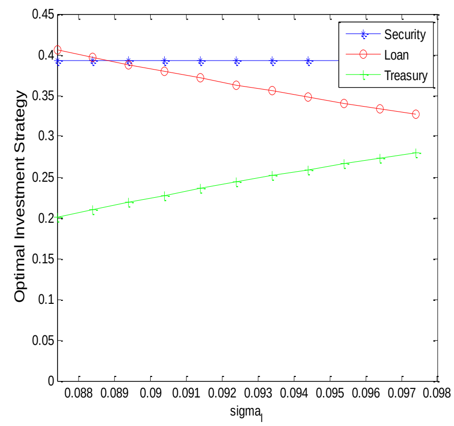

Fig. 4: The effect of the parameter $\sigma_l$on the optimal investment strategy

Table 1 shows how the trends of the optimal amount of the capital invested in the three assets change with respect to the risk adverseparameter θ_1. From the Table 1, the investor distributes his/hercapital among the three assets such that the investor invests more in security and loan, and less in treasury when $\theta_1 = 1$. While the reverse is the case when$\theta_1 = 5$. Hence for the case of exponential utility function, the investor invests more in security and loan, and invests less in treasury to reach the optimal investment strategy when $\theta_1 = 1$ (when$\theta_1$is small), invests less in the risky assets and invests more in the riskless asset when $\theta_1 = 5$ (when$\theta_1$is big) to reach the optimal investment strategy. This can be explaining by the risk tolerance of the investor. For instance, a risk adverse investor is an investor who prefers lower returns with known risks rather than higher returns with unknown risks. In other words, among various investments giving the same return with different level of risks, this investor always prefers the alternative with least interest. Therefore, an investor with low risk averse policy invests more in the risky assets (when$\theta_1 = 1$) and investor with high risk averse policy (more risk averse policy) invests more in the riskless asset when $\theta_1 = 5$.

Figure 1 shows the relationship between the parameter λ_s and the optimal investment strategy. Note that from equation (2), $r(t) + \lambda_s$ is the appreciation rate of the security. Therefore, from Figure 1 the optimal amount invested in the security increases as the parameter λ_s increases while the optimal amount invested in the loan remains constant and the optimal amount invested in the treasury decreases as shown respectively.

Figure 2 also illustrates the relationship between the parameter λ_l and the optimal investment strategy. Observe that from equation (3), $r(t) + \lambda_l$ is the appreciation rate of the loan. Therefore, from Figure 2 the optimal amount invested in the security remains constant as the parameter λ_l increases and the optimal amount invested in the treasury decreases as the result of increment in the amount invested in loan while the optimal amount invested in the loan increases as the parameter λ_l increases as shown respectively.

Figure 3 also illustrates the relationship between the parameter σ_s and the optimal investment strategy. As the parameter σ_s increases the optimal amount invested in the security decreases, the optimal amount in the loan remains same while the optimal amount invested in treasury increases as shown in Figure 3.

Similarly, Figure 4 also illustrates the relationship between the parameter σ_l and the optimal investment strategy. As the parameter σ_l increases the optimal amount invested in the loan decreases, the optimal amount invested in the security remains same while the optimal amount invested in treasury increases as shown in Figure 4.

5. Conclusion

We considered capital optimization problem of the financial institution where we assumed that the interest rate and volatility are constant. The investor is allowed to invest in the financial market consisting of three assets namely; a treasury, a marketable security and a loan. Using the method of stochastic optimal control, we derived the optimal investment strategy for the case of CARA utility function (exponential utility function), obtained the explicit solution of the capital optimization problem and present numerical examples to illustrate the effect of the model parameters on the optimal investment strategy. Some of the results obtained show that:

i. The optimal investment strategy is to diversify the financial institution investment away from the riskless asset (treasury) and toward the risky assets namely; security and loan when $\theta_1$is small (for investor with less risk averse policy).
ii. The optimal investment strategy is to diversify the financial institution investment away from the risky assets (security and loan) and toward the riskless asset (treasury) when $\theta_1$is big for investor with more risk averse policy.
iii. As the appreciation rate or income of security and loan increases the investor increases investment in the security and loan, and decreases investment in treasury to reach the optimal investment strategy.
iv. As the volatilities of the security and loan increases, the investor invests less in the risky assets (security and loan) and invests more in the riskless asset (treasury) to arrive at the optimal investment strategy.

In order to obtain the explicit solutions for the formulated capital optimization problem, we only considered a special utility function namely; exponential utility function and assumed that the interest rate is constant, the volatilities of the security and loan are also constant. In further research work, we can consider the same problem under Vasicek interest rate structure and the volatility of the security model will be consider to follow Cox – Ingersoll – Rose model. The explicit solution of the formulated capital optimization problem can be considered under hyperbolic utility function and logarithmic utility function.

References

[1] Peter, J. W., Garth, J. V. S. and Grant, E. M. (2011). An Optimal Investment Strategy in Bank Management. *Mathematical Methods in the Applied Sciences.* 34: 1606 - 1617.

[2] Fatma, C. and Fathi, A. (2014). A Methodology to Estimate the Interest Rate Yield Curve in Illiquid Market the Tunisian Case. *Journal of Emerging Market Finance*. 13(5): 1 – 29.

[3] Grant, E. M. and Peter, J. W. (2014). An Optimal Portfolio and Capital Management Strategy for Basel III Compliant Commercial Banks. *Journal of Applied Mathematics.* 2014: 1 – 11.

[4] Fouche, C. H., Mukuddem – Petersen, J. and Petersen, M. A. (2006). Continuous – time Stochastic Modeling of Capital Adequacy Ratio for Banks. *Applied Stochastic Model in Business and Industry.* 22(1): 41 – 71.

[5] Astic, F. and Tourin, A. (2014). Optimal Bank Management Under Capital and liquidity constraints. *Journal of Financial Engineering*. 1(3): 1450022-1 – 1450022-21.

[6] Mulaudzi, M. P., Petersen, M.A. and Schoeman, I. M. (2008). Optimal Allocation Between Bank Loans and Treasuries with Regret. *Optimization Letters*. 2(4): 555 – 566.

[7] Borio, C., Furfine, C. and Lowe, P. (2001). Procyclicality of the Financial system and Financial Stability: Issues and Policy options. Bank of International Settlements Conference Marrying the Macro and Micro prudential Dimensions of Financial Stability, 1 – 57.

[8] Lowe, P. (2002). Credit Risk Measurement and Procyclicality. Bank for International Settlements Working Paper. 116: 1 – 17.

[9] Nigeria Stock Exchange 2000 – 2010. Fact Book. Lagos. Nigeria.

[10] Yanli, Z. (2014). Computational Methods for Various Stochastic Differential Equation Models in Finance. PHD Thesis, Curtin University Australia.