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PRODUCTION–INVENTORY MODEL OF DELAYED DETERIORATING ITEMS WITH QUADRATIC AND PRICE DEPENDENT DEMANDS AND BACKORDERING

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Abstract

This paper develops an inventory model of delayed deteriorating items in which demand is a deterministic function of selling price. Depletion of the items will depend on demand before the deterioration starts and at the time the deterioration begins, it will depend on both demand and deterioration. Before the deterioration sets in, demand is assumed to be quadratic time dependent function and when deterioration begins, the demand depends on selling price. The model is developed with the assumption of finite replenishment. The model is constructed to minimize the total inventory cost and maximize the total profit by determining the optimal cycle length, the optimal time length of replenishment, and the optimal production quantity. Numerical examples are given to test the application of the model.

Keywords**:** Inventory, Delayed Deterioration, Quadratic Demand, Backordering.

1. Introduction

The maiden research work in the area first modeled negative exponential decaying inventory [1,2]. The study of inventory system generally assumes that the product is stored at stockroom for a long time to fulfill the future customer demand. The traditional EOQ models assume that products can be stored indefinitely to meet future demand [3,4]. However, many perishable products such as fruits, vegetables, medicines, and volatile liquids degrade or deteriorate continuously due to evaporation, spoilage and obsolescence, among other reasons. [5], developed a model on deteriorating items that do not start deteriorating immediately they were held in stock until later. Depletion of these items as soon as they are stored would depend on demand only, and when deterioration begins, depletion depends on both demand and deterioration. Several authors [6-10] have developed inventory models under constant deterioration rate.

Abubakar and Babangida [5] assume that If the deterioration starts, the demand of the item will differ from the situation before deterioration sets in. In this paper, we developed an EPQ model for delayed deteriorating items where we assumed two types of demand rate. One is time dependent quadratic demand and the other is price dependent demand.

2. Assumptions and Notations

The following Assumptions and Notations are considered in developing the mathematical model:

2.1 Assumption

(i) We consider a single item over an infinite planning horizon

(ii) All items are inspected and defective ones are discarded

(iii) Demand of product exceeds its supply

(iv) Demand before deterioration begins (D_1) is assumed to be quadratic and defined by $D = a + bt + ct^2, c \geq 0$

(v) We assume that the demand rate " D_2 " is proportional to an exponential function of the price S . Thus " D_2 " is proportional to $\alpha e^{-\lambda s}$ where α is the maximum number of potential consumers $\Rightarrow D_2 = \alpha e^{-\lambda s}$

(vi) Backordering is allowed

2.2 Notation

D_1 The demand rate in $t_1 \leq t \leq t_2$	D_2 The demand rate in $t_2 \leq t \leq T$
i The inventory carrying charge (excluding interest charge)	p The unit cost of items
S Selling price per unit after deterioration sets in	s_1 Selling price per unit before deterioration sets in
C_1 Deterioration cost	A The set-up cost per production run
θ Deterioration rate of the stock	$q_1(t)$ The inventory level at time t , $t_1 \leq t \leq t_2$
$q_2(t)$ The inventory level at time t , $t_2 \leq t \leq T$	b_1 The maximum shortages (backorder) level permitted
c_b The backorder cost per unit time	C_B The total backorder cost per unit cycle
T The production cycle length	t_1 The production build-up period
t_2 The time deterioration sets in	k Production rate of the items
I_0 The initial inventory	

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The inventory level

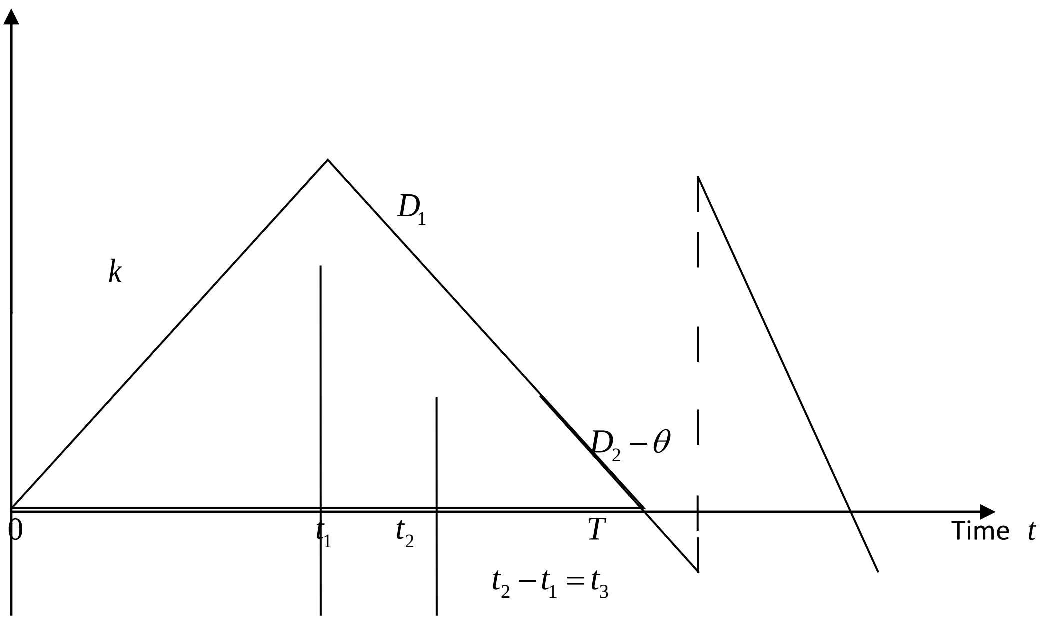

Figure 1: Inventory depletion in a delayed deterioration situation with shortages

3. The Mathematical Model

The differential equation that represents the depletion of inventory due to demand only before deterioration sets in is given by:

$$\frac{dq_1(t)}{dt} = k \qquad 0 \le t \le t_1 \tag{1}$$

With boundary condition $dq_1(0) = 0$

Integrating equation (1) and applying the boundary condition, we get

$$\int \frac{dq_1(t)}{dt} dt = \int k dt$$

$$\Rightarrow q_1(t) = kt + c \tag{2}$$

Where c is the constant of integration and applying the boundary condition $q_1(0) = 0$ we have

$$q_1(t) = kt \qquad 0 \le t \le t_1 \tag{3}$$

The differential equation that represents the depletion of inventory after deterioration sets in is given by:

$$\frac{dq_2(t)}{dt} = -D_1 \qquad t_1 \le t \le t_2 \tag{4}$$

With boundary condition $q_1(t) = q_2(t)$

Re-writing equation (4) gives $\frac{dq_2(t)}{dt} = -a - bt - ct^2$, we solve equation (4) as thus;

$$\int \frac{dq_2(t)}{dt} dt = \int (-a - bt - ct^2) dt \qquad \to q_2(t) = -at - \frac{bt^2}{2} - \frac{ct^3}{3} + \beta \tag{5}$$

Where β is the constant of integration, using the boundary condition $q_1(t) = q_2(t)$,equation (5) becomes:

$$\beta = q_2(t_1) + at_1 + \frac{bt_1^2}{2} + \frac{ct_1^3}{3} \tag{6}$$

From equation (3) $q_1(t) = kt$, since $q_1(t) = q_2(t)$ $\to q_2(t_1) = kt_1$

Rewriting equation (6), yields:

$\beta = kt_1 + at_1 + \frac{bt_1^2}{2} + \frac{ct_1^3}{3}$, We put in equation (5) to obtain

$q_2(t) = -at - \frac{bt^2}{2} - \frac{ct^3}{3} + kt_1 + at_1 + \frac{bt_1^2}{2} + \frac{ct_1^3}{3}$, simplifying, yields

$$q_2(t) = kt_1 + a(t_1 - t) + \frac{b(t_1^2 - t^2)}{2} + \frac{c(t_1^3 - t^3)}{3} \qquad t_1 \le t \le t_2 \tag{7}$$

Deterioration starts at the time t_2 due to the demand and deterioration, the inventory status in this interval * $t_1 \le t \le T$ is represented by the differential equation.

$$\frac{dq_3(t)}{dt} = -\theta q_3(t) - D_2 \qquad t_1 \le t \le T \tag{8}$$

Re-writing equation (8) as $\frac{dq_3(t)}{dt} + \theta q_3(t) = -D_2$ with boundary condition $q_3(T) = 0$, we then solve the equation to have:

Let $q_3(t) = y$ we have $\frac{dy}{dt} + \theta y = -D_2$ which is a linear first order differential equation.

Here: $P = \theta, Q = -D_2 \Rightarrow$ the integrating factor $= e^{\int \theta dt} = e^{\theta t}$, we then multiply through by the integrating factor, $e^{\theta t}$

$$\Rightarrow e^{\theta t}\frac{dy}{dt} + \theta y e^{\theta t} = -D_2 e^{\theta t} \Rightarrow \frac{d(ye^{\theta t})}{dt} = -D_2 e^{\theta t}$$

Integrating both sides we have $ye^{\theta t} = -D_2\left[\frac{1}{\theta}e^{\theta t}\right] + c \Rightarrow y = \frac{-D_2}{\theta} + ce^{\theta t}$ but $y = q_3(T)$

we have $\Rightarrow q_3(t) = \frac{-D_2}{\theta} + ce^{\theta t}$ (9)

With boundary condition $q_3(T) = 0$ we have $c = \frac{D_2}{\theta}e^{-\theta t}$ (10)

Put in equation (9) and applying the boundary condition, gives:

$$q_3(t) = \frac{D_2}{\theta}\left[e^{\theta(T-t)} - 1\right] \tag{11}$$

At t_2, $q_2(t) = q_3(t)$ we have

$$k = \frac{1}{t_1}\left[\frac{D_2}{\theta}\left(e^{\theta(T-t)} - 1\right) + a(t_2 - t_1) + \frac{b(t_2^2 - t_1^2)}{2} + \frac{c(t_2^3 - t_1^3)}{3}\right] \tag{12}$$

4. Computation of the Total Inventory Costs

The total inventory or variable cost is the sum of the inventory ordering cost, cost due to deterioration of inventory items, the total inventory carrying cost and the total backorder cost. The costs are computed individually before they are added together:

(i) The cost for ordering is given as A

(ii) The cost for purchasing is given as $PC = \int_0^{t_1} PKdt = PKt_1$

(iii) Inventory holding cost $HC = C_1\left[\int_0^{t_1} q_1(t)dt + \int_{t_1}^{t_2} q_2(t)dt + \int_{t_2}^{T} q_3(t)dt\right]$

$$= iP\left\{Kt_1t_2 - \frac{Kt_1^2}{2} - \frac{a(t_2 - t_1)^2}{2} + \frac{b(3t_1^2t_2 - t_2^3 - 2t_1^3)}{6} + \frac{c\left(4t_1^3t_2 - t_2^4 - 3t_1^4\right)}{3} + \frac{D_2}{\theta}\left[t_2 - T + \frac{1}{\theta}(e^{\theta(T-t)} - 1)\right]\right\} \tag{13}$$

The number of deteriorated items $d(t_2)$ is calculated as:

$$d(t_2) = \int_0^{t_1} kdt - \int_{t_1}^{t_2} D_1 dt - \int_{t_2}^{T} D_2 dt = kt_1 - a(t_2 - t_1) - \frac{b(t_2^2 - t_1^2)}{2} - \frac{c(t_2^3 - t_1^3)}{3} - D_2(T - t_2) \tag{14}$$

(iv) The deterioration cost is given as:

$$DC = C_1\left[kt_1 - a(t_2 - t_1) - \frac{b(t_2^2 - t_1^2)}{2} - \frac{c(t_2^3 - t_1^3)}{3} - D_2(T - t_2)\right] \tag{15}$$

(v) The total backorder cost per cycle is given as:

$$C_B = C_b\int_0^{T-t_2} D_2 t dt = \frac{C_bD_2}{2}\left[t^2\right]_0^{T-t_2} = \frac{C_bD_2}{2}(T - t_2)^2$$

The total inventory or variable cost per production run is given as:

$$TC = A + PC + HC + DC + C_B$$

$$\Rightarrow TC = A + PKt_1 + iP\left[kt_1t_2 - \frac{kt_1^2}{2} - \frac{a(t_2 - t_1)^2}{2} + \frac{b(3t_1^2t_2 - t_2^3 - 2t_1^3)}{6} + \frac{c\left(4t_1^3t_2 - t_2^4 - 3t_1^4\right)}{3} + \frac{D_2}{\theta}\left(t_2 - T + \frac{1}{\theta}\left(e^{\theta(T-t_2)} - 1\right)\right)\right]$$
$$+ C_1\left[kt_1 - a(t_2 - t_1) - \frac{b(t_2^2 - t_1^2)}{2} - \frac{c(t_2^3 - t_1^3)}{3} - D_2(T - t_2)\right] + \frac{C_bD_2}{2}(T - t_2)^2 \tag{16}$$

The total inventory cost per unit time is given as: $TC(T) = \frac{TC}{T} = \frac{1}{T}\left[A + PC + HC + DC + C_B\right]$

$$\Rightarrow TC(T) = \frac{1}{T}\left\{A + PKt_1 + iP\left[kt_1t_2 - \frac{kt_1^2}{2} - \frac{a(t_2 - t_1)^2}{2} + \frac{b(3t_1^2t_2 - t_2^3 - 2t_1^3)}{6} + \frac{c\left(4t_1^3t_2 - t_2^4 - 3t_1^4\right)}{3} + \frac{D_2}{\theta}\left(t_2 - T + \frac{1}{\theta}\left(e^{\theta(T-t_2)} - 1\right)\right)\right]\right.$$
$$\left. + C_1\left[kt_1 - a(t_2 - t_1) - \frac{b(t_2^2 - t_1^2)}{2} - \frac{c(t_2^3 - t_1^3)}{3} - D_2(T - t_2)\right] + \frac{C_bD_2}{2}(T - t_2)^2\right\} \tag{17}$$

The revenue obtained is computed using the relation

Revenue $= S_1 D_1 (t_2 - t_1) + S D_2 (t_2 - T)$ (18)

The profit is obtained using the relation

Profit $= TC - revenue$ (19)

Equation (17) is differentiated to determine the value of T which minimizes the total variable cost per unit time as follows:

$$\frac{dTC(T)}{dT} = \frac{-1}{T^2}\left\{A + PKt_1 + iP\left[kt_1t_2 - \frac{kt_1^2}{2} - \frac{a(t_2-t_1)^2}{2} + \frac{b(3t_1^2t_2 - t_2^3 - 2t_1^3)}{6} + \frac{c\left(4t_1^3t_2 - t_2^4 - 3t_1^4\right)}{3} + \frac{D_2}{\theta}\left(t_2 - T + \frac{1}{\theta}\left(e^{\theta(T-t_2)} - 1\right)\right)\right]\right.$$
$$\left. + C_1\left[kt_1 - a(t_2-t_1) - \frac{b(t_2^2-t_1^2)}{2} - \frac{c(t_2^3-t_1^3)}{3} - D_2\left(T-t_2\right)\right] + \frac{C_bD_2}{2}\left(T-t_2\right)^2\right\}$$
$$+\frac{1}{T}\left(\frac{iPD_2}{\theta}\left(\frac{e^{\theta(T-t_2)}}{\theta} - 1\right) - D_2C_1\right) = 0 \tag{20}$$

We can use equation (20) after simplification with other parameters provided to determine the best cycle length T which minimizes the total variable cost per unit time.

5.0 Computation of the Economic Production Quantity (EPQ)

The EPQ corresponding to the best cycle length T can be obtained thus:

$$\text{EPQ} := D_1T_1 + D_2\left(T_2 - T_1\right) + KT_1 - a\left(T_2 - T_1\right) - \frac{b}{2}\left(T_3^2 - T_1^2\right) - \frac{c}{3}\left(T_2^3 - T_1^3\right) \tag{21}$$

6. Numerical Example

In this section, we provide numerical examples to illustrate the application of the above model.

For the numerical illustration of the developed model, the values of various parameters in proper units can be taken as follows:

$A = 500, C_1 = 95, C_y = 350, D_1 = 950, D_2 = 700, i = 0.15, T_1 = 0.1534,$

$\theta = 0.25, a = 0.09, b = 2.00, c = 0.10, T_2 = 0.1726, K = 350, P = 150,$

Solving Equation (20) with the above parameters, we obtain T^*. Substitution of the optimal value T^* in Equations (17) and (21), we obtain the maximum total cost per unit time TC^* and the Economic production quantity EPQ respectively. The solutions of five different numerical examples representing the application of the EPQ model are given in table 1 below.

Table 1: EPQ and optimal cycle length of delayed deteriorating inventory model with quadratic, price dependent demand and backorder

S/N	A (N)	C_1	C_b	D_1	D_2	I	t_1	θ	a	b	c	t_2	K	P	T^*	$TC(T)$	EPQ	I_0	b_1
1	200	25	160	600	300	0.05	0.0333	0.60	0.05	8.00	0.09	0.5750	100	70	0.9635	4084.46	22	12	10
2	160	30	200	700	400	0.07	0.0576	0.40	1.00	7.00	0.09	0.0767	150	80	0.2618	22557.21	47	36	11
3	250	55	250	600	500	0.09	0.0777	0.30	0.60	6.00	0.08	0.0967	200	90	0.3997	7450.68	61	49	12
4	350	75	300	900	600	0.12	0.1151	0.20	0.03	4.00	0.05	0.1343	250	100	0.6633	28497.93	131	118	13
5	500	95	350	950	700	0.15	0.1534	0.25	0.09	2.00	0.10	0.1726	350	150	0.6903	41503.72	199	185	14

Table 1 above gives the different cycle length T and economic production quantity (EPQ) for different parameter values. The EPQ in this case is made up of the initial inventory, I_0 and the maximum backorder level permitted which is always filled first when a new replenishment is received. The cycle length in all the five examples corresponds to the least which corresponds to the overall inventory cost.

7. Conclusion

In this paper, we developed an Economic production quantity model for delayed deteriorating items with quadratic time dependent demand and selling price dependent demand. Where we determined the optimal cycle length T, the total variable cost per unit time TC(T) and the Economic Production Quantity (EPQ) and maximum profit. This paper could be extended to consider various other assumptions such as time-dependent holding cost and rate of deterioration dependent on time. The optimal cycle length T that gives the minimum total inventory or variable cost, the maximum backorder level allowed and the backorder cost were determined in each of the five examples given in table 1.

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