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EFFECT OF DISEASE TRANSMISSION COEFFICIENT ON SEIR EPIDEMIC MODEL WITH SATURATED INCIDENCE RATE

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Abstract

In this paper, the numerical simulation of a Susceptible-Exposed-Infected-Recovered (SEIR) epidemic model with saturated incidence rate including saturation term for the susceptible individuals was analyzed.
The local and global stabilities of both disease free and endemic equilibrium were also investigated. The effect of transmission coefficient was also analyzed using Runge Kutta of order 4 for the numerical analysis.

Keywords: Basic Reproduction Number, Local and Global Stabilities, Disease free equilibrium, Disease transmission Coefficient.

1. INTRODUCTION

In [1] (2016), analysis of SEIRS Epidemic model with disease induced death was modified. Also in [2] effect of Disease transmission coefficient was also investigated with a saturated incidence rate i.e. $\frac{\beta SI}{m_1 s + m_2 I}$. The paper covers when δ *and* $m_2 = 0$ which changes the incidence rate to $\frac{\beta SI}{m_1 s}$ and the new model is now SEIR i.e. there is a permanent immunity. Any individual cured will no longer goes into the susceptible class again. In [3] numerical simulation on saturated term on SEIRS epidemic model was also investigated. Analysis of initial state and treatment are discussed in [4], [5] and [6].

2. THE BASIC MATHEMATICAL MODEL

In this paper, SEIRS model [1] was adopted and modified by making δ *and* m_2 equals zero

Existing model (SEIRS) KOLAWOLE (2017)

$$\left.\begin{aligned}
\frac{dS}{dt} &= \wedge - \frac{\beta SI}{1 + m_1 S + m_2 I} - \mu S + \delta R \\
\frac{dE}{dt} &= \frac{\beta SI}{1 + m_1 S + m_2 I} - (\mu + \varepsilon) E \\
\frac{dI}{dt} &= \varepsilon E - (\mu + \gamma) I \\
\frac{dR}{dt} &= \gamma I - (\mu + \delta) R
\end{aligned}\right\} \qquad (1)$$

when $\delta = 0$ **and** $m_2 = 0$ **gives the proposed model in (1)**

Proposed Model (SEIR)

$$\left.\begin{aligned}
\frac{dS}{dt} &= \wedge - \frac{\beta SI}{1 + m_1 S} - \mu S \\
\frac{dE}{dt} &= \frac{\beta SI}{1 + m_1 S} - (\mu + \varepsilon) E \\
\frac{dI}{dt} &= \varepsilon E - (\mu + \gamma) I \\
\frac{dR}{dt} &= \gamma I - \mu R
\end{aligned}\right\} \qquad (2)$$

Therefore at disease free equilibrium (DFE) = $(S^\circ, E^\circ, I^\circ, R^\circ) = \left(\frac{\wedge}{\mu}, 0, 0, 0\right)$.

Also at endemic equilibrium i.e. when $I = 0$. The endemic equilibrium points are $S, E, I, R = \left(S^{**}, E^{**}, I^{**}, R^{**}\right)$

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Therefore the Endemic Equilibrium Points are (S,E,I,R) = $\left(S^{**}, E^{**}, I^{**}, R^{**}\right)$

$$\therefore \text{ R} = \frac{(\mu+\varepsilon)(\mu+\gamma)(\mu+\delta)(\mu+\wedge m_1) - \wedge\varepsilon\beta(\mu+\delta)\gamma\varepsilon}{\left(\varepsilon\beta(\mu+\delta) - m_1(\mu+\varepsilon)(\mu+\gamma)(\mu+\delta)\right)\left(\gamma\varepsilon\delta - (\mu+\varepsilon)(\mu+\gamma)(\mu+\delta)\right)} = R^{**}$$

$$I = \frac{(\mu+\delta)}{\gamma}\left(\frac{(\mu+\varepsilon)(\mu+\gamma)(\mu+\delta)(\mu+\wedge m_1) - \wedge\varepsilon\beta(\mu+\delta)\gamma\varepsilon}{\left(\varepsilon\beta(\mu+\delta) - m_1(\mu+\varepsilon)(\mu+\gamma)(\mu+\delta)\right)\left(\gamma\varepsilon\delta - (\mu+\varepsilon)(\mu+\gamma)(\mu+\delta)\right)}\right) = \text{I}^{**}$$

$$E = \frac{(\mu+\gamma)(\mu+\delta)}{\varepsilon\gamma}\left(\frac{(\mu+\varepsilon)(\mu+\gamma)(\mu+\delta)(\mu+\wedge m_1) - \wedge\varepsilon\beta(\mu+\delta)\gamma\varepsilon}{\left(\varepsilon\beta(\mu+\delta) - m_1(\mu+\varepsilon)(\mu+\gamma)(\mu+\delta)\right)\left(\gamma\varepsilon\delta - (\mu+\varepsilon)(\mu+\gamma)(\mu+\delta)\right)}\right) = \text{E}^{**}$$

$$S = \frac{(\mu+\varepsilon)(\mu+\gamma)(\mu+\delta)}{\varepsilon\gamma\beta - m_1(\mu+\varepsilon)(\mu+\gamma)(\mu+\delta)} = S^{**}$$

3. DERIVATIVE OF $R_\circ$

There are two diseases state but only one way to create new infections. Hence, we are concerned with E and I compartment of the model. Thus, from equation (2) gives

$$\frac{dE}{dt} = \frac{\beta SI}{1+m_1 S} - (\mu+\varepsilon)E$$

And (3a)

$$\frac{dI}{dt} = \varepsilon E - (\mu+\gamma)I$$

From equation (3a) we obtain the characteristics equation of matrix G as

$$|G - \lambda I| = 0$$

$$\begin{pmatrix} \frac{\beta\Lambda\varepsilon}{(\mu+m_1\wedge)(\mu+\gamma)(\mu+\varepsilon)} - \lambda & \frac{\beta\Lambda}{(\mu+m_1\wedge)(\mu+\lambda)} \\ 0 & -\lambda \end{pmatrix} = 0 \quad (3b)$$

$$= \left(\frac{\beta\Lambda\varepsilon}{(\mu+m_1\wedge)(\mu+\gamma)(\mu+\varepsilon)} - \lambda\right)(-\lambda) = 0 \quad (4)$$

$$\lambda = 0 \quad or \quad \frac{\beta\Lambda\varepsilon}{(\mu+m_1\wedge)(\mu+\gamma)(\mu+\varepsilon)} \quad (5)$$

The dominant eigenvalue is our $R_\circ$, therefore

$$R_\circ = \frac{\beta\Lambda\varepsilon}{(\mu+m_1\wedge)(\mu+\gamma)(\mu+\varepsilon)} \quad (6)$$

4. LOCAL STABILITY OF DISEASE FREE EQUILIBRUM

If we let

$S - S_1 = x,\ E = E,\ I = I,\ R = R$

Therefore, the resulting linearized equations are;

$$\left.\begin{aligned} \frac{dx}{dt} &= -\mu x - \beta S_1 I + non-linear\ terms \\ \frac{dE}{dt} &= -(\mu+\varepsilon)E + \beta S_1 I + non-linear\ terms \\ \frac{dI}{dt} &= \varepsilon E - (\mu+\gamma)I \\ \frac{dR}{dt} &= \gamma I - \mu R \end{aligned}\right\} \quad (7)$$

The resulting Jacobian Matrix is

$$\begin{bmatrix} x^I \\ E^I \\ I^I \\ R^I \end{bmatrix} = \begin{bmatrix} -\mu & 0 & -\beta S_1 & 0 \\ 0 & -(\mu+\varepsilon) & \beta S_1 & 0 \\ 0 & \varepsilon & -(\mu+\gamma) & 0 \\ 0 & 0 & \gamma & -\mu \end{bmatrix} \begin{bmatrix} x \\ E \\ I \\ R \end{bmatrix} + \text{non-linear terms} \quad (8)$$

So at the Disease Free Equilibrium point $S = \frac{\wedge}{\mu}$ therefore by substituting $S_1 = \frac{\wedge}{\mu}$ yields

$$(-\mu-\lambda)(-\mu-\lambda)\left[\lambda^2+(2\mu+\varepsilon+\gamma)\lambda+(\mu+\varepsilon)(\mu+\gamma)-\frac{\beta\wedge\varepsilon}{\mu}\right]=0$$

solving the equation above

$\lambda_1=-\mu,\ \lambda_2=-\mu$ *and if*

$D=4\wedge\beta\mu\varepsilon+\gamma^2\mu^2-2\gamma\mu^2\varepsilon+\mu^2\varepsilon^2$

$$\lambda_3=-\frac{1}{2}\frac{-\gamma\mu-2\mu^2-\varepsilon\mu+\sqrt{D}}{\mu}\quad \lambda_4=-\frac{1}{2}\frac{\gamma\mu+2\mu^2+\varepsilon\mu+\sqrt{D}}{\mu}$$

which are all negative. Hence the disease free equilibrium is unstable.

LOCAL STABILITY OF ENDEMIC EQUILIBRIUM

Let S-S*= w, E-E*= x, I-I*= y, R-R*= z

We linearize the system in matrix form we have

$$\begin{pmatrix} w^* \\ x^* \\ y^* \\ z^* \end{pmatrix}=\begin{pmatrix} (-\beta I^*+\beta I^* m_1 S-\mu) & 0 & (\beta S^*+\beta S^* m_1) & \delta \\ (\beta I^*-2\beta I^* m_1 S-\mu) & -(\mu+\varepsilon) & (\beta S^*-\beta S^* m_1) & 0 \\ 0 & \varepsilon & -(\mu+\gamma) & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}+\text{non linear terms} \qquad (9)$$

The equation gives

$$-((\mu)+\lambda)\left(-(\mu+\varepsilon)+\lambda\right)\left((-\mu+\gamma)\left((-\beta I^*+2\beta I^* m_1 S^*-\mu)-\lambda\right)\right.$$
$$\left.-\varepsilon(\beta S^*-\beta S^* m_1)\left((-\beta I^*+2\beta^{-*}+m_1 S^*-\mu)-\lambda\right)+(\beta I^*+2\beta I^* m_1 S^*)(+\beta S^*-\beta S^* m_1)\right)=0$$
$$-((\mu)+\lambda)\left(-(\mu+\varepsilon)+\lambda\right)\left((-\mu+\gamma)\left((-\beta I^*+2\beta I^* m_1 S^*-\mu)-\lambda\right)\right.$$
$$\left.-\varepsilon(\beta S^*-\beta S^* m_1)(\beta I^*+2\beta I^* m_1 S^*)-\left((\beta I^*-2\beta I^* m_1 S^*-\mu)+\lambda\right)\right)=0 \qquad (10)$$

Let $A_0=0$

$$A_1=(-\beta I^*+2\beta I^* m_1 S^*-\mu) \qquad (11)$$
$$A_2=\beta S^*-\beta S^* m_1$$

By Descartes's Rule of Sign, let

$A_3=\gamma+A_2-\mu<0$

$A_4=\left[(\mu+\varepsilon)(\mu+\gamma+A_1)+A_2(\mu+\varepsilon)-(\mu+\gamma)A_1-A_2A_1+\mu(\gamma+A_1-\varepsilon+A_2)\right]<0$

$$A_5=\begin{bmatrix}-(\mu+\varepsilon)(\mu+\gamma)A_1-(\mu+\varepsilon)-A_2A_1(\mu+\varepsilon)\mu(\mu+\varepsilon)(\mu+\gamma+A_1)+A_2\mu(\mu+\varepsilon)-\mu(\mu+\gamma)A_1 \\ -(\mu+\varepsilon)-\mu A_2A_1\end{bmatrix}<0$$

$A_6=\left[-\mu(\mu+\varepsilon)(\mu+\gamma)A_1-A_2A_1(\mu+\varepsilon)\mu\right]<0$

$\therefore f(\lambda)=\lambda^4-A_3\lambda^3-A_4\lambda^2-A_5\lambda-A_6=0$

Let $A_3<0, A_4<0, A_5<0, A_6<0$ .Then $f(\lambda)$ have no change in sign meaning there are no positive roots of $f(\lambda)$

Also if λ is replaced by - λ

$f(-\lambda)=\lambda^4+A_3\lambda^3-A_4\lambda^2-A_5\lambda-A_6=0$

So if $A_3<0, A_4<0, A_5<0, A_6<0$

$f(-\lambda)$, have four sign changes which implies, there are exactly four negative roots of $f(-\lambda),$. Since there is no positive roots for $A_3<0, A_4<0, A_5<0, A_6<0$.

That is all Eigen values are negatives, then the endemic or Disease free equilibrium is locally asymptotically stable if $A_3<0, A_4<0, A_5<0, A_6<0$.

Global Stability of the disease-free equilibrium

$$L=(\mu+\varepsilon)I+\varepsilon E$$
$$L'=(\mu+\varepsilon)I'+\varepsilon E'$$

At disease free equilibrium we have,

$$(S_1, E_1, I_1, R_1) = \left(\frac{\wedge}{\mu}, 0, 0, 0\right)$$

$$= \left[\left(\frac{\varepsilon\beta \wedge}{\mu + m_1 \wedge} - (\mu+\varepsilon)(\mu+\gamma)\right) I\right]$$

$$= (\mu+\varepsilon)(\mu+\gamma)\left[R_0 - 1\right] I$$

If $R_0 \leq 1$

$L' \leq 0$

Hence the disease free equilibrium is globally asymptotically stable.

RESULTS AND DISCUSSION

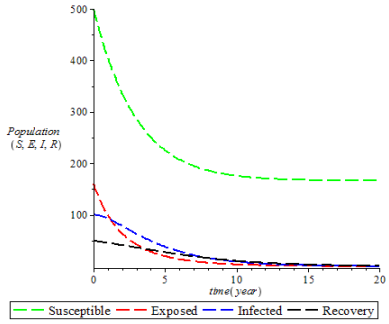

Fig. 1: Graph of SEIR against t with $\beta = 0.01,\ \phi = 50, \mu = 0.3, \gamma = 0.1, \varepsilon = 0.25, m_1 = 0.1$

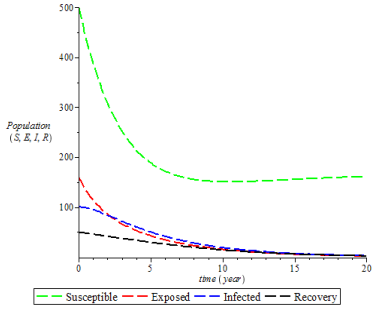

Fig. 2: Graph of SEIR against t when $\beta = 0.03,\ \phi = 50, \mu = 0.3, \gamma = 0.1, \varepsilon = 0.25, m_1 = 0.1$

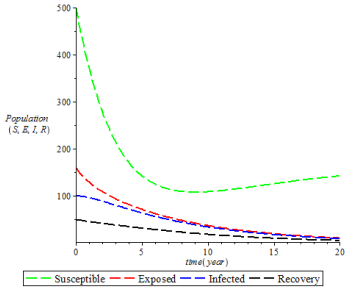

Fig. 3: Graph of SEIR against t when $\beta = 0.05,\ \phi = 50, \mu = 0.3, \gamma = 0.1, \varepsilon = 0.25, m_1 = 0.1$

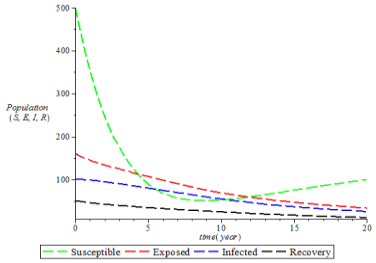

Fig. 4: Graph of SEIR against t when $\beta = 0.07,\ \phi = 50, \mu = 0.3, \gamma = 0.1, \varepsilon = 0.25, m_1 = 0.1$

4. RESULTS AND DISCUSSION

From Figures 1-4, the simulation result considering $\delta = 0$ on the effect of disease transmission coefficient on SEIR model indicates that, the lower the disease transmission coefficient the better asymptotic stability of disease free and endemic equilibrium. Hence for better eradication β should be so small in an infected environment.

$\delta = 0$ shows that, there is a kind of permanent immunity i.e. any individual cured is no longer going to the susceptible class. Hence SEIR epidemic model. The result is in line with [1] and [2].

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