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MATHEMATICAL ANALYSIS OF A ONE-DIMENSIONAL, ADVECTION-DISPERSION EQUATION WITH A CONTINUOUS REACTION, INPUT POINT SOURCE DISCHARGE IN A MOVING STREAM WITH VARIABLE TRANSPORT PROPERTIES

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Abstract

In this work, semi analytical solution of a reactive Advection-Dispersion Equation (ADE) in one dimension describing the dispersion of a continuous type, input point source discharge in a moving stream, through a semi-infinite medium is presented using the He's Homotopy Perturbation Method (HPM). Advection and dispersion coefficients are both considered as functions of distance and the solute concentration distribution as revealed in the graphs agree with the linearly time dependent assumption in real life.

***Keywords*:** Reactive, Advection, Dispersion, Continuous-type, Semi-infinite medium

1. Introduction:

In recent years, the need to understand the behavior of harmful waste and toxic chemicals in soils has been of great concern. The primary concern is the fact that these contaminants find their way through the unsaturated and saturated zones to contaminate the ground water.

Degradation of ground water quality can take place over large areas from plane or diffuse sources like intensively farmed fields, or it can be caused by line sources of poor quality water like seepage or intrusion of salt water. It can also be caused by point sources such as garbage disposal sites, mines and oil spoils, to mention but a few.

Analysis and predictions of solute transport (contaminants) in hydro geological systems generally involve the use of some form of partial differential equations of parabolic type usually referred to as Advection and Dispersion Equation. Although the usefulness of the mass balance equation on which the Advection and Dispersion Equation (ADE) is based to describe transport in natural soils due to the variability of flow and transport properties in the field is still under scientific scrutiny, it is likely, that Advection and Dispersion Equation will remain relevant as a potent tool for research purposes thus it will be used in this work.

Most works on Advection and Dispersion equation studies of late, presented the advection and dispersion coefficients as constants determined by laboratory experiments. In [1] solutions to two dimensional ADE, with time – dependent dispersion was examined. Pore flow velocity was assumed to be a non – divergence free, unsteady and non stationary random function of space and time in [2].

Analytical solution describing transport of dissolved substances in heterogeneous porous media with a distance dependent dispersion was developed in [3]. A method to solve the transport equation for a kinetically adsorbing solute in a porous medium with spatially varying velocity and dispersion coefficients was presented in [4] while a description of solute transport in a radially convergent flow field with scale- dependent dispersion was given in [5]. Other contributors to the study of ADE with variable coefficients include, [6, 7, 8].

In this work, semi analytical solution of a reactive ADE in one dimension describing the dispersion of a continuous type, input point source discharge in a moving stream through semi-infinite medium is presented using the He's Homotopy Perturbation Method [9]. The Advection and Dispersion coefficients are both considered as functions of the distance.

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2. Mathematical Formulation of the Problem:

We consider a general ADE in one-dimension as follows:

$$\frac{\partial C}{\partial t}=D_L\frac{\partial^2 C}{\partial x^2}-U_L\frac{\partial y}{\partial x}-\alpha C+\beta t \qquad (1)$$

where,

$C(x,t)$ is the concentration at any time t, in a horizontal plane, $D_L[\frac{L^2}{T}]$ is the coefficient of Dispersion, $U_L[\frac{L}{T^2}]$ is the coefficient of Advection, , $\beta[\frac{1}{T}]$ is the size of the pollutant at source, $\alpha[\frac{1}{T}]$ is the first order decay constant, x [L] is the distance and t [T] is time .We assume that at the point of injection, x = 0.

Let $U_L = U(x) = sx$, (3)

and $D_L = D(x) = kx^2$ (4)

where s and k are constants of dimension, $[\frac{1}{T}]$.

Thus equations (1) and (2) are transformed into:

$$\frac{\partial C}{\partial t}=(kx^2)\frac{\partial^2 C}{\partial x^2}-(sx)\frac{\partial C}{\partial x}\quad-\lambda C-\beta t \qquad (5)$$

Here we need to specify a source β (per unit height and per unit time). The relationship between the two is:

$$\beta=\frac{\textit{amount}\text{ released}}{\textit{height}\times\textit{time}}=\frac{\textit{amount}\text{ released}}{\textit{height}\times\textit{downstream}\text{ length}}\times\frac{\textit{downstream}\text{ length}}{\textit{time}}=mU \qquad (6)$$

m (kg) is the amount of contaminant released per unit height and unit downstream direction, and U is the velocity in the direction of flow.

From equation (3) and (6), it follows that;

$$\beta = Msx \qquad (7)$$

Hence equation (5) becomes;

$$\frac{\partial C}{\partial t}=(kx^2)\frac{\partial^2 C}{\partial x^2}-(sx)\frac{\partial C}{\partial x}-\lambda C-mtsx \qquad (8)$$

Equations (8) will be considered in this study as the one-dimension ADE problem with reaction.

An analytical solution of equation (8) using the Homotopy Perturbation method is sought under the following initial and boundary conditions:

$C(x,0) = e^{-\lambda x}$, $C_t(x,0) = 0$ and $C(\infty,t) = 0$ (9)

$\lambda[\frac{1}{T}]$ is the flow resistance coefficient.

In order to solve equation (8) we employ He's Homotopy Perturbation method (HPM) which is constructed as follows:

$$(1-p)[\frac{\partial C}{\partial t}]+p[\frac{\partial C}{\partial t}-(kx^2)\frac{\partial^2 C}{\partial x^2}+(sx)\frac{\partial C}{\partial x}+\lambda C+mtsx]=0 \qquad (10)$$

It is assumed that the solution of equation (8) can be written as power series of p as follows:

$$V=V^{(0)}+pV^{(1)}+p^2V^{(2)}+... \qquad (11)$$

and the best approximation for the solution of equation (8) is :

$$C=\lim_{p\to 1}V=V^{(0)}+V^{(1)}+V^{(2)}+... \qquad (12)$$

Substituting equation (11) into (10) and comparing like terms on both sides gives:

$$P^0:\frac{\partial V^{(0)}}{\partial t}=0 \qquad (13)$$

$$p^1:\frac{\partial V^{(1)}}{\partial t}-(kx^2)\frac{\partial^2 V^{(0)}}{\partial x^2}+(sx)\frac{\partial V^{(0)}}{\partial x}+\lambda V^{(0)}+mtsx=0 \qquad (14)$$

$$p^2:\frac{\partial V^{(2)}}{\partial t}-(kx^2)\frac{\partial^2 V^{(1)}}{\partial x^2}+(sx)\frac{\partial V^{(1)}}{\partial x}+\lambda V^{(1)}=0 \qquad (15)$$

Solutions of equations (13) , (14) and (15) by the method of integrating factor yields :

$$V^{(0)} = e^{-\lambda x} \tag{16}$$

$$V^{(1)} = -\frac{1}{2}mst^2x + (kx^2\lambda^2e^{-\lambda x} + sx\lambda e^{-\lambda x} - \lambda e^{-\lambda x})t + f_1(x) \tag{17}$$

$$V^{(2)} = (\frac{1}{6}(s^2xm + \lambda msx))t^3 + \frac{1}{4}(4k^2x^2\lambda^2e^{-\lambda x} - 8k^2x^3\lambda^3e^{-\lambda x} + 2k^2x^4\lambda^4e^{-\lambda x} - 8kx^2s\lambda^2e^{-\lambda x} + 4kx^3s\lambda^3e^{-\lambda x} - 4kx^2\lambda^3e^{-\lambda x} - 2s^2x\lambda e^{-\lambda x} + 2s^2x^2\lambda^2e^{-\lambda x} - 4sx\lambda^2e^{-\lambda x} + 2\lambda^2e^{-\lambda x})t^2 + f_2(x) \tag{18}$$

The leading equation inherits the auxiliary conditions of the initial problem while all higher terms satisfies homogenous auxiliary condition.
From equation (11) and (12), the solution of equation (8) becomes:

$$C(x,t) = e^{-\lambda x} + \lambda e^{-\lambda x}(kx^2\lambda + sx - 1)t + \frac{1}{4}(4k^2x^2\lambda^2e^{-\lambda x} - 8k^2x^3\lambda^3e^{-\lambda x} + 2k^2x^4\lambda^4e^{-\lambda x} - 8kx^2s\lambda^2e^{-\lambda x} + 4kx^3s\lambda^3e^{-\lambda x} - 4kx^2\lambda^3e^{-\lambda x} - 2s^2x\lambda e^{-\lambda x} + 2s^2x^2\lambda^2e^{-\lambda x} - 4sx\lambda^2e^{-\lambda x} + 2\lambda^2e^{-\lambda x} - \frac{1}{2}msx)t^2 + (\frac{smx}{6}(s+\lambda))t^3 + o(p^3) \tag{19}$$

This is the solution of a reactive ADE in one dimension describing the dispersion of a continuous type, input point source discharge in a moving stream through semi-infinite medium.

3. Result and discussions:

The analytical solutions are illustrated with the help of a set of input data to understand the behavior of the concentration distribution. The different variables are assigned numerical values, as follows; λ =1.0 (1/day), k =0.01(1/day), s =0.1 (1/day) $m=1$(kg). The dispersion coefficient is considered as the square of the advection coefficient. The following graphs are obtained using Maple software for the various parameters in (19).

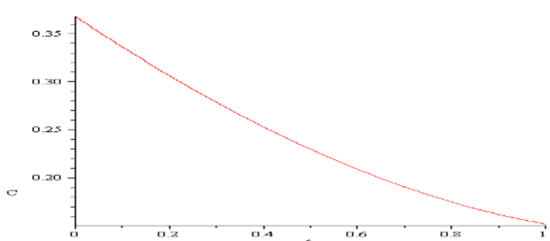

Figure1: Behaviour of concentration $C(x,t)$ against time(t) for a 1D advection-dispersion equation with reaction $(x=1)$.

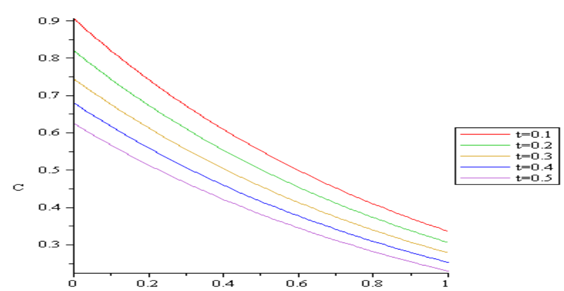

Figure2: Behaviour of concentration $C(x,t)$ against distance(x for a 1D advection-dispersion equation with reaction for time t=0.1 to 0.5.

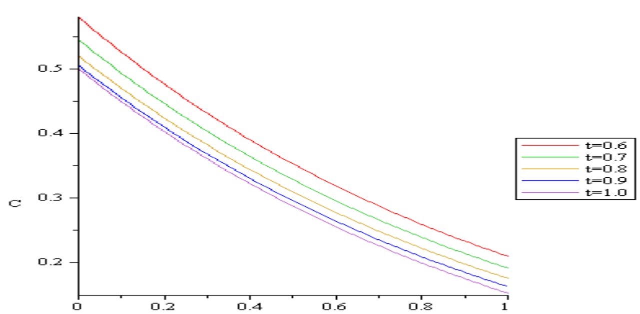

Figure3: Behaviour of concentration $C(x,t)$ against distance(x) for a 1D advection-dispersion equation with reaction for time t=0.6 to 1.0.

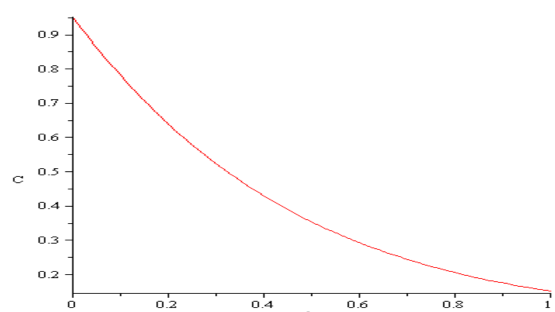

Figure4: Behaviour of concentration $C(x,t)$ against decay constant (λ) for a 1D advection-dispersion equation with reaction $(x=1, t=1)$.

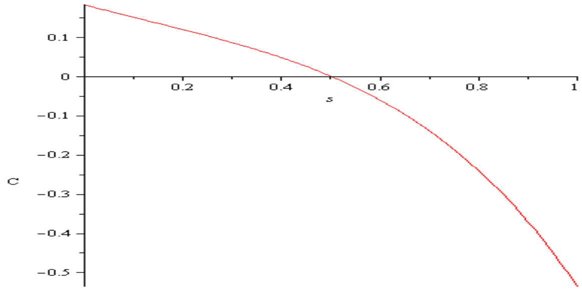

Figure5: Behaviour of concentration $C(x,t)$ against the velocity component (s) for a 1D advection-dispersion equation with reaction $(k = s^2, x = 1, t = 1)$.

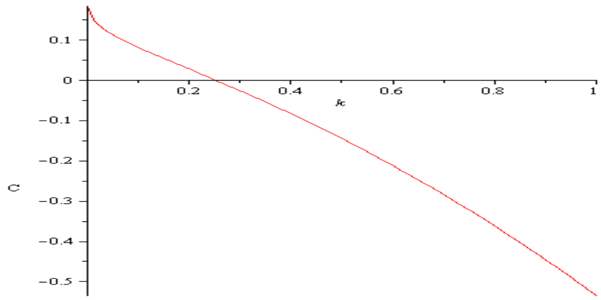

Figure 6: Behaviour of concentration $C(x,t)$ against the dispersion component (k) for a 1D advection-dispersion equation with reaction $(s = k^{0.5}, x = 1, t = 1)$.

Conclusion:

The governing solute transport equation of a reactive Advection and Dispersion Equation (ADE) in one-dimension describing the dispersion of a continuous type, input point source discharge in a moving stream, through semi-infinite medium with variable coefficients is solved analytically using He's Homotopy Perturbation Method. The Advection and Dispersion coefficients are both considered as functions of distance.

The proposed solution can be applied to field problems where the hydrological properties of the medium and prevailing boundary and initial conditions are the same or an approximation of the ones considered in this study. The behavior of the solute concentration as revealed in the graphs agree with the linearly time dependent assumption in real life. Our graphs also agree with the established fact that solute concentration disappears faster with higher decay constants.

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