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NEW CLASS OF THIRD DERIVATIVE BOUNDARY VALUE METHODS FOR STIFF IVPS

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Abstract

We present a new class of Third Derivative boundary value methods (TDBVMs) for the numerical solution of stiff initial value problems (IVPs). The proposed method is a modification of the third derivative generalized Adams-type methods (TDGAMs). The constructed methods are all $0_{v,k+1-v}$-stable and $A_{v,k+1-v}$-stable with $(v, k+1-v)$-boundary conditions for all values of the steplength $k \geq 1$ with high order up to $p = 3k+4$. Numerical results show that the class of methods is of good accuracy when applied to standard stiff problems.

1. Introduction

Let us consider the stiff initial value problem (IVP)

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0, \tag{1.1}$$

on the finite interval $I = [x_0, x_N]$ where $y : [x_0, x_N] \to R^m$ *and* $f : [x_0, x_N] \, x \, R^m \to R^m$ is continuous and differentiable.

Nwachukwu and Okor [1] developed a class of third derivative generalized backward differentiation formulas (TDGBDF) based on the linear multistep formulas (LMF) and applied as boundary value methods (BVMs) for stiff initial value problems (IVPs) in ordinary differential equations (ODEs). The TDGBDF are $A_{v,k-v}$ −stable and $0_{v,k-v}$ −stable with $(v, k-v)$ −boundary conditions for all values of $k \geq 2$ with order $p = k+2$ where k is the steplength. The formulas have the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_v f_{n+v} + h^2 \gamma_v f'_{n+v} + h^3 f''_{n+v} \tag{1.2}$$

where

$$v = \begin{cases} \frac{k+2}{2} & for\ even\ k \\ \frac{k+3}{2} & for\ odd\ k \end{cases} \tag{1.3}$$

The methods (1.2) are used alongside the following additional initial methods which are defined generally as:

$$\sum_{j=0}^{k} \alpha_j^* y_{n+j}^* = h\beta_i f_i + h^2 \gamma_i f'_i + h^3 f''_i; \qquad i = 1, 2, \dots, v-1 \tag{1.4}$$

and final methods given generally as:

$$\sum_{j=0}^{k} \alpha_j^* y_{n+j}^* = h\beta_i f_i + h^2 \gamma_i f'_i + h^3 f''_i; \qquad i = v+1, \dots, N \tag{1.5}$$

In [2] Nwachukwu and Mokwunyei constructed a family of third derivative generalized Adams-type methods (TDGAMs) which is an extension of the second derivative generalized Adams methods (SDGAMs) proposed by [3]. The methods in [2] have higher order and smaller error constants than the methods in [1] and are defined by

$$y_{n+v} - y_{n+v-1} = h\sum_{i=0}^{k} \beta_i f_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i g_{n+i} + h^3 \sum_{i=0}^{k} \phi_i r_{n+i} \tag{1.6}$$

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where v is given as

$$v=\begin{cases}\frac{k+1}{2} & for\ odd\ k\\ \frac{k}{2} & for\ even\ k\end{cases} \quad (1.7)$$

The family of methods in [2] is used with the following additional initial methods

$$y_j-y_{j-1}=h\sum_{i=0}^{k}\beta_{i,j}f_i+h^2\sum_{i=0}^{k}\gamma_{i,j}g_i+h^3\sum_{i=0}^{k}\phi_{i,j}r_i \quad j=1,\dots,v-1\ , \quad (1.8)$$

and final methods

$$y_{N+j}-y_{N+j-1}=h\sum_{i=0}^{k}\beta_{i+v,j}f_{N+i-1}+h^2\sum_{i=0}^{k}\gamma_{i+v,j}g_{N+i-1}+h^3\sum_{i=0}^{k}\phi_{i+v,j}r_{N+i-1}\ ,\ j=0,\dots,k-v-1 \quad (1.9)$$

For more details on BVMs see [1-12].

In this work we seek to improve the accuracy of the methods (1.6) by adding a future point to the first derivative of the solution. In section 2, the construction of the methods is given. Section 3 is on the stability analysis of the proposed methods. Section 4 shows the choice of the additional methods. Section 5 is devoted to numerical experiments while the conclusion is in section 6.

2. Construction of the methods

The third Derivative Boundary Value Methods (TDBVMs) for solving the IVP (1.1) is given by

$$y_{n+v}-y_{n+v-1}=h\sum_{i=0}^{k+1}\beta_i f_{n+i}+h^2\sum_{i=0}^{k}\gamma_i g_{n+i}+h^3\sum_{i=0}^{k}\phi_i m_{n+i} \quad (2.1)$$

where

$$v=\begin{cases}\frac{k+1}{2} & for\ odd\ k\\ \frac{k}{2} & for\ even\ k\end{cases} \quad (2.2)$$

and $\beta_i,\ \gamma_i\ ,\phi_i$ are parameters to be determined.

In polynomial notation, we can write the TDBVMs (2.1) as

$$\rho(E)=h\sigma(E)f_n+h^2\eta(E)g_n+h^3\delta(E)m_n, \quad (2.3)$$

$$f_{n+i}=f(x_{n+i},y_{n+i})=y'(x_{n+i})\ ,\quad g_{n+i}=g(x_{n+i},y_{n+i})=y''(x_{n+i})\ ,\quad m_{n+i}=m(x_{n+i},y_{n+i})=y'''(x_{n+i})$$

where $\rho(E)=E^{v-1}(E-1)$, $\sigma(E)=\sum_{i=0}^{k+1}\beta_i E^i$, $\eta(E)=\sum_{i=0}^{k}\gamma_i E^i$ and $\delta(E)=\sum_{i=0}^{k}\phi_i E^i$ are the first, second, third and fourth characteristic polynomials respectively. The linear difference operator associated with the TDBVMs (2.1) can be written as

$$L[y(x_n);h]=y(x_n+vh)-y(x_n+(v-1)h)-h\sum_{i=0}^{k+1}\beta_i y'(x_n+ih) \quad (2.4)$$

$$-h^2\sum_{i=0}^{k}\gamma_i y''(x_n+ih)-h^3\sum_{i=0}^{k}\phi_i y'''(x_n+ih)$$

Assuming that $y(x_n)$ is sufficiently differentiable, Taylor series expansion of (2.4) about the point x_n yields

$$L[y(x_n);h]=C_0y(x_n)+C_1hy'(x_n)+C_2h^2y''(x_n)+\cdots+C_qh^qy^q(x_n)+\cdots \quad (2.5)$$

$$where\quad \left.\begin{aligned} C_0&=\sum_{i=0}^{k}\alpha_i\\ C_1&=1-\sum_{i=0}^{k+1}\beta_i\\ C_2&=\frac{1}{2}\left(v^2-(v-1)^2\right)-\sum_{i=1}^{k+1}i\beta_i-\sum_{i=0}^{k}\gamma_i\\ &\ \vdots\\ C_q&=\frac{1}{q!}\left(v^q-(v-1)^q\right)-\frac{1}{(q-1)!}\sum_{i=1}^{k+1}i^{q-1}\beta_i\\ &-\frac{1}{(q-2)!}\sum_{i=1}^{k}i^{q-2}\gamma_i-\frac{1}{(q-3)!}\sum_{i=1}^{k}i^{q-3}\phi_i\end{aligned}\right\} \quad (2.6)$$

The TDBVMs (2.1) is said to be of order p if
$C_0 = C_1 = C_2 = \cdots = C_p = 0, \quad C_{p+1} \neq 0.$
C_{p+1} is called the error constant (EC). The coefficients are chosen so that the TDBVMs (2.1)
s of order $p = 3k + 4$. The coefficients, the error constant and the order of the TDBVMs
at different values of k are shown in Table 1.

3. Stability Analysis

To analyze the stability properties of the method, the class of TDBVMs (2.1) is applied to the test equation (3.1)

$$y' = \lambda y \tag{3.1}$$

to yield the stability polynomial

$$\pi(w,z) = w^{v} - w^{v-1} - w^{i}\sum_{i=0}^{k+1} z\beta_i - w^{i}\sum_{i=0}^{k}(z^2\gamma_i + z^3\phi_i), \quad z = \lambda h \tag{3.2}$$

where v is defined as in (2.2).

Then inserting $w = e^{t\theta}, \quad t = 0(1)k, \ \theta \in [0, 2\pi]$ in (3.2) and equating to zero, a polynomial of degree three in z is obtained with three roots describing the stability region of the TDBVMs (2.1). The stability regions for odd and even values of k are given in Figures 1 and 2 respectively. The methods (2.1) are $0_{v,k+1-v}$-stable and $A_{v,k+1-v}$-stable with $(v, k+1-v)$-boundary conditions for $k \geq 1$, see Figures 1 and 2 .

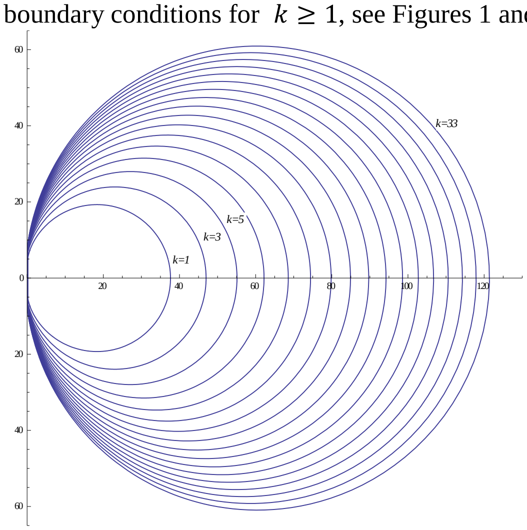

Figure 1: Stability regions (exterior of closed curves) of the TDBVMs (2.1) for k =1(2)33

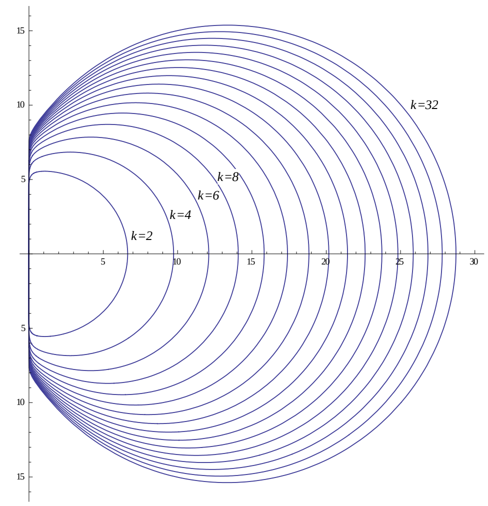

Figure 2: Stability regions (exterior of closed curves) of the TDBVMs (2.1) for k =2(2)32

Table 1: The Coefficients, Error Constant (EC) and Order p of the TDBVMs (2.1) for $k = 1(1)6$

k	β_0	β_1	β_2	β_3	β_4	β_5
1	$\frac{529}{1120}$	$\frac{37}{10}$	$-\frac{1}{1120}$			
2	$\frac{24653}{60480}$	$\frac{4369}{6720}$	$-\frac{389}{6720}$	$\frac{1}{8640}$		
3	$-\frac{8850277}{2490808320}$	$\frac{25351121}{51891840}$	$\frac{336241}{640640}$	$-\frac{1532147}{155675520}$	$\frac{673}{118609920}$	
4	$-\frac{3672432059}{2988969984000}$	$\frac{1214315581}{2717245440}$	$\frac{25616383}{42577920}$	$-\frac{4214947}{84913920}$	$\frac{279738859}{119558799360}$	$-\frac{139}{193536000}$
5	$\frac{816870806863}{10648205568000000}$	$-\frac{13004985674323}{1609062174720000}$	$\frac{9608927383141}{19308746096640}$	$\frac{10817681969}{20570397120}$	$-\frac{102136855307}{6436248698880}$	$\frac{3756449903251}{10970878464000000}$
6	$\frac{1890276114750377}{98251811868672000000}$	$-\frac{1333709114908309}{362038989312000000}$	$\frac{17958768069925399}{38617492193280000}$	$\frac{5832016531723}{10025695088640}$	$\frac{8832430287701}{185363962527744}$	$\frac{559240627672711}{120679663104000000}$

Table 2: Table 1 continued

k	β_6	β_7	γ_0	γ_1	γ_2	γ_3
1			$\frac{7}{80}$	$-\frac{4}{35}$		
2			$\frac{631}{10080}$	$-\frac{19}{160}$	$\frac{3}{140}$	
3			$-\frac{7633}{7687680}$	$\frac{1622459}{17297280}$	$-\frac{201211}{1921920}$	$\frac{163409}{51891840}$
4			$-\frac{62580263}{199264665600}$	$\frac{194904181}{2490808320}$	$-\frac{34063}{322560}$	$\frac{2766377}{155675520}$
5	$\frac{366855859}{7240779786240000}$		$\frac{28790961007}{1508495788800000}$	$-\frac{8471282549}{3155023872000}$	$\frac{30327693499}{321812434944}$	$-\frac{120790825907}{1206796631040}$
6	$-\frac{139559789721529}{2005139017728000000}$	$\frac{134627}{21123970560000}$	$\frac{9187700429513}{2027418340147200000}$	$-\frac{1264580310817}{1097087846400000}$	$\frac{108255927056579}{1287249739776000}$	$-\frac{5411279}{53887680}$

Table 3: Table 1 continued

k	γ_4	γ_5	γ_6	ϕ_0	ϕ_1	ϕ_2
1				$\frac{11}{1680}$	$\frac{1}{84}$	
2				$\frac{37}{10080}$	$\frac{319}{10080}$	$-\frac{13}{5040}$
3				$-\frac{8003}{103783680}$	$\frac{18617}{1921920}$	$\frac{39703}{2882880}$
4	$-\frac{5643877}{7970586624}$			$-\frac{437551}{19926466560}$	$\frac{8054143}{1245404160}$	$\frac{2206123}{92252160}$
5	$\frac{89552507}{16253153280}$	$-\frac{385790814583}{4022655436800000}$		$\frac{64659047}{50283192960000}$	$-\frac{1740891127}{5363540582400}$	$\frac{1806125023}{160906217472}$
6	$\frac{215701915379}{14042724433920}$	$-\frac{6395172938153}{4022655436800000}$	$\frac{256941897587}{13791961497600000}$	$\frac{2265373361}{7899032494080000}$	$-\frac{41100991}{336061440000}$	$\frac{1060746776461}{128724973977600}$

Table 4: Table 1 continued

k	v	ϕ_3	ϕ_4	ϕ_5	ϕ_6	EC	p
1	1					$-\frac{1}{470400}$	7
2	1					$\frac{59}{3353011200}$	10
3	2	$-\frac{15779}{51891840}$				$\frac{673}{21371135385600}$	13
4	2	$-\frac{2076097}{622702080}$	$\frac{1249277}{19926466560}$			$-\frac{52183}{249126378209280000}$	16
5	3	$\frac{5491769}{380933280}$	$-\frac{215444107}{268177029120}$	$\frac{237327953}{30943503360000}$		$-\frac{366855859}{674239865134024949760000}$	19
6	3	$\frac{1772742329}{83547459072}$	$-\frac{13538289871}{3677856399360}$	$\frac{408110291}{1924715520000}$	$-\frac{121990911323}{86889357434880000}$	$\frac{14698690211}{447779550432134319759360000}$	22

4. Choice of Additional Methods

The TDBVMs (2.1) are used alongside the following additional initial methods

$$y_j - y_{j-1} = h\sum_{i=0}^{k+1}\beta_{i,j} f_i + h^2\sum_{i=0}^{k}\gamma_{i,j} g_i + h^3\sum_{i=0}^{k}\phi_{i,j} m_i, \quad j = 1,\ldots v-1 \tag{4.1}$$

and final methods

$$y_{N+j} - y_{N+j-1} = h\sum_{i=0}^{k+1}\beta_{i+v,j} f_{N+i-v-1} + h^2\sum_{i=0}^{k}\gamma_{i+v,j} g_{N+i-v-1} + h^3\sum_{i=0}^{k}\phi_{i+v,j} m_{N+i-v-1} \quad (4.2)$$
$$j = 0,\ldots k - v$$

The methods (2.1) require $v - 1$ initial and $k + 1 - v$ final additional methods for the implementation since y_0 is already provided by the problem to be solved.

For $k = 2$, the TDBVMs (2.1)

$$y_n - y_{n-1} = \frac{h}{60480}(24653 f_{n-1} + 39321 f_n - 3501 f_{b+1} + 7 f_{n+2}) + \frac{h^2}{10080}(631 g_{n-1} - 1197 g_n + 216 g_{n+1}) + \frac{h^3}{10080}(37 m_{n-1} + 319 m_n - 26 m_{n+1})$$

Can be used with the following two final methods

$$y_N - y_{N-1} = \frac{h}{60480}(-1037 f_{N-2} + 34407 f_{N-1} + 27117 f_N - 7 f_{N+1}) + \frac{h^2}{10080}(-55 g_{N-2} + 1197 g_{N-1} - 792 g_N) + \frac{h^3}{10080}(-5 m_{N-2} + 193 m_{N-1} + 58 m_N)$$

and

$$y_{N+1} - y_N = \frac{h}{6720}(7717 f_{N-2} - 36623 f_{N-1} + 34555 f_N + 1071 f_{N+1}) + \frac{h^2}{1120}(431 g_{N-2} - 1701 g_{N-1} - 1880 g_N) + \frac{h^3}{10080}(373 m_{N-2} - 7073 m_{N-1} + 4342 m_N)$$

For $k = 3$, the TDBVMs (2.1)

$$y_n - y_{n-1} = \frac{h}{2490808320}(-8850277 f_{n-2} + 1216853808 f_{n-1} + 1307305008 f_n - 24514352 f_{n+1} + 14133 f_{n+2}) + \frac{h^2}{207567360}(-206091 g_{n-2} + 19469508 g_{n-1} - 21730788 g_n + 653636 g_{n+1}) + \frac{h^3}{103783680}(-8003 m_{n-2} + 1005318 m_{n-1} + 1429308 m_n - 31558 m_{n+1})$$

Can be used with the following Initial method,

$$y_1 - y_0 = \frac{h}{2490808320}(926414053 f_0 + 2221537488 f_1 - 755395632 f_2 + 98328368 f_3 - 75957 f_4) + \frac{h^2}{207567360}(10594699 g_0 - 8278212 g_1 + 20431332 g_2 - 2749764 g_3) + \frac{h^3}{103783680}(270275 m_0 + 6086394 m_1 - 3111228 m_2 + 140614 m_3)$$

and two final additional methods

$$y_N - y_{N-1} = \frac{h}{830269440}(4714231 f_{N-3} - 89756944 f_{N-2} + 578470896 f_{N-1} + 336866576 f_N - 25319 f_{N+1}) + \frac{h^2}{207567360}(344459 g_{N-3} - 8278212 g_{N-2} + 20431332 g_{N-1} - 13000004 g_N) + \frac{h^3}{34594560}(4673 m_{N-3} - 277506 m_{N-2} + 1269228 m_{N-1} + 132290 m_N)$$

$$y_{N+1} - y_N = \frac{h}{2490808320}\begin{pmatrix}-3984662117 f_{N-3} + 57213914928 f_{N-2} - 85870986192 f_{N-1} + 34775050448 f_N \\ + 357491253 f_{N+1}\end{pmatrix} + \frac{h^2}{207567360}(-98864651 g_{N-3} + 1772260548 g_{N-2} + 808538652 g_{N-1} - 1024370364 g_N) + \frac{h^3}{103783680}(-4108099 m_{N-3} + 201653766 m_{N-2} - 475975620 m_{N-1} + 84789178 m_N)$$

5. Numerical Experiments

To examine the performance of the TDBVMs (2.1) for $k = 2$ *and* 3 two stiff problems are considered. MATLAB is used to carry out all numerical computations.

Problem 1: Singularly Perturbed Problem [13]

$$y_1' = -(2 + 10^4) y_1 + 10^4 y_2^2, \qquad y_2' = y_1 - y_2 - y_2^2$$

$y_1(0) = 1, \quad y_2(0) = 1$

The exact solution is $y_1 = e^{-2x}$, $y_2 = e^{-x}$

It is obvious from the numerical results in Tables 5 and 6 that our method for $k = 2$ performs better than the methods in [1] and [2] for $k = 3$. From Tables 7 and 8, for $k = 3$ our method performs excellently compared with the methods given in [1] and [2] for $k = 4$. Details of the numerical results are given in Tables 5-8.

Problem 2: Robertson's equation [13] (nonlinear problem)

$y_1' = -0.04y_1 + 10^4 y_2 y_3 , \quad y_2' = 0.04y_1 - 3 * 10^7 y_2^2 - 10^4 y_2 y_3 , \quad y_3' = 3 * 10^7 y_2^2$

$y_1(0) = 1, \quad y_2(0) = 0, \quad y_3(0) = 0.$

This problem is solved using our methods for $k = 2$ and $k = 3$. The results are reproduced in Tables 9 and 10 and compared with the results given in [1] and [2]. It is seen from Tables 9 and 10 that our methods for $k = 2$ and $k = 3$ compare favourably with the methods in [1] and [2] for $k = 3$ and $k = 4$ respectively.

Table 5: Absolute error for problem 1 using TDBVMs (2.1) for $k = 2$, $h = 0.01$.

x	y_i	Error in TDBVMs (2.1) ($k = 2$)	Error in TDGAMs [2] ($k = 3$)	Error in TDGBDF [1] ($k = 3$)
1.0	y_1	4.625444333106188e-10	2.808114851760024e-11	1.840463059732400e-10
	y_2	9.619527396864669e-13	3.850492147350337e-11	2.638990692638288e-10
2.0	y_1	1.003195790327816e-11	3.834006029324044e-12	2.729638043375005e-11
	y_2	3.851086116668512e-13	1.430763840737370e-11	1.010463102080195e-10
3.0	y_1	1.680101370526987e-13	4.985126894618830e-13	3.416046959192620e-12
	y_2	1.365019208776630e-13	5.158935778570850e-12	3.571543755187534e-11
4.0	y_1	5.245532740810743e-15	7.308140637790617e-14	5.009221119497975e-13
	y_2	5.227762667203706e-14	1.975204722004520e-12	1.367520333084293e-11
5.0	y_1	2.913319000104331e-16	9.503370855014348e-15	6.258246687800700e-14
	y_2	1.848434599827087e-14	7.123000106412647e-13	4.833577982310544e-12
6.0	y_1	3.621658772575212e-17	1.312224288919309e-15	9.175104083338891e-15
	y_2	7.079840186330344e-15	2.646940826245281e-13	1.850751323029254e-12
7.0	y_1	4.403724292775108e-18	1.706409282132464e-16	1.146249982849842e-15
	y_2	2.503205975834533e-15	9.545532766996878e-14	6.541577175778190e-13

Table 6: Table 5 continued

X	y_i	Error in TDBVMs (2.1) ($k = 2$)	Error in TDGAMs [2] ($k = 3$)	Error in TDGBDF [1] ($k = 3$)
8.0	y_1	6.437450399132683e-19	2.356190225888307e-17	1.680476971405580e-16
	y_2	9.588141912375559e-16	3.547146300644788e-14	2.504715727359719e-13
9.0	y_1	8.039037501157855e-20	3.252517181950358e-18	2.099441635557973e-17
	y_2	3.390300193362172e-16	1.317769135998625e-14	8.853112470549873e-14
10.0	y_1	1.178897809010575e-20	4.228332224411814e-19	3.077922929287871e-18
	y_2	1.298603152094513e-16	4.750831618296342e-15	3.389787907835673e-14

Table 7: Absolute error for problem 1 using TDBVMs (2.1) for $k = 3$, $h = 0.01$.

X	y_i	Error in TDBVMs (2.1) ($k = 3$)	Error in TDGAMs [2] ($k = 4$)	Error in TDGBDF [1] ($k = 4$)
1.0	y_1	6.383782391594650e-16	6.067449320745766e-11	1.772000601807378e-10
	y_2	0.0000	1.069209720760966e-10	2.550769595544011e-10
2.0	y_1	2.567390744445675e-16	9.647916146549029e-12	2.512047514446891e-11
	y_2	5.551115123125783e-17	4.013037124828145e-11	9.573458692457848e-11
3.0	y_1	1.604619215278547e-17	1.380842922643621e-12	3.541552034275197e-12
	y_2	4.857225732735060e-17	1.506143820773076e-11	3.592972447341580e-11
4.0	y_1	7.480994990149981e-18	1.949373279984401e-13	4.989202952686289e-13
	y_2	2.775557561562891e-17	5.652738693795456e-12	1.348481395990753e-11
5.0	y_1	8.063753657860939e-19	2.746697454984043e-14	6.109777397553251e-14
	y_2	1.387778780781446e-17	2.121521708309260e-12	4.718898015398931e-12
6.0	y_1	2.168404344971009e-19	3.869097425258927e-15	8.606149511569336e-15
	y_2	1.344410693882026e-17	7.962246313664156e-13	1.771054223415058e-12
7.0	y_1	2.339928516790005e-20	5.449947625488097e-16	1.053875957754077e-15
	y_2	8.348356728138384e-18	2.988299711847997e-13	6.197570668470265e-13

Table 8: Table 7 continued

X	y_i	Error in TDBVMs (2.1) ($k = 3$)	Error in TDGAMs [2] ($k = 4$)	Error in TDGBDF [1] ($k = 4$)
8.0	y_1	5.307190810140226e-21	5.801905696940020e-17	1.484485458811755e-16
	y_2	4.282598581317743e-18	9.750165085031792e-14	2.326036498828676e-13
9.0	y_1	1.644435057755419e-21	8.172577539248416e-18	2.091033510067494e-17
	y_2	2.005774019098183e-18	3.659392396309497e-14	8.729919998701208e-14
10.0	y_1	7.279189390466644e-23	1.151190567212499e-18	2.945455330861485e-18
	y_2	9.147955830346444e-19	1.373427332149527e-14	3.276500978085378e-14

Table 9: Errors for Problem 2 using the modulus of the solution of the TDBVMs (2.1) for $k = 2$ minus the solution of Ode15s, $h = 0.0001$. Error $y_i = |y_{i(k=2)} - y_{iode15s}|$, $i = 1,2,3$.

X	y_i	Error in TDBVMs (2.1) ($k = 2$)	Error in TDGAMs [2] ($k = 3$)	Error in TDGBDF [1] ($k = 3$)
1.0	y_1	4.420997700149698e-7	4.427653609306859e-7	4.419958079537878e-7
	y_2	7.074174519826684e-11	7.095899130055931e-11	7.072967809584971e-11
	y_3	4.421707629972960e-7	4.428363210018382e-7	4.420709879965346e-7
3.0	y_1	3.911970628989181e-6	2.174455346992676e-5	3.911872170969666e-6
	y_2	4.932816919995706e-10	2.749392642898443e-9	4.932710674992948e-10
	y_3	3.912464227998069e-6	2.174730293100224e-5	3.912369955005879e-6
5.0	y_1	4.195570756926337e-6	4.195637292925269e-6	4.195654775940305e-6
	y_2	8.502273995999034e-10	8.502153913019016e-10	8.502353418992010e-10
	y_3	4.196420379992683e-6	4.196487151997275e-6	4.196500206998799e-6
10.0	y_1	7.192487407403636e-5	7.983002028799646e-5	7.192481450302157e-5
	y_2	5.640575355201477e-9	6.265406187400455e-9	5.640571071999809e-9
	y_3	7.193051470599787e-5	7.983628550597977e-5	7.193045938400089e-5

Table10: Errors for Problem 2 using the modulus of the solution of the TDBVMs (2.1) for $k = 3$ minus the solution of Ode15s, $h = 0.0001$. Error $y_i = |y_{i(k=3)} - y_{iode15s}|$, $i = 1,2,3$.

X	y_i	Error in TDBVMs (2.1) ($k = 3$)	Error in TDGAMs [2] ($k = 4$)	Error in TDGBDF [1] ($k = 4$)
1.0	y_1	4.421416069932960e-7	4.427860279543339e-7	4.419921140197403e-7
	y_2	7.074886439499203e-11	7.096249069859627e-11	7.072920270030213e-11
	y_3	4.422129029971189e-7	4.428569910022717e-7	4.420680979957958e-7
3.0	y_1	3.911993783023426e-6	2.174456425096949e-5	3.911869656980649e-6
	y_2	4.932847973002823e-10	2.749394077697717e-9	4.932708290967896e-10
	y_3	3.912487682000698e-6	2.174731371300254e-5	3.912368173000780e-6
5.0	y_1	4.195554960007009e-6	4.195630176950793e-6	4.195656818972715e-6
	y_2	8.502256262991588e-10	8.502146024025195e-10	8.502354907025610e-10
	y_3	4.196404280995547e-6	4.196480035995043e-6	4.196501482991999e-6
10.0	y_1	7.192488285001630e-5	7.983002409195361e-5	7.192481282003449e-5
	y_2	5.640576084500233e-9	6.265406495801763e-9	5.640571001499562e-9
	y_3	7.193052378498543e-05	7.983628931199083e-5	7.193045868697512e-5

6. Conclusion

In this paper, a new class of third derivative boundary value methods for the numerical solution of stiff initial value problems is presented. The inclusion of a future point in the first derivative of the solution of the TDGAMs has improved the accuracy of the methods. The numerical results displayed in Tables 5-10 show that the new TDBVMs (2.1) give a better approximation compared to the TDGAMs.

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