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THE WEIBULL-HALF LOGISTIC DISTRIBUTION

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Abstract

This paper focuses on the generalization of the Weibull distribution. We follow an approach to generate a Weibull-Half logistics distribution. Some properties of the new distribution are studied as well as the estimation of the parameters using the method of maximum likelihood. An application of the new distribution to real life data is carried out. The results show that the new distribution is more flexible than the Weibull distribution.

1. Introduction

The Weibull distribution is a very flexible and highly studied probability distribution introduced by Swedish physicist, Waloddi Weibull [1], who first published its application to fly ash and strength of material in 1951. It was further popularized by Kao [2] when he applied it to the failure of electronic components and systems, and from that time, it has been extensively used for analyzing lifetime data.

Johnson et al. [3] gave the probability density function (pdf) of two- parameters Weibull distribution as

$$f(x)=\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{\left(\frac{x}{\lambda}\right)k} \qquad (1.1)$$

Where λ >0 is the scale parameter and k>0 is the shape parameter and the corresponding cumulative distribution (CDF) to (1.1) is

$$F(x)=1-e^{-\left(\frac{x}{\lambda}\right)^k},\ x>0 \qquad (1.2)$$

However, for higher flexibility, a lot of classical distributions have been modified and extended in view of obtaining good statistical models capable of fitting real life data more effectively. In many applied areas of biology, lifetime analysis, finance and insurance, the classical families of distributions may not be able to represent the behavior of some phenomenon very well or less adequately, hence the desire and drive to derive new models. Extending the classical families of distribution has been an area of focus over the last few decades. Many generalization of the classical distribution have been undertaken by many authors in the literature [4-7]

2.0 Materials and Method

Consider the Weibull-G family as defined in [8].

$$F(x,\alpha,\beta;\varepsilon)=\int_0^{H(x;\varepsilon)}\alpha\beta t^{\beta-1}e^{-\alpha t^{\beta}}dt \qquad (2.1)$$

Where $H(x;\varepsilon)=\frac{G(x;\varepsilon)}{1-G(x;\varepsilon)}$ (2.2)

is called the odd ratio, and $G(x;\varepsilon)$ is a continuous distribution.

The Half- logistic distribution is given by

$$f(x)=\frac{2e^{-x}}{(1+e^{-x})^2}, \qquad 0<x>\infty \qquad (2.3)$$

with the cumulative function given by

$$F(x)=\frac{1-e^{-x}}{1+e^{-x}}, \qquad 0<x>\infty \qquad (2.4)$$

The distribution (2.3) has a positive domain and monotonic increasing hazard function. "It is very useful for modelling lifetime distributions" Balakrishnan and Cohen [9].

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Again considering (2.1), where $H(x;\varepsilon)=\frac{G(x;\varepsilon)}{1-G(x;\varepsilon)}$, then we have

$$F(x)=\int_0^{\frac{G(x;\varepsilon)}{1-G(x;\varepsilon)}}\alpha\beta t^{\beta-1}\exp(-\alpha t^{\beta})dt$$

$F(x)=1-\exp[-\alpha(\frac{G(x;\varepsilon)}{1-G(X;\varepsilon)})^{\beta}]$; x ε R, α , β >0

and its corresponding pdf is

$$F(x,\alpha,\beta;\varepsilon)=\alpha\beta g(x;\varepsilon)\frac{G(x;\varepsilon)^{\beta-1}}{1-G(x;\varepsilon)^{\beta+1}}\exp[-\alpha(\frac{G(x;\varepsilon)^{\beta}}{1-G(x;\varepsilon)})] \quad (2.5)$$

Where g(x;ε) is the derivative of G(x;ε).

The interpretation of the Weibull-G family of distribution is given as follows: let Y be a lifetime random variable having a certain continuous G distribution. The odd ratio that an individual (or component) following Y will die (failure) at time x is G(x;ε)/ 1-G(x;ε). Consider that the variability of the odds of death is represented by the random variable X and assume that it follows the Weibull model with scale α, and slope β. We can now write

$\Pr(Y\le x)=\Pr(x\le G(x;\varepsilon)/1-G(x;\varepsilon)$

$=F(x,\alpha,\beta;\varepsilon)$

which is the cdf given by (2.1)

Suppose we take $G(x;\varepsilon)$ to be the CDF of the Half-logistic distribution given by

$$G(x;\varepsilon)=\frac{1-e^{-x}}{1+e^{-x}}\ , \qquad x>0 \quad (2.6)$$

Olapade, [10]. Then the odd ratio

$$\frac{G(x;\varepsilon)}{1-G(x;\varepsilon)}=\frac{\frac{1-e^{-x}}{1+e^{-x}}}{1-[\frac{1-e^{-x}}{1+e^{-x}}]}$$

$$\frac{G(x;\varepsilon)}{1-G(x;\varepsilon)}=\frac{e^{x}-1}{2} \quad (2.7)$$

Substituting (2.7) into (2.1) gives the CDF of the Weibull-Half Logistic Distribution, defined by

$$F(x;\alpha,\beta)=1-\exp\left\{-\alpha\left[\frac{e^{x}-1}{2}\right]^{\beta}\right\} \quad (2.8)$$

$x>0,\beta>0,\alpha>0$

Differentiating (2.8) w.r.t. x gives the pdf of the proposed distribution given by

$$f(x;\alpha,\beta)=2^{-\beta}\alpha\beta\left(e^{x}-1\right)^{\beta+1}\exp\left\{-\alpha\left[\frac{e^{x}-1}{2}\right]^{\beta}\right\} \quad (2.9)$$

$x>0,\beta>0,\alpha>0$

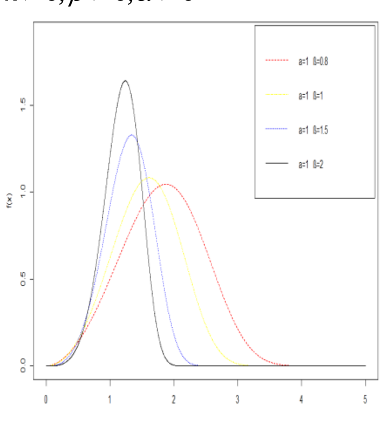

Figure 1: PDF of Weibull- Half logistic distribution.

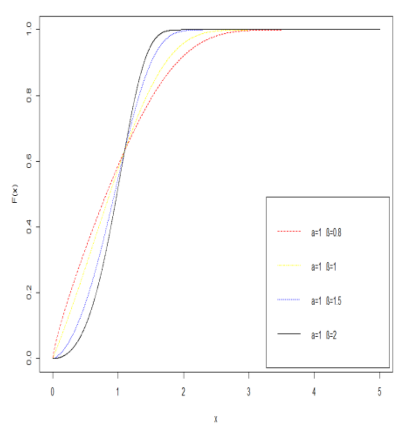

Figure 2: CDF of the Weibull- half logistic distribution.

2.1 PROPERTIES OF THE WEIBULL-HALF LOGISTIC DISTRIBUTION

(a) Maximum Likelihood Estimation of the Parameters of Weibull-Half Logistic Distribution

For a random independent sample $X_1, X_2, \ldots, X_n$ of size n, the maximum likelihood estimates of the parameters of the Weibull-Half Logistic distribution is obtain by maximizing the log-likelihood function

$$L = \sum_{i=0}^{n} \log\left(f(x_i) \right)$$

$$L = \sum_{i=1}^{n} \log\left[\alpha\beta \left[\frac{e^x - 1}{2} \right]^{\beta-1} \left(e^x - 1\right)^{\beta+1} \exp\left\{ -\alpha \left[\frac{e^x - 1}{2} \right]^{\beta} \right\} \right]$$

$$= \sum_{i=1}^{n} \left(\log\alpha + \log\beta + \beta \log\left[\frac{e^{x_i} - 1}{2} \right] + \log\left(e^{x_i} - 1\right)^{\beta+1} - \alpha \left[\frac{e^{x_i} - 1}{2} \right]^{\beta} \right)$$

$$L = n\log\alpha + n\log\beta + \beta\sum_{i=1}^{n} \log\left(\frac{e^{x_i} - 1}{2} \right) + \sum_{i=1}^{n} \log\left(\frac{e^{x_i} - 1}{2} \right) - \alpha \sum_{i=1}^{n} \left[\frac{e^{x_i} - 1}{2} \right]^{\beta}$$

To maximize the function L, we take the partial derivative of L w.r.t. α and β and set the derivative to zero as follows;

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \left(\frac{e^{x_i} - 1}{2} \right)^{\beta}$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log\left(\frac{e^{x_i} - 1}{2} \right) - \alpha \sum_{i=1}^{n} \left(\frac{e^{x_i} - 1}{2} \right) \log\left(\frac{e^{x_i} - 1}{2} \right)$$

Setting the derivatives to zero we have

$$\Rightarrow \frac{n}{\alpha} = \sum_{i=1}^{n} \left(\frac{e^{x_i} - 1}{2} \right)^{\beta}$$

$$\alpha = \frac{n}{\sum_{i=1}^{n} \left(\frac{e^{x_i} - 1}{2} \right)^{\beta}} \qquad (2.10)$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log\left(\frac{e^{x_i} - 1}{2} \right) - \alpha \sum_{i=1}^{n} \left(\frac{e^{x_i} - 1}{2} \right) \log\left(\frac{e^{x_i} - 1}{2} \right) = 0 \qquad (2.11)$$

Substituting the value of α in (2.1.1) into the expression above gives

$$\frac{n}{\beta} + \sum_{i=1}^{n} \log\left(\frac{e^{x_i} - 1}{2} \right) - \frac{n}{\sum_{i=1}^{n} \left(\frac{e^{x_i} - 1}{2} \right)^{\beta}} \sum_{i=1}^{n} \left(\frac{e^{x_i} - 1}{2} \right) \log\left(\frac{e^{x_i} - 1}{2} \right) = 0 \qquad (2.12)$$

The estimate $\hat{\beta}$ of β can be obtained numerically by finding the root of the expression (2.12) above, and using the estimate, we obtain the value of α from (2.10).

(b) Survival Function

The survival function of the Weibull-Half Logistic distribution is given by $S(x) = 1 - F(x; \alpha, \beta)$

$$S(x) = \exp\left\{ -\alpha \left[\frac{e^x - 1}{2} \right]^{\beta} \right\} \qquad (2.13)$$

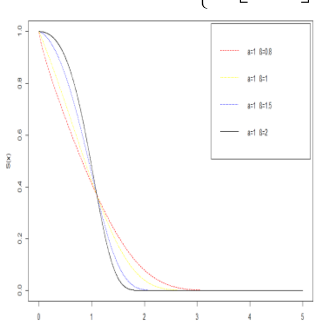

Figure 3: Survival function of the Weibull- half logistic distribution

(c) Hazard Function

The hazard function is given by

$$h(x) = \frac{f(x;\alpha,\beta)}{1-F(x;\alpha,\beta)}$$

$$h(x) = \alpha\beta\left[\frac{e^x-1}{2}\right]^{\beta}\left(e^x-1\right) \quad (2.14)$$

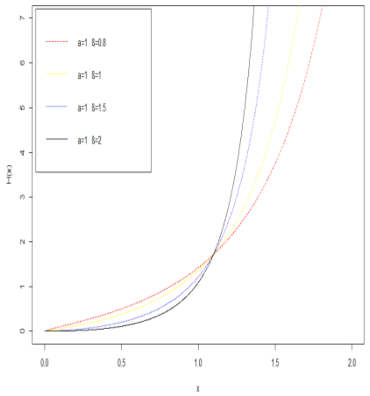

Figure 4: Hazard function of the Weibull- half logistic distribution.

(d) Quantile Function

The quantile function of the distribution is obtained from the CDF by solving for x in the equation $F(x;\alpha,\beta) = p$.

$$1-\exp\left\{-\alpha\left[\frac{e^x-1}{2}\right]^{\beta}\right\} = p$$

$$\Rightarrow \ln\left[2\left(-\frac{\ln(1-p)}{\alpha}\right)^{\frac{1}{\beta}}+1\right] = x$$

Thus, the quantile function of the Weibull-Half Logistic distribution is given as

$$Q(p) = \ln\left[2\left(-\frac{\ln(1-p)}{\alpha}\right)^{\frac{1}{\beta}}+1\right] \quad (2.15)$$

(e) Median of the Weibull-Half Logistic Distribution

The median of the Weibull-Half Logistic distribution is obtained by setting $p = 0.5$ in the quantile function given in (2.15). Thus the median m of the Weibull-Half Logistic distribution is given as

$$m = Q(0.5) = \ln\left[2\left(-\frac{\ln(1-0.5)}{\alpha}\right)^{\frac{1}{\beta}}+1\right]$$

$$m = Q(0.5) = \ln\left[2\left(-\frac{0.693147}{\alpha}\right)^{\frac{1}{\beta}}+1\right] \quad (2.16)$$

(f) Moments of the Weibull-Half Logistic Distribution

The r^{th} non-central moment of a random variable X is expressed as

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx$$

where the random variable X is continuous with a probability density function $f(x)$ for the Weibull-Half Logistic distribution, we express the relation for calculating the r^{th} moments as follows;

Consider the pdf of the Weibull-G family given by

$$f(x;\alpha,\beta,\varepsilon) = \alpha\beta g(x;\varepsilon)\frac{G(x;\varepsilon)^{\beta-1}}{1-G(x;\varepsilon)^{\beta+1}}\exp\left\{-\alpha\left[\frac{G(x;\varepsilon)}{1-G(x;\varepsilon)}\right]^{\beta}\right\}$$

for the Weibull-Half Logistic distribution we have

$$g(x)=\frac{2e^{-x}}{\left(1+e^{-x}\right)^2}$$

which is the PDF of the Half Logistic distribution, and

$$G(x)=\frac{1-e^{-x}}{1+e^{-x}},$$

Which is the CDF of the half-Logistic distribution. Thus, we have that

$$f(x;\alpha,\beta)=\alpha\beta\left[\frac{2e^{-x}}{\left(1+e^{-x}\right)^2}\right]\left[\frac{\left(\frac{1-e^{-x}}{1+e^{-x}}\right)^{\beta-1}}{\left(\frac{2e^{-x}}{1+e^{-x}}\right)^{\beta+1}}\right]\exp\left\{-\alpha\left[\frac{\left(\frac{1-e^{-x}}{1+e^{-x}}\right)^{\beta-1}}{\left(\frac{2e^{-x}}{1+e^{-x}}\right)^{\beta+1}}\right]^{\beta}\right\} \qquad (2.17)$$

where

$$1-G(x)=1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)=\frac{2e^{-x}}{1+e^{-x}}$$

Now, consider the term

$$\exp\left\{-\alpha\left[\frac{\left(\frac{1-e^{-x}}{1+e^{-x}}\right)^{\beta-1}}{\left(\frac{2e^{-x}}{1+e^{-x}}\right)^{\beta+1}}\right]^{\beta}\right\}$$

where $e^{-\alpha}$ is expressed as

$$e^{-\alpha}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\alpha^k$$

which implies that

$$\exp\left\{-\alpha\left[\frac{\left(\frac{1-e^{-x}}{1+e^{-x}}\right)^{\beta-1}}{\left(\frac{2e^{-x}}{1+e^{-x}}\right)^{\beta+1}}\right]^{\beta}\right\}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\alpha^k\left[\frac{\left(\frac{1-e^{-x}}{1+e^{-x}}\right)^{\beta-1}}{\left(\frac{2e^{-x}}{1+e^{-x}}\right)^{\beta+1}}\right]^{\beta k}$$

Substituting into (2.17) gives

$$f(x;\alpha,\beta)=\alpha\beta\left[\frac{2e^{-x}}{\left(1+e^{-x}\right)^2}\right]\left[\frac{\left(\frac{1-e^{-x}}{1+e^{-x}}\right)^{\beta-1}}{\left(\frac{2e^{-x}}{1+e^{-x}}\right)^{\beta+1}}\right]\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\alpha^k\left[\frac{\left(\frac{1-e^{-x}}{1+e^{-x}}\right)^{\beta-1}}{\left(\frac{2e^{-x}}{1+e^{-x}}\right)^{\beta+1}}\right]^{\beta k}$$

$$=\alpha\beta\left[\frac{2e^{-x}}{\left(1+e^{-x}\right)^2}\right]\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\alpha^k\left(\frac{1-e^{-x}}{1+e^{-x}}\right)^{\beta(k+1)-1}\left(\frac{2e^{-x}}{1+e^{-x}}\right)^{-[\beta(k+1)+1]} \qquad (2.18)$$

Also, the term

$$\left(\frac{2e^{-x}}{1+e^{-x}}\right)^{-[\beta(k+1)+1]}=\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right]^{-[\beta(k+1)-1]}$$

where

$$\left|\left(\frac{1-e^{-x}}{1+e^{-x}}\right)\right|<1$$

The Generalized Binomial theorem holds that

$$(1-Z)^{-b}=\sum_{j=0}^{\infty}\frac{\Gamma(b+j)}{j!\Gamma(b)}Z^j, \quad |Z|<1$$

where

$$\Gamma(n)=\int_0^{\infty}t^{n-1}e^{-t}dt,\ t>0$$

is the complete gamma function. Thus we have

$$\left(\frac{2e^{-x}}{1+e^{-x}}\right)^{-[\beta(k+1)+1]}=\sum_{j=0}^{\infty}\frac{\Gamma(\beta(k+1)+j+1)}{j!\Gamma(\beta(k+1)+1)}\left(\frac{1-e^{-x}}{1+e^{-x}}\right)^j$$

Substituting into (2.18) gives

$$f(x;\alpha,\beta)=\alpha\beta\left[\frac{2e^{-x}}{\left(1+e^{-x}\right)^{2}}\right]\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!}\alpha^{k}\left(\frac{1-e^{-x}}{1+e^{-x}}\right)^{\beta(k+1)-1}$$
$$\times\sum_{j=0}^{\infty}\frac{\Gamma(\beta(k+1)+j+1)}{j!\Gamma(\beta(k+1)+1)}\left(\frac{1-e^{-x}}{1+e^{-x}}\right)^{j}$$

Which implies that

$$f(x;\alpha,\beta)=\beta\left[\frac{2e^{-x}}{\left(1+e^{-x}\right)^{2}}\right]\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!j!}\alpha^{k+1}$$
$$\times\frac{\Gamma(\beta(k+1)+j+1)}{j!\Gamma(\beta(k+1)+1)}\left(\frac{1-e^{-x}}{1+e^{-x}}\right)^{\beta(k+1)-1} \quad (2.19)$$

which we can write in mixture representation as

$$f(x;\alpha,\beta)=\sum_{k=0}^{\infty}W_{j,k}h_{p}(x) \quad (2.20)$$

where

$$W_{j,k}=\frac{(-1)^{k}\alpha^{k+1}\beta\Gamma(\beta(k+1)+j+1)}{k!j![\beta(k+1)+j-1]\Gamma(\beta(k+1)+1)}$$

and

$$h_{p}(x)=p\left(\frac{2e^{-x}}{\left(1+e^{-x}\right)^{2}}\right)\left(\frac{1-e^{-x}}{1+e^{-x}}\right)^{p}$$

with

$$p=\beta(k+1)+j-1$$

The expression given in (2.20) is another way of representing the PDF of Weibull-Half Logistic distribution. The r^{th} moment of Weibull-Half distribution can then be evaluated, from this representation in particular,

$$\mu_r'=E(X^r)=\sum_{k,j=0}^{(X)}W_{j,k}E(Z_{j,k}^{r})$$

$$E(Z_{j,k}^{r})=\int_{0}^{(X)}Z_{j,k}^{r}h_{p}(x)dx$$

Thus

$$\mu_r'=\sum_{k,j=0}^{(X)}W_{j,k}\int_{0}^{(X)}Z_{j,k}^{r}h_{p}(x)dx \quad (2.21)$$

Obviously, the r^{th} moment of the Weibull-Half Logistic distribution expressed in (2.21) is not in closed form but numerical approximations implemented in many software packages is possible.
The mean of the Weibull-Half Logistic distribution is given by

$$E(X)=\mu_1'$$

The variance is

$$Var(x)=\mu_2'-(\mu_1')^{2}$$

3.0 APPLICATION AND RESULTS

In this paper, we present an application of the proposed Weibull-half logistic distribution to a real data set. We further compare the fit of the proposed distribution with the Weibull distribution, with probability density function

$$f(x)=\frac{\beta}{\alpha}\left(\frac{x}{c}\right)^{\beta-1}\exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right), \quad (3.1)$$

$x,\alpha,\beta>0.$

For the analysis, the strength of 1.5cm glass fibres, obtained by workers at the UK National Physical Laboratory is used. The data set was also used in [8] to test the flexibility of the Weibull-exponential distribution. The proposed Weibull-half logistic distribution and the Weibull distribution are used to fit the data. The maximum likelihood method is used to estimate the parameters of the distributions and the Akaike information criterion (AIC) and the Kolmogorov-Smirnov (k-s) statistic is used to assess the performance of the distributions. The model with the smallest AIC value is taken as the best model for the data. The K-S value obtained from the fitted distribution is compared to the critical value at $\theta = 0.05$ level of significance. The R package **fitdistrplus** used for maximum likelihood fits of distributions, reports both the k-s statistics and AIC value. The critical k-s statistic value is computed using

$$critical\ value(k-s) = \frac{1.35}{\sqrt{n}}, \qquad (3.2)$$

where n is the sample size of the data. When the reported k-s statistic value from the maximum likelihood fit of the data is smaller than the critical value computed using (3.2), it is taken that the distribution used in fitting the data, is a good model for the data.

Breaking strength of 1.5cm glass fibres.

0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24

Source: [8]

3.1 ANALYSIS AND RESULTS

The maximum likelihood fit of the two distributions to the data are contained in Table 1.The density plot of the fitted distributions is presented in Figure 5. We also present the P-P plots in Figure 6 and 7.

Table1: Maximum Likelihood estimates of the parameters of the Weibull-half logistic distribution and the Weibull distribution

Distribution	Weibull-half logistic	Weibull
Parameter Estimates	$\alpha = 0.1179$ (0.0324) $\beta = 2.9099$ (0.2768)	$\alpha = 1.6281$ (0.0371) $\beta = 5.7806$ (0.5761)
Log-Likelihood	-14.4022	-15.2068
AIC	32.8044	34.4137
k-s statistic	0.1367	0.1522

(Standard error of estimates in parenthesis)

The critical value for the k-s statistic is given as

$$critical\ value(k-s) = \frac{1.35}{\sqrt{63}} = 0.1701$$

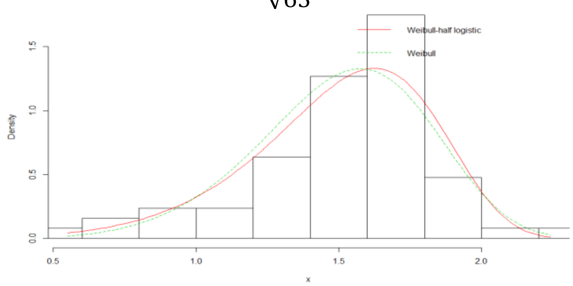

Figure 5:Density plot of fitted distributions.

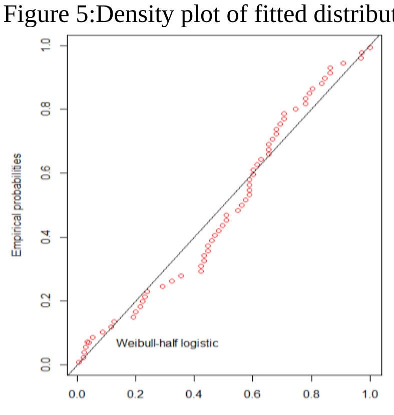

Figure 6: P-P plots of the fitted Weibull Half-logistic

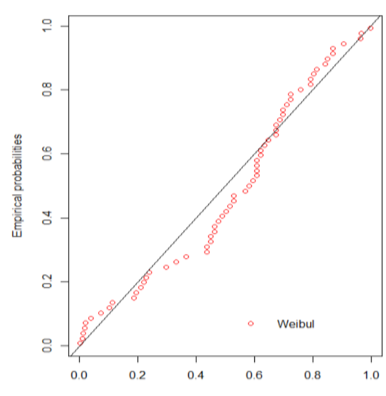

Figure 7: p-p plots of the fitted distribution. Weibull distribution.

3.2 DISCUSSION OF RESULTS

Results from the analysis clearly show that the fit of the proposed Weibull-half logistic distribution was better than that of the Weibull distribution in fitting the data. This is sustained by the fact that the AIC and the k-s statistic value of the Weibull-half logistic distribution is smaller than that of the Weibull distribution. The k-s statistic further suggest the two distributions as good models for the data, since the critical k-s statistic value is larger than the empirical ones obtained from the estimates of the parameters of the distributions. The density plot and the P-P plots further suggest the superiority of the Weibull-half logistic distribution over the Weibull distribution in fitting the data.

4.0 CONCLUSION

In this study a new probability distribution has been proposed by combining the Weibull distribution and the half logistic distribution. The goal was to add more flexibility to the Weibull distribution when fitting real data. Some properties of the new distribution have been investigated and a real data set has been used to assess the performance of the new distribution, in comparison with the Weibull distribution

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